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Autoprojectivities and Automorphisms of Certain Algebraic Groups (**)

SUMMARY. — We prove that the group of autoprojectivities of a simple algebraic group G over the algebraic closure of a finite field acts in a natural way on the building $\Delta(G)$ canonically associated to G, and that for every autoprojectivity φ of G there exists a unique automorphism α of G acting on $\Delta(G)$ in the same way as φ.

Autoproiettività ed automorfismi di certi gruppi algebrici

SOMMARIO. — Si dimostra che il gruppo delle autoproiettività di un gruppo algebrico semplice G sulla chiusura algebrica di un campo finito opera in modo naturale sul building $\Delta(G)$ canonicamente associato a G, e che per ogni autoproiettività φ di G esiste un unico automorfismo α di G che opera su $\Delta(G)$ come φ.

1. - INTRODUCTION-NOTATION

In this paper we are concerned with the group of lattice automorphisms of a simple algebraic group G over the algebraic closure of a finite field. Following Metelli [9, 10] and Völklein [16], the main idea is to show that every lattice automorphism of φ of G induces an automorphism of the building $\Delta(G)$ canonically associated to G, then to show that there exists an (abstract) automorphism of G inducing φ on $\Delta(G)$. We shall then study in a forthcoming paper the group of lattice automorphisms of G fixing every face of $\Delta(G)$ (i.e. every parabolic subgroup of G).

Given a group G, the set $\mathcal{L}(G)$ of all subgroups of G partially ordered by inclusion is well known to be a complete algebraic lattice. A *projectivity* of a group G onto a group $\overline{G}$ is any lattice isomorphism form $\mathcal{L}(G)$ onto $\mathcal{L}(\overline{G})$, and an *autoprojectivity* of G is any projectivity of G onto itself. We shall denote by $\mathrm{Aut}\,\mathcal{L}(G)$ the group of all autoprojectivities of G. Two groups G, $\overline{G}$ will be called *projective* if there exists a projectivity of G onto $\overline{G}$. We shall use the usual abuse of notation φ: $G \to \overline{G}$ to denote a projectivity φ of G onto $\overline{G}$. φ is said to be *index-preserving* if given $H \leq K \leq G$ and $[K: H] =$

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$= n$, we have $[K^{\varphi}: H^{\varphi}] = n$. It is well known ([18] Corollario 3) that to prove that φ is index-preserving it is enough to prove that $H \leq K \leq G$, K cyclic and $[K: H] = n$ implies $[K^{\varphi}: H^{\varphi}] = n$.

Let G, $\overline{G}$ be groups, and let α be an isomorphism of G onto $\overline{G}$. We can define in a natural way the projectivity α^* of G onto $\overline{G}$ given by $X^{\alpha^*} = X^{\alpha}$ for every $X \leq G$. α^* is called the projectivity *induced* by the isomorphism α. If $\overline{G} = G$, then we have a homomorphism $*: \operatorname{Aut} G \to \operatorname{Aut} \mathcal{L}(G)$ given by $\alpha \mapsto \alpha^*$ for every α in $\operatorname{Aut} G$.

An interesting problem is to study in which cases a projectivity of G onto a group $\overline{G}$ is induced by an isomorphism. A group G is said to be *strongly lattice determined* if every projectivity of G onto a group $\overline{G}$ is induced by an isomorphism. It is clear that G is strongly lattice determined if and only if the following two conditions are satisfied:

G projective to $\overline{G}$ implies G isomorphic to $\overline{G}$,

the homomorphism $*: \operatorname{Aut} G \to \operatorname{Aut} \mathcal{L}(G)$ is surjective.

Studying the latter problem for a finite simple group G of Lie type, Völklein ([16]) showed that the answer is positive if the absolute Weyl group of G has non-trivial center (and the characteristic of the base field is sufficiently large). The problem whether for a finite simple group of Lie type for which the absolute Weyl group has trivial center, the map $*$ is surjective, seems to be much harder. For the groups $PSL_3(q)$ and $PSU_3(q^2)$ $*$ is not in general surjective ([16], [7]). It is in this connection that we are going to study the behaviour of the map $*$ for simple algebraic groups over the algebraic closure of a finite field.

For algebraic groups, we use the standard notation ([3], [4], [12]). For every root α, X_α is the *root subgroup* corresponding to α, and x_α is a fixed algebraic isomorphism $x_\alpha: (K, +) \to X_\alpha$.

Let p be any prime. We shall always denote by K the algebraic closure of the field F_p with p elements. Also, for every natural n, we shall denote by K_n the unique subfield of K of order $p^{n!}$. Hence we have $K_n \leq K_{n+1}$ for every n in N, and $\bigcup_{n \in \mathrm{N}} K_n = K$.

The main result of this paper (Theorem 4.4, Corollary 4.9) is that *if G is a simple algebraic group over $\overline{F}_p$ then for every autoprojectivity φ of G there exists a unique automorphism α of G acting on the building associated to G in the same way as φ.*

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2. - The action of $\operatorname{Aut} \mathcal{L}(G)$ on $\Delta(G)$

Our aim is too show that if G is a simple algebraic group over K, then the group $\operatorname{Aut} \mathcal{L}(G)$ acts in a natural way on the building $\Delta(G)$. We make use of the fact that every projectivity of G onto a group $\overline{G}$ is index-preserving. We prove the result in the more general context of *reductive* algebraic groups.

Let G be a simple algebraic group over K. Every simple algebraic group can be defined over the prime subfield. Hence, without loss of generality, we may assume G defined over F_p, so that we can consider the finite subgroup $G(K_n)$ of the K_n-rational points of G for every n in $\mathbb{N}$. There exists $\bar{n}$ such that the derived group $G(K_n)'$ is *perfect* for every $n \geq \bar{n}$ ([14] Main theorem and Proposition 1,4). We define $L_n = G(K_{\bar{n}+n-1})'$ for every n in $\mathbb{N}$. We get a family of finite perfect subgroups of G such that $L_n \leq L_{n+1}$ for every n in $\mathbb{N}$. Also, if x is in G, there exists elements $a_1, \dots, a_r$, $b_1, \dots, b_r$ in G such that $x = [a_1, b_1] \dots [a_r, b_r]$ as G is perfect. From the well known fact that $\bigcup_{n \in \mathbb{N}} G(K_n) = G$, there exists M in $\mathbb{N}_0$ such that $a_1, \dots, a_r$, $b_1, \dots, b_r$ are all in $G(K_{\bar{n}+M})$. But then x lies in $G(K_{\bar{n}+(M+1)-1})' = L_{M+1}$, and so $\bigcup_{n \in \mathbb{N}} L_n = G$.

Suppose now that G is connected semisimple. Then we have $G = N_1 \dots N_t$, where N_i's are simple algebraic groups with $[N_r, N_s] = \{1\}$ for every r, s in $\{1, \dots, t\}$, $r \neq s$. From the previous discussion, for every $i = 1, \dots, t$ we have a family $(H_{i,j})_{j \in \mathbb{N}}$ of finite perfect subgroups of N_i, such that $H_{i,j} \leq H_{i,j+1}$ for every j in $\mathbb{N}$ and $\bigcup_{j \in \mathbb{N}} H_{i,j} = N_i$. Let us denote by H_j the product $H_{1,j} \dots H_{t,j}$, for every j in $\mathbb{N}$. Then H_j is a finite perfect subgroup of G. We have $H_n \leq H_{n+1}$ for every n and $\bigcup_{n \in \mathbb{N}} H_n = G$.

We finally assume that G is a connected reductive non-commutative algebraic group over K. Then we have $G = TG'$, where T is the connected component of the centre $Z(G)$ of G, and the derived subgroup G' of G is semisimple. For G' we consider the family $(H_j)_{j \in \mathbb{N}}$ above constructed. On the other hand T is a torus, and so it is isomorphic to the direct product of a finite number, k say, of copies of the multiplicative group $K^\times$ of K. Let n be any natural number coprime to p, and let D_n be the unique subgroup of $K^\times$ of order n. For every j in $\mathbb{N}$, let $p^{s} r_j$ be the cardinality of H_j, where $(p, r_j) = 1$. Let T_j be the unique subgroup of T isomorphic to the subgroup $D_{r_j} \times \dots \times D_{r_j}$ of $K^\times \times \dots \times K^\times$ (k copies). T_j is a finite subgroup of $Z(G)$, so that $G_j = T_j H_j$ is a finite subgroup of G.

Proposition 2.1: *Let G be a connected reductive non-commutative algebraic group over K, and let $(G_j)_{j \in \mathbb{N}}$ be the family of subgroups of G defined above. Then we have $G_j \leq G_{j+1}$ for every n in $\mathbb{N}$ and $\bigcup_{j \in \mathbb{N}} G_j = G$.*

Proof: It is clear that $G_j \leq G_{j+1}$ for every j in $\mathbb{N}$. To show that $\bigcup_{j \in \mathbb{N}} G_j = G$, we only need to show that $\bigcup_{j \in \mathbb{N}} G_j \geq T$. T is a torsion group whose p-component is the identity. Hence we only need to show that for every natural number r coprime to p, there exists j in $\mathbb{N}$ such that $r \mid |H_j|$. For then we get $r \mid r_j$, and so if t is an element of order r of T then, by construction, t lies in T_j. Now as G is non-commutative, G' contains a one-dimensional torus S, and so it contains an element s of order r. But then there exists j in $\mathbb{N}$ such that s lies in H_j, so that $t \mid |H_j|$, as we required. ∎

Proposition 2.2: *Let j be any natural number, and let φ be any projectivity of the group G_j defined above. Then φ is index-preserving.*

Proof: Suppose, for a contradiction, that φ is not index-preserving. Then there exists a Sylow r-subgroup R of G_j and a normal complement N of R in G_j, with R

cyclic or elementary abelian ([13] Theorem 8 page 45). In particular the group G_j/N is abelian, and so $N \geqslant G_j'$. Hence $N \geqslant H_j$, and so $r \nmid |H_j|$. Thus we must have $r \mid |T_j|$. But this is a contradiction, because every prime divisor of $|T_j|$ is also a divisor of $|H_j|$ by construction. Therefore φ is index-preserving. ■

We can now prove the following

Theorem 2.3: *Let G be a connected reductive algebraic group over K. Then every projectivity φ of G onto a group $\overline{G}$ is index-preserving if and only if G is not a 1-dimensional torus.*

Proof: Let φ be a projectivity of G onto a group $\overline{G}$. Suppose G non-commutative. As G is a torsion group, it is enough to show that $|\langle g\rangle^{\varphi}| = |\langle g\rangle|$ for every g in G ([18] Corollario 3). Then we are done by 2.1, 2.2. Suppose now that G is abelian, hence a torus of dimension k, say. For every prime $q \neq p$ the q-component G_q of G is isomorphic to $C_{q^\infty} \times \ldots \times C_{q^\infty}$ (k copies) (C_{q^∞} denotes the Prüfer group relative to q). Therefore, if $k \geqslant 2$, every projectivity of G_q is index-preserving ([13] ch. I, Th. 18, ch. II, Prop. 2.1), and so is every projectivity of G. Suppose finally that $k = 1$. Then $G \cong \mathrm{Dr}\, C_{q^\infty}$ (product over all primes q different from p). Hence $\mathcal{L}(T) \cong \mathrm{Cr}\, \mathcal{L}(C_{q^\infty})$ ([13] Theorem 4 page 5), where $\mathrm{Cr}\, \mathcal{L}(C_{q^\infty})$ denotes the cartesian product of the lattices $\mathcal{L}(C_{q^\infty})$. For every q, C_{q^∞} has only the trivial autoprojectivity. Therefore, for every permutation σ of the set of all primes different from p, we have a unique autoprojectivity φ_σ of $\mathrm{Dr}\,(C_{q^\infty})$ (product over all primes q different from p) such that $C_{q^\infty}^{\varphi_\sigma} = C_{\sigma(q)^\infty}$, and these are all the autoprojectivities of $\mathrm{Dr}\, C_{q^\infty}$ (thus $\mathrm{Aut}\, \mathcal{L}(\mathrm{Dr}\, C_{q^\infty}) \cong S_{\mathbb{N}}$). In particular every non-trivial autoprojectivity of $\mathrm{Dr}\, C_{q^\infty}$ is not index-preserving, and so the same holds for G. ■

We recall that if G is an algebraic group over K, then an element x of G is *unipotent* (resp. *semisimple*) if and only if x has order a power of p (resp. x has order coprime to p). We have the following

Proposition 2.4: *Let G be a connected reductive algebraic group over K, and let φ be an autoprojectivity of G. Let x, x_1 be elements of G such that $\langle x\rangle^{\varphi} = \langle x_1\rangle$. Then x is unipotent (resp. semisimple) if and only if x_1 is unipotent (resp. semisimple).*

Proof: If G is abelian, then the result is obvious. If G is non-commutative, then it follows from 2.3. ■

To prove that the image of a Borel subgroup of G under an autoprojectivity of G is still a Borel subgroup of G, we first consider the behaviour of maximal unipotent subgroups and of maximal tori of G. φ denotes always an autoprojectivity of G. The following result is immediate.

Proposition 2.5: *If U is a unipotent subgroup of G, then U^{φ} is unipotent and U is maximal unipotent if and only if U^{φ} is maximal unipotent.* ■

LEMMA 2.6: *Let A be an abelian subgroup of G such that each element of A is semisimple and such that A has no proper subgroup of finite index. Then the closure* $\mathrm{cl}(A^\varphi)$ *of A^φ in G is a torus of G.*

PROOF: Let C be the closure of A^φ in G. We show that C is connected. As the connected component C^0 of C has finite index in C, it follows that $A^\varphi \wedge C^0$ has finite index, n say, in A^φ. Suppose that $n > 1$. Then there exists a maximal subgroup M_1 of A^φ containing $A^\varphi \wedge C^0$. Hence $M = M_1^{\varphi^{-1}}$ is a maximal subgroup of A. But A is abelian, and so M must have finite index in A, which is a contradiction, as A has no proper subgroup of finite index by hypothesis. So we get $n = 1$, and C is connected. C is soluble because the projective image of a soluble group is soluble ([17]) and the closure of a soluble subgroup is soluble ([1] Cor. 2 on page 110). Hence we get $C = C_u \rtimes T$, where C_u is the set of all unipotent elements of C, and T is a maximal torus of C. We have $A^\varphi \wedge C_u = \{1\}$, as A^φ consists only of semisimple elements by 2.4. Hence we get $A^\varphi \cong A^\varphi C_u/C_u \leqslant TC_u/C_u \cong T$, which implies that A^φ is abelian (we could also get this result from the structure of locally finite modular groups). Therefore C is itself abelian ([1] (2.1) d). So, if we denote by C_s the set of all semisimple elements of C, we get $C = C_u \times C_s$, where C_s is a torus. As the elements of A^φ are semisimple, we have $A^\varphi \leqslant C_s$, and so $C \leqslant C_s$. Hence $C = C_s$, and C is a torus. ∎

PROPOSITION 2.7: *If T is a maximal torus of G, then T^φ is a maximal torus of G.*

PROOF: T is a divisible group, hence T has no proper subgroup of finite index. But 2.6 the closure T_1 of T^φ in G is a torus of G. Again by 2.6, $\mathrm{cl}(T_1^{\varphi^{-1}})$ is a torus of G, and it contains T. But T is a maximal torus of G, thus we have $\mathrm{cl}(T_1^{\varphi^{-1}}) = T$. Also $T = \mathrm{cl}(T_1^{\varphi^{-1}}) \geqslant T_1^{\varphi^{-1}} \geqslant T$ gives $T_1^{\varphi^{-1}} = T$, so that $T_1 = T^\varphi$. Hence T^φ is closed in G, and it is a torus. Now suppose S is a maximal torus of G containing T^φ. Then $S^{\varphi^{-1}}$ is a torus of G containing T, and so we must have $S^{\varphi^{-1}} = T$, as T is maximal. Hence we get $S = T^\varphi$, and T^φ is a maximal torus of G. ∎

We finally consider the behaviour of Borel subgroups under autoprojectivities.

THEOREM 2.8: *Let B be a Borel subgroup of G. Then B^φ is a Borel subgroup of G.*

PROOF: We have $B = UT$, where U is the unipotent radical of B, and T is any maximal torus of B. Then U is a maximal unipotent subgroup of G and T is a maximal torus of G. By 2.5 U^φ is a maximal unipotent subgroup of G and so it is the unipotent radical of a certain Borel subgroup of G ([8] Theorem 30.4(b)). In particular U^φ is a closed and connected subgroup of G. On the other hand T^φ is a maximal torus of G by 2.7, hence T^φ is closed and connected as well. Therefore $B^\varphi = U^\varphi \vee T^\varphi$ is, by a well known theorem of Chevalley, a closed and connected subgroup of G. By definition, a Borel subgroup of G is a maximal closed connected soluble subgroup of G, and it is well known that a Borel subgroup of G is maximal even in the family of soluble subgroups of G. From this it follows that B^φ is a maximal soluble subgroup of G. In particular B^φ is a Borel subgroup of G. ∎

We are now in the position to prove

COROLLARY 2.9: *Let G be a connected reductive algebraic group over K. Then the group* Aut $\mathcal{L}(G)$ *of all autoprojectivities of G acts in a natural way on the building $\Delta(G)$ canonically associated to G, in the sense that every autoprojectivity φ of G induces an automorphism of $\Delta(G)$.*

PROOF: We recall that the faces of the building $\Delta(G)$ are all the parabolic subgroups of G and that the partial order on $\Delta(G)$ is the reverse of the set-inclusion. Let now φ be an autoprojectivity of G. From 2.8, it is clear that φ induces a permutation on the set of all Borel subgroups of G. It follows that φ permutes all the faces of $\Delta(G)$ and that it induces an automorphism of $\Delta(G)$. ■

At this stage we could follow two different procedures. One is to make use of a deep result by Tits ([15] Theorem 5.8) about isomorphisms between buildings of *adjoint* groups of rank at least 2. The idea would be as follows: given the simple group G and an autoprojectivity φ of G, we get an automorphism τ of $\Delta(G)$ and so an automorphism σ of $\Delta(G_{ad})$ (G_{ad} is the adjoint group isogeneous to G, hence $\Delta(G)$ and $\Delta(G_{ad})$ are isomorphic). By Tits' theorem there exists an automorphism α of G_{ad} inducing σ on $\Delta(G_{ad})$. The final step would be to lift α to an automorphism of G inducing τ on $\Delta(G)$. The only difficulty arises with the case when G is of type D_l, l even, G neither simply-connected nor adjoint.

We shall follow a more elementary approach, first because the structure of $\mathcal{L}(G)$ is obviously richer than that of $\Delta(G)$, and also because even to deal only with the case D_l, above mentioned (and with the case A_1, of course), we need to known how φ acts on the root subgroups of G: it is then possible to use this information to reduce the problem to study autoprojectivities inducing type-preserving automorphisms of $\Delta(G)$. We shall then use a procedure enterely similar to a procedure used by Shangzhi ([11]) for finite simple groups of Lie type (and which is itself an elementary version of Tits' theorem).

We conclude this paragraph with another corollary of 2.8.

COROLLARY 2.10: *For every Borel subgroup B of G we have $R_u(B)^\varphi = R_u(B^\varphi)$. We also have $Z(G)^\varphi = Z(G)$.*

PROOF: The first part is clear. To prove that $Z(G)^\varphi = Z(G)$, we just observe that $Z(G)$ is the intersection of all Borel subgroups of G. ■

3. - REDUCTION TO TYPE-PRESERVING AUTOMORPHISMS OF $\Delta(G)$

We consider a reductive algebraic group G over K, and we study how certain autoprojectivities of G act on the Weyl group and on the Dynkin diagram of G. We start with the following

PROPOSITION 3.1: *Let T be any maximal torus of G and let φ be an autoprojectivity of G. Then we have $\mathcal{N}(T)^\varphi = \mathcal{N}(T^\varphi)$.*

PROOF: As $\mathcal{N}(T)/T$ is generated by involutions, it follows that $T^\varphi \triangleleft \mathcal{N}(T)^\varphi$. Thus we have

$$(*) \qquad \mathcal{N}(T)^\varphi \leq \mathcal{N}(T^\varphi).$$

Let $S = T^\varphi$. If we apply $(*)$ to the pair (S, φ^{-1}), we get $\mathcal{N}(S)^{\varphi^{-1}} \leq \mathcal{N}(S^{\varphi^{-1}})$. Hence $\mathcal{N}(T)^\varphi = \mathcal{N}(T^\varphi)$. ■

REMARK: From the previous proposition, given any maximal torus T of G and any autoprojectivity φ of G, we can define a projectivity $\overline{\varphi}$: $\mathcal{N}(T)/T \to \mathcal{N}(T^\varphi)/T^\varphi$, by $(L/T)^{\overline{\varphi}} = L^\varphi/T^\varphi$, for every subgroup L such that $T \leq L \leq \mathcal{N}(T)$. It is clear that $\overline{\varphi}$ is index-preserving.

LEMMA 3.2: *Let B be a Borel subgroup of G and T be a maximal torus of B. Then there exists an element g in G such that $B^\varphi = B^g$ and $T^\varphi = T^g$.*

PROOF: From 2.8, B^φ is a Borel subgroup of G. Hence there exists x in G such that $B^\varphi = B^x$. Now $(T^\varphi)^{x^{-1}}$ is a maximal torus of G contained in $(B^\varphi)^{x^{-1}} = B$. So there exists b in B such that $(T^\varphi)^{x^{-1}} = T^b$. If we put $g = bx$, then we have $B^\varphi = B^g$ and $T^\varphi = T^g$ as required. ■

For any pair (B, T), where B is a Borel subgroup of G and T is a maximal torus of B, we define $\Gamma_{B,T}$ to be the group of all autoprojectivities of G fixing B and T. By 3.2, for every pair (B, T) and for every autoprojectivity φ of G, there exists g in G such that $\varphi(\iota_g^{-1})^*$ lies in $\Gamma_{B,T}$.

We fix a pair (B, T). We have the set Φ of the roots of G relative to T. Also, the choice of B determines the set Φ^+ of positive roots and the set $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ of simple roots. The Weyl group $W = \mathcal{N}(T)/T$ has the presentation

$$W = \langle s_1, \ldots, s_l \mid s_i^2 = 1 \ \forall i, \ (s_i s_j)^{m_{ij}} = 1 \text{ for } i \neq j \rangle$$

as a Coxeter group ([4] § 1.9). If φ lies in $\Gamma_{B,T}$, from 3.1 and the following remark, for every $i = 1, \ldots, l$, there exists a unique involution $\bar{s}_i$ in W such that $\langle s_i \rangle^{\overline{\varphi}} = \langle \bar{s}_i \rangle$.

PROPOSITION 3.3: *Let i, j be in $\{1, \ldots, l\}$. Then $|\bar{s}_i \bar{s}_j| = |s_i s_j|$ $(= m_{ij})$.*

PROOF: We already have the result for $i = j$. So assume $i \neq j$. We have

$$2|s_i s_j| = |\langle s_i, s_j \rangle| = |\langle s_i, s_j \rangle^{\overline{\varphi}}| = |\langle \bar{s}_i, \bar{s}_j \rangle| = 2|\bar{s}_i \bar{s}_j|,$$

and we are done. ■

We consider the minimal parabolic subgroups of G containing B. For every $i = 1, \ldots, l$ let us fix a representative n_i in $\mathcal{N}(T)$ of s_i. Then the minimal parabolic subgroups of G containing B are $P_1, \ldots, P_l$, where $P_i = \langle B, n_i \rangle$ for every $i = 1, \ldots, l$. Also, for every $i = 1, \ldots, l$ let $\bar{n}_i$ be an element of $\mathcal{N}(T)$ such that $\langle n_i \rangle^\varphi = \langle \bar{n}_i \rangle$, so that $P_i^\varphi = \langle B, \bar{n}_i \rangle$ for every $i = 1, \ldots, l$.

Proposition 3.4: *For every φ in $\Gamma_{B,T}$ there exists a permutation σ of the set $\{1, \ldots, l\}$ such that for every $i = 1, \ldots, l$ we have $P_i^{\varphi} = P_{\sigma_i}$ and $\bar{s}_i = s_{\sigma_i}$.*

Proof: Let i be an element $\{1, \ldots, l\}$. Let us denote by w_i the element $T\bar{n}_i$ of W, and let $w_i = s_{j_1} \ldots s_{j_k}$ be a reduced expression for w_i. Then if we denote by J the set of distinct elements of $\{j_1, \ldots, j_k\}$, we have $\langle B, \bar{n}_i \rangle = P_J$. But $\langle B, \bar{n}_i \rangle$ is a minimal parabolic subgroup of G containing B, hence J must consist of a single element, σ_i say. Hence we obtain $P_i^{\varphi} = \langle B, \bar{n}_i \rangle = P_{\sigma_i}$. Also, the previous reduced form of w_i must be $w_i = s_{\sigma_i}$. Hence $Tn_{\sigma_i} = s_{\sigma_i} = w_i = T\bar{n}_i$, and so $\langle s_i \rangle^{\bar{\varphi}} = \langle s_{\sigma_i} \rangle$. But we had $\langle s_i \rangle^{\bar{\varphi}} = \langle \bar{s}_i \rangle$, which leaves us with $\bar{s}_i = s_{\sigma_i}$. σ is clearly a bijection. ■

By the previous results we can prove that every φ in $\Gamma_{B,T}$ induces a symmetry of the *Dynkin diagram* of G if arrows are disregarded. In fact the nodes corresponding to the roots α_i, α_j with $i \neq j$ are joined by n_{ij} bonds, where n_{ij} depends only on the order m_{ij} of $s_i s_j$. If φ lies in $\Gamma_{B,T}$, then, from 3.4, we have a permutation σ of $\{1, \ldots, l\}$ such that $\langle s_i \rangle^{\bar{\varphi}} = \langle s_{\sigma_i} \rangle$ for every $i = 1, \ldots, l$. We denote by $\bar{\sigma}$ the bijection of the set of nodes of Dynkin diagram defined by $\bar{\sigma}(\alpha_i) = \alpha_{\sigma_i}$ for every $i = 1, \ldots, l$. Also, from 3.3, we get $m_{\sigma_i \sigma_j} = |s_{\sigma_i} s_{\sigma_j}| = |\bar{s}_i \bar{s}_j| = |s_i s_j| = m_{ij}$. In particular we have $n_{\sigma_i \sigma_j} = n_{ij}$, and so $\bar{\sigma}$ is a symmetry of the Dynkin diagram of G if arrows are disregarded.

Remark: If instead of considering an autoprojectivity of a given connected reductive algebraic group G over K, we consider a projectivity φ: $G \to G_1$, where G, G_1 are connected reductive algebraic groups over K, then with a similar argument it is possible to show that if U, T, B are resp. a maximal unipotent subgroup, a maximal torus and a Borel subgroup of G, then U^{φ}, T^{φ} and B^{φ} are resp. a maximal unipotent subgroup, a maximal torus and a Borel subgroup of G_1. Also we still have $\mathcal{N}(T)^{\varphi} = \mathcal{N}(T^{\varphi})$ for every maximal torus T of G.

Now let B be a Borel subgroup of G, and T be a maximal torus of B. We denote by B_1 the Borel subgroup B^{φ} of G_1 and by T_1 the maximal torus T^{φ} of B_1. We can then define a projectivity $\bar{\varphi}$: $W \to W_1$, where $W = \mathcal{N}(T)/T$ is the Weyl group of G and $W_1 = \mathcal{N}(T_1)/T_1$ is the Weyl group of G_1. Let

$$W = \langle s_1, \ldots, s_l \,|\, s_i^2 = 1 \ \forall i,\ (s_i s_j)^{m_{ij}} = 1 \text{ for } i \neq j \rangle$$

be the presentation of W as a Coxeter group relative to the choice of B, and

$$W_1 = \langle t_1, \ldots, t_m \,|\, t_i^2 = 1 \ \forall i,\ (t_i t_j)^{p_{ij}} = 1 \text{ for } i \neq j \rangle$$

be the presentation of W_1 as a Coxeter group relative to the choice of B_1. Let $\bar{s}_i$ be the unique involution of W_1 such that $\langle s_i \rangle^{\bar{\varphi}} = \langle \bar{s}_i \rangle$, for every $i = 1, \ldots, l$. If $\mathcal{F}$ is the free group on the set $\{x_1, \ldots, x_l\}$ we define the homomorphism τ: $\mathcal{F} \to W_1$ by extending the map $x_i \mapsto \bar{s}_i$ for every $i = 1, \ldots, l$. Hence we can define the epimorphism $\bar{\tau}$: $W \to W_1$ such that $\bar{\tau}(s_i) = \bar{s}_i$ for every i. As W and W_1 have the same order, $\bar{\tau}$ is an isomorphism. In particular G and G_1 have isomorphic Weyl groups. ■

Our aim is to show that if G is simple, for every φ in $\Gamma_{B,T}$, there exists a graph automorphism ϑ of G which induces the same symmetry on the Dynkin diagram as φ does. We need to known the behaviour of root subgroups under φ.

Proposition 3.5: *Let φ be an element of $\Gamma_{B,T}$. Then φ induces a permutation of the set of all Borel subgroups of G containing T, and it fixes the opposite B^- of B with respect to T. Moreover, if we denote by U, U^- resp. the unipotent radical of B and of B^- then φ fixes both U and U^-.*

Proof: The result comes from 2.8, 2.10 and from the uniqueness of the opposite. ■

We shall now show that every element φ of $\Gamma_{B,T}$ permutes the set of root subgroups of G relative to T. We recall that the root subgroups X_α are the minimal closed proper subgroups contained in U and U^- which are normalized by T ([4] page 18). It turns out that the X_α's are in fact the minimal proper subgroups contained in U and U^- which are normalized by T, because every subgroup of U normalized by T must be closed and connected (and then the product of the X_α's it contains) ([5] Exp. 13, Th. 1*d*).

Lemma 3.6: *Let φ be in $\Gamma_{B,T}$, and let V be a unipotent subgroup of G such that T is contained in $\mathcal{N}(V)$. Then T is also contained in $\mathcal{N}(V^\varphi)$.*

Proof: Let t be in T and let v be any element of V^φ. There exists v_1 in V and t_1 in T such that $\langle v_1\rangle^\varphi = \langle v\rangle$ and $\langle t_1\rangle^\varphi = \langle t\rangle$. By 2.1, there exists n in $\mathbb{N}$ such that v_1, t_1 are in G_n. We put $V_n = V \wedge G_n$. As $V_n \lhd \langle V_n, t_1\rangle$, V_n is the unique p-Sylow subgroup of $\langle V_n, t_1\rangle$. It follows that V_n^φ must be the unique p-Sylow subgroup of $\langle V_n^\varphi, t\rangle$, so that $V_n^\varphi \lhd \langle V_n^\varphi, t\rangle$. In particular v^t lies in V_n^φ. Hence we have $(V^\varphi)^t \leq V^\varphi$ for every t in T, and T is contained in $\mathcal{N}(V^\varphi)$. ■

Proposition 3.7: *Let X_α be a root subroup of G with respect to the choice of the maximal torus T. Then, for every φ in $\Gamma_{B,T}$, X_α^φ is a root subgroup of G.*

Proof: Let φ be in $\Gamma_{B,T}$. We have $T \leq \mathcal{N}(X_\alpha)$, and so, by 3.6, $T \leq \mathcal{N}(X_\alpha^\varphi)$. Also, from $X_\alpha \leq U$ or $X_\alpha \leq U^-$, it follows by 3.5, that $X_\alpha^\varphi \leq U$ or $X_\alpha^\varphi \leq U^-$. Now suppose V is a non-trivial subgroup of X_α^φ such that T is contained in $\mathcal{N}(V)$. Then we have $V^{\varphi^{-1}} \leq X_\alpha$ and, by 3.6, $T \leq \mathcal{N}(V^{\varphi^{-1}})$. Hence we must have $V^{\varphi^{-1}} = X_\alpha$, as X_α is a root subgroup. So we get $V = X_\alpha^\varphi$, and this means that X_α^φ is a root subgroup of G. ■

By the previous proposition, given an autoprojectivity φ in $\Gamma_{B,T}$, we can define a map $\tau\colon \Phi \to \Phi$ such that $X_\alpha^\varphi = X_{\tau(\alpha)}$ for every α in Φ. τ is clearly a bijection. From 3.5, we also have that $\tau(\Phi^+) = \Phi^+$ and $\tau(\Phi^-) = \Phi^-$.

So far, for every φ in $\Gamma_{B,T}$ we have defined a permutation σ of the set $\{1, \ldots, l\}$ and a bijection $\tau\colon \Phi \to \Phi$ such that $\langle s_i\rangle^{\bar\varphi} = \langle s_{\sigma_i}\rangle$ and $P_i^\varphi = P_{\sigma_i}$ for every $i = 1, \ldots, l$, and $X_\alpha^\varphi =$ $= X_{\tau(\alpha)}$ for every α in Φ, $\tau(\Phi^+) = \Phi^+$. We show how σ and τ are related.

Proposition 3.8: *Let φ be in $\Gamma_{B,T}$. Then we have $(B^{n_i})^\varphi = B^{n_{\bar{i}}}$ for every $i = 1, \ldots, l$.*

Proof: For every i in $\{1, \ldots, l\}$, we have $P_i = \langle B, B^{n_i} \rangle$ ([4] Proposition 2.1.5). From $B^{n_i} \leq P_i$, we get $(B^{n_i})^\varphi \leqslant P_i^\varphi = P_{\bar{i}}$. But $T \leq B^{n_i} \Rightarrow T \leq (B^{n_i})^\varphi$, and so $(B^{n_i})^\varphi$ lies in the thin chamber complex Σ_0 of all parabolic subgroups of G containing T. In particular $P_{\bar{i}}$ contains only two Borel subgroups in Σ_0, namely B and $B^{n_{\bar{i}}}$. Hence we must have $(B^{n_i})^\varphi = B^{n_{\bar{i}}}$, as $B^\varphi = B$. ■

Proposition 3.9: *Let φ be in $\Gamma_{B,T}$. Then we have $(X_{-\alpha_i})^\varphi = X_{-\alpha_{\bar{i}}}$ and $X_{\alpha_i}^\varphi = X_{\alpha_{\bar{i}}}$ for every $i = 1, \ldots, l$.*

Proof: For every $i = 1, \ldots, l$, we have $X_{-\alpha_i} = U^- \wedge U^{n_i}$ ([3] page 101, 115 and [4] page 50, 58). From 3.8, we have $(B^{n_i})^\varphi = B^{n_{\bar{i}}}$, and so we get $(U^{n_i})^\varphi = U^{n_{\bar{i}}}$, by 2.10. Therefore $(X_{-\alpha_i})^\varphi = (U^- \wedge U^{n_i})^\varphi = U^- \wedge U^{n_{\bar{i}}} = X_{-\alpha_{\bar{i}}}$. We now consider the Borel subgroup B^-. We have $B^- = B^{n_0}$, where n_0 is any representative of the longest element w_0 of W. $B^{n_0 n_i}$ is the opposite to B^{n_i} in Σ_0. Hence $(B^{n_0 n_i})^\varphi$ is opposite to $(B^{n_i})^\varphi = B^{n_{\bar{i}}}$ in Σ_0. Therefore we must have $(B^{n_0 n_i})^\varphi = B^{n_0 n_{\bar{i}}}$, and finally $X_{\alpha_i}^\varphi = (U \wedge U^{n_0 n_i})^\varphi = U \wedge U^{n_0 n_{\bar{i}}} = X_{\alpha_{\bar{i}}}$. ■

Corollary 3.10: *Let φ be in $\Gamma_{B,T}$. Then, for every $i = 1, \ldots, l$, we have $\tau(\alpha_i) = \alpha_{\bar{i}}$ and $\tau(-\alpha_i) = -\alpha_{\bar{i}}$, where τ and $\bar{\tau}$ are the maps previously defined.*

Proof: By the definition of τ we have $X_{\alpha_i}^\varphi = X_{\tau(\alpha_i)}$ and $X_{-\alpha_i}^\varphi = X_{\tau(-\alpha_i)}$. Hence, from Proposition 3.9, it follows that $\tau(\alpha_i) = \alpha_{\bar{i}}$ and $\tau(-\alpha_i) = -\alpha_{\bar{i}}$ ■

From now on we shall assume that G is a simple algebraic group over K. Given φ in $\Gamma_{B,T}$, for the existence of the graph automorphism δ we must consider separately the case when G has type B_2, F_4, G_2, and D_l, l even, G neither simply-connected nor adjoint and K of odd characteristic. We begin with the cases when G has type B_2, F_4 or G_2.

Proposition 3.11: *Let φ be in $\Gamma_{B,T}$. Suppose the symmetry $\bar{\varphi}$ of the Dynkin diagram of G induced by φ is non-trivial. If G has type B_2 or F_4, then the characteristic p of the field K must be 2. If G has type G_2, then we must have $p = 3$.*

Proof: Assume first that G has type B_2 or F_4, and suppose for a contradiction that p is not 2. There exist two simple roots $a = \alpha_i$, $b = \alpha_j$ interchanged by $\bar{\varphi}$, such that the set $\lambda(a, b)$ of positive roots which are linear combinations of a, b, are: $a, b, a + b, 2a + b$ ([3] page 214). We show that the bijection τ of Φ induced by φ fixes the set $\lambda(a, b)$. This is clear if G has type B_2, for then we have $\lambda(a, b) = \Phi^+$. So suppose G has type F_4. We have $[X_a, X_b] = X_{a+b}X_{2a+b}$ by Proposition 2.5 and the following remark on page 66 in [2]. Hence we get $X_{a+b}X_{2a+b} \leqslant U_1$, where we denote by U_1 the subgroup $\langle X_a, X_b \rangle$ of U. Let Ψ be the set of roots γ in Φ such that X_γ is contained in U_1. We have $\lambda(a, b) = \Psi$, by 3.4 in [2]. Also, by 3.10, we have $\tau(a) = b$ and $\tau(b) = a$. Hence we obtain $U_1^\varphi = X_a^\varphi \vee X_b^\varphi = X_{\tau(a)} \vee X_{\tau(b)} = X_b \vee X_a = U_1$. Therefore, if γ is a root in Φ, we have $\gamma \in \lambda(a, b) \Leftrightarrow X_\gamma \leqslant U_1 \Leftrightarrow X_\gamma^\varphi \leqslant U_1^\varphi = U_1 \Leftrightarrow X_{\tau(\gamma)} \leqslant U_1 \Leftrightarrow \tau(\gamma) \in \lambda(a, b)$. Hence τ

fixes $\lambda(a, b)$. In particular τ fixes the set $\kappa(a, b) = \lambda(a, b) \setminus \{a, b\}$. For every root s in Φ, let us denote by $c(s)$ the number of roots r in $\kappa(a, b)$ such that $[X_r, X_s] = \{1\}$. We prove that $c(a) = c(b)$. Let $\gamma_1, \dots, \gamma_k$ be the roots r in $\kappa(a, b)$ such that $[X_a, X_r] = \{1\}$. Then we have $[X_a^\tau, X_{\gamma_i}^\tau] = \{1\}$ for $i = 1, \dots, k$, as the root subgroups are elementary abelian p-groups. Hence we have $[X_b, X_{\tau(\gamma_i)}] = \{1\}$. But the $\tau(\gamma_i)$'s are pairwise distinct, and so we have $c(b) \geqslant k = c(a)$, as τ fixes $\kappa(a, b)$. By symmetry we have $c(a) \geqslant c(b)$, so that $c(a) = c(b)$. From Chevalley's Commutator Formula, we have

$$[x_a(1), x_{a+b}(1)] = x_{2a+b}(\pm 2),$$

$$[X_a, X_{2a+b}] = \{1\},$$

$$[X_b, X_{a+b}] = \{1\},$$

$$[X_b, X_{2a+b}] = \{1\}.$$

As the characteristic p is not 2, we get $c(b) = 2$ and $c(a) = 1$. This is a contradiction. Hence if G has type B_2 or F_4, the characteristic of the field must be 2.

Now assume that G has type G_2, and suppose that p is not 3. In this case $\bar{\tau}$ interchanges the simple roots α_1 and α_2. We choose the notation $(\alpha_1, \alpha_2) = (a, b)$ or $(\alpha_1, \alpha_2) = (a, b)$ in order to have $\Phi^+ = \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$. As in the previous case, the bijection τ fixes the set $\lambda(a, b) \setminus \{a, b\} = \Phi^+ \setminus \{a, b\}$, and again, if for every root s we denote by $c(s)$ the number of roots r in $\lambda(a, b) \setminus \{a, b\}$ such that $[X_r, X_s] = \{1\}$, we must have $c(a) = c(b)$. From Chevalley's Commutator formula, we have the following relations:

$$[X_b, X_{a+b}] = \{1\},$$

$$[X_b, X_{2a+b}] = \{1\},$$

$$[X_b, X_{3a+2b}] = \{1\},$$

$$[x_a(1), x_{a+b}(1)] = x_{2a+b}(\pm 2)x_{3a+b}(\pm 3)x_{3a+2b}(\pm 3),$$

$$[x_a(1), x_{2a+b}(1)] = x_{3a+b}(\pm 3),$$

$$[X_a, X_{3a+2b}] = \{1\},$$

$$[X_r, X_{3a+b}] = \{1\}.$$

As p is not 3, we have $c(b) \geqslant 3$, while $c(a) = 2$. This is a contradiction. Therefore we must have $p = 3$. ■

We now consider the case when G is of type D_l, l even, G neither simply-connected nor adjoint and K of odd characteristic.

Let φ be in $\Gamma_{\beta,\gamma}$, and let $\bar{\tau}$ be the symmetry of the Dynkin diagram induced by φ. It is proved in [12] corollary on page 156, that if we denote by ρ the isometry of the Euclidean space $X \otimes \mathbb{R}$ induced by $\bar{\tau}$ (recall that X is the character group of T and that the Dynkin diagram of G has only single bonds), then there exists a graph automorphism ϑ inducing $\bar{\tau}$ if and only if $\rho X = X$. The point is the following. We consider the universal covering $\pi\colon G_u \to G$. Given $\bar{\varphi}$, there exists an automorphism γ of G_u inducing $\bar{\varphi}$ (Theorem 28, 29 in [12]). Therefore there exists an automorphism δ of G inducing $\bar{\varphi}$ if and only if $(\ker \pi)^\gamma = \ker \pi$ (and this is equivalent to $\rho X = X$).

PROPOSITION 3.12: *Let G be of type D_l, l even, G neither simply-connected nor adjoint and K of odd characteristic. Let φ be in $\Gamma_{B,T}$, and let $\bar{\varphi}$ be the symmetry of the Dynkin diagram induced by φ. Then there exists a graph automorphism δ of G such that δ induces $\bar{\varphi}$.*

PROOF: By the previous discussion we are left to prove the following. Let G be a simply-connected simple algebraic group of type D_l, l even, over K of odd characteristic, and let C be a subgroup of order 2 of $Z(G)$ (in our case $Z(G)$ is isomorphic to $C_2 \times \times C_2$). If φ is an autoprojectivity of G/C inducing the symmetry of the Dynkin diagram $\bar{\varphi}$, then the graph automorphism γ of G inducing $\bar{\varphi}$ fixes C.

We need a description of the three involutions of $Z(G)$. We just write Z for $Z(G)$. For every root α we consider the elements $h_\alpha(t)$ of $\langle X_\alpha, X_{-\alpha}\rangle$ as defined in Lemma 19 page 27 in [12]. We write just $h_i(t)$ for $h_{\alpha_i}(t)$ when α_i lies in Π. Every element h of the maximal torus T is uniquely expressible as a product $h = h_1(t_1)\cdot\ldots\cdot h_l(t_l)$. By Lemma 28 page 43 in [12], we have $Z = \{h_1(t_1)\cdot\ldots\cdot h_l(t_l) \mid t_1^{\langle\beta,\alpha_1\rangle}\cdot\ldots\cdot t_l^{\langle\beta,\alpha_l\rangle} = 1$ for every β in $\Phi\}$. It follows that the three involutions of Z are

$$\sigma_1 = h_1(-1)h_3(-1)h_5(-1)\cdot\ldots\cdot h_{l-3}(-1)h_{l-1}(-1),$$

$$\sigma_2 = h_1(-1)h_3(-1)h_5(-1)\cdot\ldots\cdot h_{l-3}(-1)h_l(-1),$$

$$\sigma_3 = h_{l-1}(-1)h_l(-1),$$

where the order $\alpha_1, \ldots, \alpha_l$ is the usual for diagrams of type D_l. Let Π_1, Π_2, Π_3, be the subsets of Π defined as follows: $\Pi_1 = \{\alpha_1, \alpha_3, \ldots, \alpha_{l-3}, \alpha_{l-1}\}$, $\Pi_2 = \{\alpha_1, \alpha_3, \ldots, \alpha_{l-3}, \alpha_l\}$, $\Pi_3 = \{\alpha_{l-1}, \alpha_l\}$. We have $\langle\sigma_i\rangle = \langle X_\alpha, X_{-\alpha} \mid \alpha \in \Pi_i\rangle \wedge Z$ for every $i = 1, 2, 3$. Let $C = \langle\sigma_j\rangle$. Let φ be an autoprojectivity of G/C inducing the symmetry $\bar{\varphi}$ on the Dynkin diagram, and let γ be the graph automorphism of G inducing $\bar{\varphi}$ (note that by 2.10 we have $(Z/C)^\varphi = Z/C$). We have to show that $\sigma_j^\gamma = \sigma_j$. Let ρ be the isometry of $X \otimes \mathbb{R}$ induced by $\bar{\varphi}$. By 3.10 we have $(CX_\alpha/C)^\varphi = CX_{\rho(\alpha)}/C$ for every root α in $\Pi \cup (-\Pi)$. Suppose $\sigma_j^\gamma \neq \sigma_j$. Then there exists $i \neq j$ such that $\sigma_i^\gamma = \sigma_j$. Hence we get $\Pi_i^\rho = \Pi_j$. Therefore

$$\begin{aligned}(Z/C)^\varphi &= (\langle\sigma_i\rangle C/C)^\varphi = ((\langle X_\alpha, X_{-\alpha} \mid \alpha \in \Pi_i\rangle \wedge Z)C/C)^\varphi = \\ &= (((\langle CX_\alpha, CX_{-\alpha} \mid \alpha \in \Pi_i\rangle \wedge Z)/C)^\varphi = (((\langle CX_{\rho(\alpha)}, CX_{-\rho(\alpha)} \mid \alpha \in \Pi_i\rangle \wedge Z)/C) = \\ &= (((\langle CX_\alpha, CX_{-\alpha} \mid \alpha \in \Pi_j\rangle \wedge Z)/C) = \langle\sigma_j\rangle C/C = C/C\end{aligned}$$

which is a contradiction, as $(Z/C)^\varphi = Z/C$. Hence we have $\sigma_j^\gamma = \sigma_j$, and we are done. ∎

LEMMA 3.13: *Let φ be in $\Gamma_{B,T}$. Then there exists a graph automorphism δ of G such that we have $P^\varphi = P^\delta$ for every parabolic subgroup P containing B.*

PROOF: Let $\bar{\sigma}$ be the symmetry pf the Dynkin diagram of G induced by φ. Then by 3.11 and 3.12, and Corollary *b* page 156 in [12], there exists a graph automorphism σ of G inducing $\bar{\sigma}$. From properties of graph atomorphisms and 3.10, we have $X_\alpha^\sigma = X_{\bar{\sigma}(\alpha)} = X_\alpha^\varphi$ for every root α in $\Pi \cup (-\Pi)$. Also $B^\sigma = B$. We are now able to prove that $P^\varphi = P^\sigma$ for every parabolic subgroup P containing B. We only need to show that this holds for the minimal parabolic subgroups $P_1, \ldots, P_l$ containing B, as the interval $[G/B]$ is a Boolean lattice. From 2.6.4 in [4], we have $P_i = \langle B, X_{-\alpha_i} \rangle$ for every $i = 1, \ldots, l$. Hence $P_i^\varphi = \langle B^\varphi, X_{-\alpha_i}^\varphi \rangle = \langle B, X_{-\bar{\sigma}(\alpha_i)} \rangle = \langle B^\sigma, X_{-\alpha_i}^\sigma \rangle = P_i^\sigma$, and we are done. ■

For every Borel subgroup B of G and every maximal torus T of G contained in B, we define the group $\Gamma_{[G/B], T}$ to be the subgroup of all elements of $\Gamma_{B,T}$ fixing every parabolic subgroup of G containing B.

Therefore, by 3.13, for every φ in $\Gamma_{B,T}$, there exists a graph automorphism σ of G such that $\varphi(\sigma^{-1})^*$ lies in $\Gamma_{[G/B], T}$.

PROPOSITION 3.14: *Let φ be in $\Gamma_{[G/B], T}$. Then the automorphism of $\Delta(G)$ induced by φ is type-preserving.*

PROOF: This follow from Proposition 3.8 nd 2.6 in [15]. ■

4. - $\operatorname{Aut} \mathcal{L}(G) = \Gamma(G) \rtimes \operatorname{Aut} G$

By 3.2, 3.13 and 3.14 we can now follow a procedure used by Shangzhi in [11], in which he gives a proof that a type-preserving automorphism of the building of a finite simple group of Lie type of rank at least 2 is induced by a special automorphism of the group. We give a sketch of the proof, stating the remain results in order to obtain also a few corollaries.

We recall some properties of $\Delta(G)$. We just write Δ for $\Delta(G)$. The set of apartments of Δ is the set $\{\Sigma_T \mid T$ is a maximal torus of $G\}$, where, for every maximal torus T of G, Σ_T is the (finite) set of all parabolic subgroups of G containing T. We make G act on Δ by left conjugation. We fix a pair (B, T) and we denote by Σ_0 the apartment Σ_T. Let U be the unipotent radical of B. For every g in G the map $u \mapsto {}^{gu}\Sigma_0$, gives rise to a bijection of U onto the set of apartments containing ${}^gB$ $(= gBg^{-1})$. We observe that if φ is an autoprojectivity of G and Σ is an apartment of Δ, then $\Sigma^\varphi = \{X^\varphi \mid X \in \Sigma\}$ is an apartment of Δ. It follows that if φ fixes B, then φ permutes the set of apartmens of Δ containing B. Hence we can define a map $\theta\colon U \to U$, by the rule ${}^{u^\theta}\Sigma_0 = ({}^u\Sigma_0)^\varphi$. θ is a bijection.

As first step, one proves that if φ lies in $\Gamma_{[G/B], T}$, then we have $X_r^\varphi = X_r$ for every simple root r. Hence, for every simple root r there exists a unique k_r in $K \setminus \{0\}$ such that $x_r(1)^\theta = x_r(k_r)$. This enables us to define an automorphism d of G such that $X^\varphi = X^d$ for every X in Σ_0 or in ${}^{x_r(1)}\Sigma_0$, where r is any simple root (actually d is inner as K is algebraically closed).

We denote by $\Gamma_{0,U}$ the group of all autoprojectivities in $\Gamma_{[G/B], T}$ such that $({}^{x_r(1)}\Sigma_0)^\varphi = {}^{x_r(1)}\Sigma_0$ for every simple root r. Hence we have

LEMMA 4.1: *For any φ in $\Gamma_{[G/B],T}$, there exists an inner automorphism d of G such that $\varphi(d^*)^{-1}$ lies in $\Gamma_{0,\Pi}$.* ∎

To get the final result, we consider an autoprojectivity φ in $\Gamma_{0,\Pi}$ and the corresponding bijection $\theta\colon U \to U$. We have the following crucial fact. Let r be a positive root, and u an element of U. Then, for every a in K, there exists a unique b in K such that $(ux_r(a))^\theta = u^\theta x_r(b)$. Furthermore, we have $b = 0$ if and only if $a = 0$ (this holds even for φ in $\Gamma_{[G/B],T}$). In particular we get $X_r^\theta = X_r$ for every positive root r. Hence, for every positive root r, we can define a bijection f_r of K onto itself by $x_r(f_r(a)) = (x_r(a))^\theta$ for every a in K.

To investigate the properties of the maps f_r, we need to know the behaviour of the bijection θ with respect to decompositions of the group U in terms of root subgroups. In [11] is proved the following. Suppose $<$ is a total order on Φ^+ compatible with the height function, and let us number the elements in Φ^+ such that $r_1 < r_2 < \ldots < r_N$. Let u be in U. u is uniquely expressible as $u = x_1 \ldots x_N$ with x_i for every X_{r_i} for every $i = 1, \ldots, N$. Then the decomposition of u^θ with respect to the same order on Φ^+ is $u^\theta = x_1^\theta \ldots x_N^\theta$. In fact it is possible to prove this for any *total* order on Φ^+. This comes from the following result, which can be proved by induction. Let X be a group, and let $X_1, \ldots, X_n$ be subgroups of X such that each element x of X can uniquely be written as product $x = x_1 \ldots x_n$, where x_i lies in X_i for every $i = 1, \ldots, n$. Suppose that we have $[X_i, X] \leq X_{i+1} \ldots X_n$ for $i = 1, \ldots, n-1$, and $[X_n, X] = \{1\}$. For every σ in the symmetric group S_n, each element x of X can be uniquely expressed as a product $x = x_{\sigma_1} \ldots x_{\sigma_n}$, where each x_i lies in X_i. We now fix i in $\{1, \ldots, n\}$. For every $j = 1, \ldots, n$, there exists a unique y_j in X_j such that $x = y_{\sigma_i} y_{\sigma_1} \cdots y_{\sigma_{i-1}} y_{\sigma_{i+1}} \cdots y_{\sigma_n}$. Then the result is that we have $y_{\sigma_i} = x_{\sigma_i}$, so that $x = x_{\sigma_1} \ldots x_{\sigma_n} = x_{\sigma_i} y_{\sigma_1} \cdots y_{\sigma_{i-1}} y_{\sigma_{i+1}} \cdots y_{\sigma_n}$. From this and what is proved in [11], we get

PROPOSITION 4.2: *Let $<$ be any total order on Φ^+, and let us number the elements in Φ^+ such that $r_1 < r_2 < \ldots < r_N$. Let φ be in $\Gamma_{0,\Pi}$, (this holds even for φ in $\Gamma_{[G/B],T}$) and θ be the bijection of U induced by φ. If $u = x_1 \ldots x_N$ is the decomposition of the element u of U with x_i in X_{r_i} for every $i = 1, \ldots, N$, then the decomposition of u^θ with respect to the same order of Φ^+ is $u^\theta = x_1^\theta \ldots x_N^\theta$ (with x_i^θ in X_{r_i} for every $i = 1, \ldots, N$).* ∎

Using 4.2 and Chevalley commutator formula, one then shows that if the rank of G is at least 2, for every r, s in Φ^+ we have $f_r = f_s (= f)$, and this map is an automorphism of K. Hence for every autoprojectivity φ in $\Gamma_{0,\Pi}$, we get an automorphism f of the field K such that $x_r(a)^\theta = x_r(f(a))$ for every r in Φ^+ and every a in K. If the rank of G is 1, one still proves that f is an automorphism of K using arguments similar to those used by Metelli in [9]. Let us denote by F the field automorphism of G induced by f. It turns out that $X^\varphi = X^F$ for every X in Δ. The final result is then

LEMMA 4.3: *Let φ be in $\Gamma_{0,\Pi}$. Then there exists a field automorphism F of G such that $X^\varphi = X^F$ for every face X of Δ.* ∎

We summarize the results so far obtained.

Theorem 4.4: *Let G be a simple algebraic group over the field $K = \overline{F}_p$, where p is any prime. Then for every autoprojectivity φ of G there exist an inner automorphism i_g, a graph automorphism δ and a field automorphism F of G such that the autoprojectivity $\varphi((F\delta i_g)^{-1})^*$ fixes all the faces of the building $\Delta(G)$ canonically associated to G.*

Proof: The result comes from the previvous discussion and by 3.2, 3.13, 4.1 and 4.3. ■

Proposition 4.5: *If α is an automorphism of G fixing every face of Δ, then α is the identity.*

Proof: First, as α fixes B and B^-, it must fix also U and U^-. Now let u be in U. We have ${}^u\Sigma_0 = ({}^u\Sigma_0)^\alpha = {}^{u^\alpha}\Sigma_0$, which gives $u^\alpha = u$. Similarly one can prove that for every u in U^- we get $u^\alpha = u$. It follows that $g^\alpha = g$ for every g in G, as $G = \langle U, U^- \rangle$. Hence α is the identity. ■

Remark: The previous result holds in the more general case when G is semisimple, as the crucial point is that $G = \langle U, U^- \rangle$ which holds in fact in the case when G is semisimple.

Corollary 4.6: *For any autoprojectivity φ of G, there exists a unique automorphism α of G such that φ and α act in the same way on the building $\Delta(G)$.*

Proof: Existence follows from 4.4, by taking $\alpha = F\delta i_g$. Uniqueness then follows from 4.5. ■

We are also able to obtain the well known structure of the automorphism group of G. If α is an automorphism of G, there exists by 4.4 an inner automorphism i_g, a graph automorphism δ and a field automorphism F of G such that α and $F\delta i_g$ act in the same way on $\Delta(G)$. Then we must have $\alpha = F\delta i_g$ by 4.5.

As a corollary of 4.5, we also obtain

Corollary 4.7: *Let G be a simple algebraic group over K. Then the homomorphism $*$: $\mathrm{Aut}\, G \to \mathrm{Aut}\, \mathcal{L}(G)$ is injective.* ■

From the previous remark, it is then clear that 4.7 holds also in the case when G is semisimple (we observe that this follows also directly from a result by Cooper ([6] 2.2.2.) that says that the homomorphism $*$ is injective for every perfect group). It does not always work for reductive groups, as one can see by taking G to be a torus, and α to be the inversion automorphism.

We shall therefore identify the groups $\mathrm{Aut}\, G$ and $(\mathrm{Aut}\, G)^*$, and, following Völklein ([16]), we finally give the following

Definition 4.8: *For every simple algebraic group G we define $\Gamma(G)$ to be the set of all autoprojectivities of G fixing every parabolic subgroup of G. We shall call the elements of $\Gamma(G)$ exceptional autoprojectivities of G.* ■

$\Gamma(G)$ is the kernel of the action of the group $\mathrm{Aut}\,\mathcal{L}(G)$ on $\Delta(G)$. Corollary 4.6 then implies

COROLLARY 4.9: *Let G be a simple algebraic group over the field K. Then we have* $\mathrm{Aut}\,\mathcal{L}(G) = \Gamma(G) \rtimes \mathrm{Aut}\,G$. ∎

REMARK: The procedure we followed gives actually a constructive way to obtain the automorphism α acting on $\Delta(G)$ as the autoprojectivity φ does, for a posteriori, one has that if φ lies in $\Gamma_{0,\Pi}$, then the field automorphism F induced by φ acts on U in the same way as the bijection θ.

In a forthcoming paper we shall prove that $\Gamma(G)$ is $\{1\}$ if the characteristic of K is odd and G is not of type A_2. Therefore in this case we have $\mathrm{Aut}\,\mathcal{L}(G) = \mathrm{Aut}\,G$.

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