Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
110° (1992), Vol. XVI, fasc. 13, pagg. 223-258

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Special Bounded Hessian and Elastic-Plastic Plate (**) (***)

SUMMARY. — A weak formulation of a free gradient discontinuity problem is proposed. The domain of the related functional (the class of functions with special bounded hessian) is introduced and studied. Necessary and sufficient conditions for the existence of equilibrium of a weak elastic-plastic plate are obtained.

Hessiano misura speciale ed equilibrio di una piastra elasto-plastica

SUNTO. — Viene proposta una formulazione debole di un problema con discontinuità libera nel gradiente. Si introduce e si studia il dominio del funzionale collegato (la classe delle funzioni con hessiano misura speciale). Vengono esplicitate condizioni necessarie e sufficienti per l'esistenza dell'equilibrio di una piastra elasto-plastica.

1. - INTRODUCTION

Recently many papers have been devoted to the study of functionals of the type

$$\int_{\Omega} f(x, v, \nabla v)\, dx + \int_{S_v} \varphi(x, v^+, v^-, n)\, d\mathcal{H}^{m-1}, \tag{1.1}$$

where v is a real function defined in the open set $\Omega \subset R^m$ ($m \geqslant 1$), S_v is the set (*a priori* unknown) of discontinuity of v, v^+ and v^- are the traces of v on the two sides of S_v, n is the normal unit vector to S_v and $\mathcal{H}^{m-1}$ is the $(m-1)$-dimensional Hausdorff measure.

Such functionals are interesting both in pure mathematics, since they connect classical calculus of variations and modern geometric measure theory (see [AM1],

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(**) Work partially supported by the MURST National Research Project «Equazioni Differenziali e Calcolo delle Variazioni» and by the CNR Project «Irregular Variational Problems».

(***) Memoria presentata il 22 settembre 1992 da Ennio De Giorgi, uno dei XL.

ISSN-0392-4106

[AM2], [PA]), and in applied mathematics (see [AT], [CL1], [CLPP], [CT], [DCL], [DMS], [MS], [MT] about computer vision theory; see [AV], [CL2], [ER], [FO], [MO], [VI] about phase transitions and liquid crystals theory). Functionals like (1.1) seem to be useful for schematization of problems in which «volume energy» and «surface energy» play a role.

A suitable subspace of $BV(\Omega)$, say the class of functions with special bounded variation $SBV(\Omega)$, introduced by E. De Giorgi and L. Ambrosio in [DGA], is a powerful framework in the study of functional (1.1). The space $SBV(\Omega)$ enjoys compactness properties for minimizing sequences of (1.1) (see [AM1]) and allows the study of strongly non convex problems, e.g. problems with non uniqueness of minimizer and non convexity of the set of minimizers.

Analogously to the functionals (1.1) it is interesting to consider functionals where also higher order derivatives appear in the «volume term» and also the traces of the derivatives of v and some kind of curvatures of the singular set appear in the «surface term».

The weak plate model for image segmentation in computer vision theory, proposed by A. Blake and A. Zisserman for piecewise smooth grey-levels functions (see [BZ], p. 98) has an energy that fits into the above framework. More precisely this energy could be expressed in the following way, by mean of our notations,

$$\int_\Omega |\nabla^2 v|^2 dx + \alpha \mathcal{H}^{m-1}(S_v) + \beta \mathcal{H}^{m-1}(S_{\nabla v} \setminus S_v) + \int_\Omega |v - g|^2 dx, \tag{1.2}$$

where α, β are positive parameters, $g \in L^\infty(\Omega)$, $v \in SBV(\Omega)$, ∇v is the absolutely continuous part of the distributional gradient Dv with respect to the Lebesgue measure and $\nabla v \in SBV(\Omega, R^m)$. The one dimensional case ($m = 1$) of (1.2) has been recently studied in [CO], [BC].

In this paper we consider functionals of the following kind

$$\int_\Omega Q(\nabla^2 v)\, dx + \mathcal{H}^{m-1}(S_{Dv}) + \int_{S_{Dv}} \left| \left[\frac{\partial v}{\partial n} \right] \right| d\mathcal{H}^{m-1} - \langle L, v \rangle, \tag{1.3}$$

where v is a real function belonging to the Sobolev space $W^{1,1}(\Omega)$ with $Dv \in SBV(\Omega, R^m)$, n and $[\partial v / \partial n]$ are respectively the normal unit vector associated to the set S_{Dv} and the jump of Dv across such set, Q is a positive definite quadratic form and L is a linear functional.

In the two dimensional case the functional (1.3) is related to the theory of thin elastic-plastic plates: assume that the bounded connected open subset $\Omega \subset R^2$ is the undeformed shape of an horizontal thin plate, v is the vertical displacement and the quadratic form Q is given by

$$Q(\nabla^2 v) = d((1 - \nu)|\nabla^2 v|^2 + \nu |\Delta^a v|^2), \tag{1.4}$$

where $\Delta^a v$ denotes the absolutely continuous part of the distributional laplacian of v. If we assume also that $v \in W^{2,2}(\Omega)$ then the second and the third term disappear in the functional (1.3) so that the functional reduces to the mechanical energy of the plate in the framework of linear elasticity (see [DL], [NH]) under the «Kirchhoff hypothesis» (*), where $d > 0$ is the stiffness coefficient, ν is the Poisson coefficient with $0 < \nu < 1/2$ and L is a dead load. If the plate is submitted to boundary conditions and, possibly, to unilateral constraints, existence and properties of the total energy minimizers in this model have been studied by many authors (see [LI], [LS], [FI1], [FI2], [ST], [BGT]).

When plastic behaviour without hardening (for instance in case of a material subject to Hencky's law) is taken into account, one cannot expect to find displacements in the Sobolev space $W^{2,2}(\Omega)$; for this reason the following deformation energy has been proposed (see [DE1], [DE2], [TE], [PE])

$$(1.5)\qquad \int_\Omega \psi(D^2 v),$$

where ψ is a convex, proper and lower semicontinuous scalar function with linear growth at infinity; in this case the domain is the space

$$BH(\Omega) = \{v \in W^{1,1}(\Omega) : D^2 v \text{ bounded matrix-valued Radon measure}\},$$

which has been thoroughly studied by F. Demengel in [DE1], [DE2] (where it is denoted $HB(\Omega)$ «à la française»). This space allows creasing without fracture: relevant examples showing gradient discontinuity along lines, in $\Omega = (-1, 1)^2$, are the dihedron $v(x_1, x_2) = |x_1|$ and the pyramid $v(x_1, x_2) = 1 - \max\{|x_1|, |x_2|\}$.

The existence of minimizers in $BH(\Omega)$ is proved in [DE2], but no regularity of the solutions or property of the discontinuity set of the gradient is shown: that is, the singular part of $D^2 v$ could be *a priori* sparse even for a minimizer. For another approach with some information about partial regularity see [SE].

The functional (1.3) studied here is defined on the subspace of $BH(\Omega)$ which we denote $SBH(\Omega)$ (functions with special bounded hessian); this space still allows creasing without fracture, on the discontinuity set of the gradient the functional (1.3) has the same kind of behaviour as (1.5) and it has the same behaviour of (1.4) in the set of continuity of the gradient. Hence our functional corresponds for $m = 2$ to a linear elastic energy density on the «elastic» set $\Omega \setminus S_{Dv}$, while yielding the plate on the «plasticity» set S_{Dv} pays for its $\mathcal{H}^1$ measure; moreover the additional energy required to crease the plate along S_{Dv} is modelled as the effect of a hinge of low elastic resistance whose energy density (with respect to $\mathcal{H}^1$) is the jump $|[Dv]|$.

We notice that the appearance of 1-dimensional set of plastic yielding is expected when dealing with thin metallic plates, contrarily to the case of 3-dimensional deformable bodies where the plasticity set is allowed to have Hausdorff dimension bigger than two; for detailed results on this subject we refer to [BS], [HK1], [HK2].

In this paper we prove existence in $SBH(\Omega)$ of a minimizer of the functional (1.3) under Neumann, Dirichlet and unilateral conditions with suitable loads L. Moreover

(*) The linear filaments of the plate perpendicular to the middle surface before deformation remain straight and perpendicular to the middle surface after deformation as well.

we study some necessary conditions and some further properties of a local minimizer.

The existence theorems in sections 3, 4 and 5 can be proved, by the same arguments, for the variational problem studied in [DCL] with an additional jump term:

$$\min\left\{\int_\Omega |\nabla v|^2 dx + \mathcal{H}^{m-1}(S_v) + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^{m-1} + \int_\Omega |v-g|^2 dx\right\},$$

and also for the variational problem

$$\min\left\{\int_\Omega |\nabla^2 v|^2 dx + \mathcal{H}^{m-1}(S_{Dv}) + \int_{S_{Dv}} |(Dv)^+ - (Dv)^-|\, d\mathcal{H}^{m-1} + \int_\Omega |v-g|^2 dx\right\}.$$

We remark also that studying (1.3) may be a preliminary step to the study of (1.2), possibly modified by adding an integral depending on the curvature of S_v.

In a forthcoming paper we show that, in the two dimensional case, for a minimizer u of (1.3) there is an open set Ω_0 (where elastic deformation takes place) such that

$$(1.6)\qquad S_{Du} \subset \Omega \setminus \Omega_0, \quad \mathcal{H}^1(S_{Du}) = \mathcal{H}^1(\overline{S_{Du}} \cap \Omega) < +\infty, \quad u \in C^0(\overline{\Omega}) \cap C^2(\Omega \setminus \overline{S_{Du}}).$$

The main results of this paper were announced in [CLT], [TO].

The outline of the paper is the following:

1) Introduction.
2) Functions with Special Bounded Hessian.
3) The Neumann problem.
4) The Dirichlet problem.
5) The obstacle problem.
6) Necessary conditions.
7) Local minimizers in R^m.

2. - Functions with Special Bounded Hessian

Given a connected open subset $\Omega \subseteq R^m$ ($m \geq 1$) we define the class of real valued functions with special bounded hessian $SBH(\Omega)$ and we point out some of its properties.

For a given set $U \subset R^m$ we denote by $\overline{U}, \mathring{U}, \partial U$ its topological closure, interior, boundary; moreover we denote by $\mathcal{H}^{m-1}(U)$ its $(m-1)$-dimensional Hausdorff measure and by $\mathcal{L}^m(U)$ (or shortly $|U|$) its Lebesgue outer measure. We indicate by $B_\rho(x)$ the open ball $\{y \in R^m : |y - x| < \rho\}$, and we set $B_\rho = B_\rho(0)$. If Ω, Ω' are open subsets in R^m, by $\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$.

We introduce the following notations: $\alpha \wedge \beta = \min\{\alpha, \beta\}$, $\alpha \vee \beta = \max\{\alpha, \beta\}$, for every $\alpha, \beta \in R$; $M_{k,m}$ stands for $k \times m$ matrices ($k \geq 1$) and I for the identity in $M_{m,m}$;

given two vectors $a = \{a_i\}$, $b = \{b_i\}$, and two matrices $A = \{A_{ij}\}$, $B = \{B_{ij}\}$, we set $a \cdot b = \sum_i a_i b_i$, $(a \otimes b)_{ij} = a_i b_j$, $(Ab)_i = \sum_j A_{ij} b_j$, $(bA)_j = \sum_i A_{ij} b_i$, $(AB)_{ij} = \sum_k A_{ik} B_{kj}$, $A : B = \sum_{ij} A_{ij} B_{ij}$.

Let $v: \Omega \to R^k$ be a Borel function; for $x \in \Omega$ and $z \in \bar{R}^k = R^k \cup \{\infty\}$ (the one point compactification of R^k) we say, following [DGA], that z is the approximate limit of v at x, and we write

$$z = \operatorname{ap} \lim_{y \to x} v(y),$$

if

$$g(z) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} g(v(y))\, dy}{|B_\rho|}$$

for every $g \in C^0(\bar{R}^k)$.

The set

$$S_v = \{x \in \Omega;\ \operatorname{ap} \lim_{y \to x} v(y) \text{ does not exist}\},$$

is a Borel set, of negligible Lebesgue measure (see e.g. [FE], 2.9.13); for brevity's sake we denote by $\tilde{v}: \Omega \setminus S_v \to \bar{R}^k$ the function

$$\tilde{v}(x) = \operatorname{ap} \lim_{y \to x} v(y).$$

Let $x \in \Omega \setminus S_v$ be such that $\tilde{v}(x) \in R^k$; we say that v is approximately differentiable at x if there exists a $k \times m$ matrix $\nabla v(x)$ such that

$$\operatorname{ap} \lim_{y \to x} \frac{|v(y) - \tilde{v}(x) - \nabla v(x)(y - x)|}{|y - x|} = 0.$$

If v is a smooth function then ∇v is the jacobian matrix. In the following with the notation $|\nabla v|$ we mean the euclidean norm of ∇v.

If $p \in [1, +\infty]$, we denote by $L^p(\Omega, M_{k,m})$ and by $W^{1,p}(\Omega, M_{k,m})$ the Lebesgue and Sobolev spaces of functions with values in $M_{k,m}$, endowed with the usual norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{1,p}}$ respectively. We denote by $\mathcal{M}(\Omega, M_{k,m})$ the space of the bounded measures on Ω with values in $M_{k,m}$ and by $|\cdot|_T$ the total variation of a measure of $\mathcal{M}(\Omega, M_{k,m})$, *i.e.*

$$|\mu|_T = \sup\left\{\int_\Omega \sum_{ij} \varphi_{ij}\, d\mu_{ij} : \varphi_{ij} \in C_0^0(\Omega), \sum_{ij} \varphi_{ij}^2 \leq 1 \text{ in } \Omega\right\}.$$

If A is any open set then $|\mu|_{T(A)}$ is defined in the same way with $\varphi_{ij} \in C_0^0(A)$ and we define a Borel measure $|\mu|$ setting for every Borel set $B \subset \Omega$

$$|\mu|(B) = \inf\{|\mu|_{T(A)} : B \subset A, A \text{ open}\}.$$

We recall the definition of the space of functions of bounded variation in Ω with values in $\boldsymbol{R}^k$:

$$BV(\Omega, \boldsymbol{R}^k) = \{v \in L^1(\Omega, \boldsymbol{R}^k);\, Dv \in \mathcal{M}(\Omega, M_{k,m})\}\,,$$

where $Dv = \{D_j v_i\}_{\substack{i=1,\dots,k \\ j=1,\dots,m}}$ denotes the distributional derivatives of v. For every $v \in BV(\Omega, \boldsymbol{R}^k)$ the following properties hold:

1) $\tilde{v}(x) \in \boldsymbol{R}^k$ for $\mathcal{H}^{m-1}$-almost all $x \in \Omega \setminus S_v$ (see [ZI], 5.9.6);

2) S_v is countably $(\mathcal{H}^{m-1}; m-1)$ rectifiable (see [ZI], 5.9.6);

3) ∇v exists a.e. on Ω and coincides with the Radon-Nikodym derivative of Dv with respect to the Lebesgue measure (see [FE], 4.5.9(26));

4) for $\mathcal{H}^{m-1}$ almost all $x \in S_v$ there exist $\boldsymbol{n} = \boldsymbol{n}_v(x) \in \partial B_1$, $v^+(x)$, $v^-(x) \in \boldsymbol{R}^k$ (outer and inner trace, respectively, of v at x in the direction $\boldsymbol{n}$) such that (see [ZI], 5.14.3)

$$\lim_{\rho \to 0} \rho^{-m} \int_{\{y \in B_\rho(x);\, y \cdot \boldsymbol{n} > 0\}} |v(y) - v^+(x)|\, dy = 0\,,$$

$$\lim_{\rho \to 0} \rho^{-m} \int_{\{y \in B_\rho(x);\, y \cdot \boldsymbol{n} < 0\}} |v(y) - v^-(x)|\, dy = 0\,,$$

and also

$$|Dv|_T \geq \int_\Omega |\nabla v|\, dx + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^{m-1} \tag{2.1}$$

(see [FE], 4.5.9(15)).

In recent papers (see [DG], [DGA]), for studying some free discontinuity problems, a class of functions with special bounded variation has been considered. Such functions are characterized by a property stronger than (2.1), as shown in the following definition.

Definition 2.1: $SBV(\Omega, \boldsymbol{R}^k)$ denotes the class of all functions $v \in BV(\Omega, \boldsymbol{R}^k)$ such that

$$|Dv|_T = \int_\Omega |\nabla v|\, dx + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^{m-1}. \tag{2.2}$$

By the previous definition it follows as in [AM1], Proposition 3.1, that $v \in SBV(\Omega, \boldsymbol{R}^k)$ if and only if $v \in BV(\Omega, \boldsymbol{R}^k)$ and

$$Dv = \nabla v\, dx + (v^+ - v^-) \otimes \boldsymbol{n}\, d\mathcal{H}^{m-1} \llcorner S_v\,,$$

where $\mathcal{H}^{m-1} \llcorner S_v(B) = \mathcal{H}^{m-1}(B \cap S_v)$ for any Borel set B.

In the theory of elastic-perfectly plastic plates developed by F. Demengel in [DE1], [DE2] the space $BH(\Omega)$ of the functions with bounded hessian in Ω has been introduced. Namely

$$BH(\Omega) = \{v \in W^{1,1}(\Omega);\, D^2 v \in \mathcal{M}(\Omega, M_{m,m})\} = \{v \in L^1(\Omega);\, Dv \in BV(\Omega, R^m)\},$$

where $D^2 v$ denotes the distributional hessian of v. The space $BH(\Omega)$ is endowed with the norm

$$\|v\|_{BH(\Omega)} = \|v\|_{L^1(\Omega)} + \|Dv\|_{L^1(\Omega)} + |D^2 v|_T, \tag{2.3}$$

and it is the dual of a Banach space.

In the present paper we introduce the space $SBH(\Omega)$ of the functions with special bounded hessian.

Definition 2.2: We define $SBH(\Omega)$ as the class of all functions $v \in L^1(\Omega)$ such that $Dv \in SBV(\Omega, R^m)$.

Remark 2.3: By the definition $SBH(\Omega)$ is a closed subspace of $BH(\Omega)$ with respect to the strong norm (2.3), while it is not closed with respect to the w^*-$BH(\Omega)$ topology. The following properties can be deduced immediately for a function $v \in SBH(\Omega)$:

1) the distributional derivative of v is absolutely continuous with respect to $\mathcal{L}^m$, hence we have for every $v \in SBH(\Omega)$

$$\nabla v = Dv, \quad \nabla Dv = \nabla^2 v, \quad \nabla \cdot Dv = \Delta^a v,$$

where $\Delta^a v$ is the Radon-Nikodym derivative with respect to $\mathcal{L}^m$ of the distributional Laplacian Δv (in the following we shall use always the notations in the right hand side of the previous equalities);

2) $S_{Dv} = \bigcup_{i=1}^{m} S_{D_i v}$;

3) $|D^2 v|_T = \int_\Omega |\nabla^2 v|\, dx + \int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^{m-1}$,

where $[Du] = (Dv)^+ - (Dv)^-$.

Now we list some embedding results for the space $BH(\Omega)$ which follow immediately from theorems proved in [DE1-2].

Theorem 2.4: Let $\Omega \subset R^m$ $(m > 1)$ be a bounded open set with the exterior cone property. Then

$$BH(\Omega) \subset W^{1,q}(\Omega), \tag{2.4}$$

with continuous embedding if $q \leq m/(m-1)$; the embedding is compact if $q < m/(m-1)$. In particular

$$BH(\Omega) \subset L^q(\Omega), \tag{2.5}$$

for $q \leq m/(m-2)$ (compactly when the inequality is strict) if $m > 2$; for any $q \geq 1$ (compactly for finite q) if $m = 2$. If $\Omega \subset R$ is a bounded interval then $BH(\Omega)$ is compactly embedded in $C^{0,1}(\Omega)$.

Since we want to consider both the case of smooth domains and polygonal ones, we introduce the following definition.

Definition 2.5: We say that a set $\Omega \subset R^m$ $(m \geq 1)$ satisfies the property $\mathcal{R}$ if it is a bounded connected open set and

Ω is strongly Lipschitz and $\partial\Omega$ is the union of finitely many C^2 curves, if $m = 2$,

Ω is C^2 uniformly regular, if $m > 2$ (see e.g. [AD], 4.5, 4.6).

Theorem 2.6: Let $\Omega \subset R^2$ be an open set. If $v \in BH(\Omega)$ has compact support in Ω then

$$\|v\|_{L^\infty(\Omega)} \leq \frac{1}{4}|D^2 v|_T. \tag{2.6}$$

If Ω satisfies property $\mathcal{R}$ then

$$BH(\Omega) \subset C^0(\overline{\Omega}) \tag{2.7}$$

(see Theorem 3.3 and Remark 3.2 in [DE1]).

The following extension theorem holds (see Theorem 2.2, Remark 2.1 in [DE1]).

Theorem 2.7: Let $\Omega \subset R^m$ with property $\mathcal{R}$ and let $\bar{\bar{x}} \in \Omega$. Then there is a constant $M = M(\Omega) > 0$ and a linear continuous map $\Pi: BH(\Omega) \to BH(R^m)$ such that

$$\Pi v = v \quad \text{a.e. in } \Omega, \quad \operatorname{spt}(\Pi v) \subset B_t(\bar{x}) \text{ where } t = 2 \operatorname{diam} \Omega,$$

$$\|\Pi v\|_{BH(R^m)} \leq M \|v\|_{BH(\Omega)},$$

$$\Pi(W^{2,1}(\Omega)) \subset W^{2,1}(R^m).$$

Remark 2.8: If Ω does not satisfy property $\mathcal{R}$ both Theorems 2.6, 2.7 may fail: take, for instance, $v(x, y) = \log \sqrt{x^2 + y^2}$ and $\Omega = \{(x, y): 0 < x < 1, |y| < x^2\}$. Then $v|_\Omega$ belongs to $BH(\Omega)$ but it is unbounded. Of course $v|_{B_1} \notin BH(B_1)$.

In the space $BH(\Omega)$ the following trace theorem holds.

Theorem 2.9: Let $\Omega \subset R^m$ with the property $\mathcal{R}$. Two bounded linear maps exist

$$\gamma_0: BH(\Omega) \to W^{1,1}(\partial\Omega),$$

$$\gamma_1: BH(\Omega) \to L^1(\partial\Omega),$$

such that

$$\gamma_0(v) = v\Big|_{\partial\Omega}, \quad \gamma_1(v) = \frac{\partial v}{\partial N}\Big|_{\partial\Omega},$$

for every $v \in C^2(\overline{\Omega})$, where N is the outward normal to $\partial\Omega$. Moreover γ_1 is onto (see [DE1], Appendix).

As an application of the previous theorem the following «matching lemma» can be proved.

Lemma 2.10: Let $A, A' \subset R^m$ be open sets, $A \cap A' = \emptyset$, and let Γ be a C^2 uniformly regular (relatively) open subset of ∂A such that $\overline{\Gamma} = \overline{A} \cap \overline{A}'$. Let $v \in BH(A)$, $v' \in BH(A')$ and set

$$\bar{v} = \begin{cases} v & \text{in } A \\ v' & \text{in } A'. \end{cases}$$

Then

$$\bar{v} \in BH(A \cup A' \cup \Gamma) \quad \text{if and only if } \gamma_0(v) = \gamma_0(v') \text{ in } \Gamma,$$

and in this case

$$D^2\bar{v} = D^2 v + D^2 v' - (\gamma_1(v) + \gamma_1(v'))\mathcal{N} \otimes \mathcal{N}\, d\mathcal{H}^{m-1} \llcorner \Gamma,$$

where $\mathcal{N}$ is the unit normal vector to Γ pointing toward A',

$$\gamma_1(v) = (Dv)\cdot\mathcal{N}, \quad \gamma_1(v') = (Dv')\cdot(-\mathcal{N}).$$

For the proof of this lemma see e.g. [DE1], Theorem 2.1 and Appendix.

Remark 2.11: We notice explicitely that $C^\infty(\Omega)$ is neither dense in $BH(\Omega)$ nor in $SBH(\Omega)$ with respect to the strong topology, nevertheless, if Ω is strongly Lipschitz, the density holds true with respect to the intermediate topology associated to the distance

$$d_2(u, v) = \|u - v\|_{L^1(\Omega)} + \left| \int_\Omega |D^2 u| - \int_\Omega |D^2 v| \right|,$$

as in the case of $BV(\Omega)$ and $SBV(\Omega)$ with

$$d_1(u, v) = \|u - v\|_{L^1(\Omega)} + \left| \int_\Omega |Du| - \int_\Omega |Dv| \right|,$$

(see [TE], III, 2.8 and I, 1.3, where the result is obtained by mollification of the trivial extension).

From now on we denote by $\mathcal{P}_1(\Omega)$ the space of the affine functions. We define the linear map $p: BH(\Omega) \to \mathscr{P}_1(\Omega)$ by

(2.8) $$(pv)(x) = v_\Omega + (\nabla v)_\Omega \cdot (x - x_\Omega) \quad \forall v \in BH(\Omega),$$

where

$$v_\Omega = |\Omega|^{-1} \int_\Omega v\,dx, \quad (\nabla v)_\Omega = |\Omega|^{-1} \int_\Omega \nabla v\,dx, \quad x_\Omega = |\Omega|^{-1} \int_\Omega x\,dx.$$

Obviously

(2.9) $$pv = v \quad \forall v \in \mathscr{P}_1,$$

(2.10) $$p(v - pv) = 0 \quad \forall v \in BH(\Omega).$$

Now we can prove a Poincaré type inequality.

THEOREM 2.12: Assume $\Omega \subset \mathbf{R}^m$ with the property $\mathscr{R}$. Then there are constants $C = C(\Omega) > 0$, $S = 1 + \sqrt{m}\,C(1 + C)$ such that

(2.11) $$\|v\|_{L^1(\Omega)} \leq C|Dv|_T \quad \forall v \in BV(\Omega) \text{ with } \int_\Omega v\,dx = 0,$$

(2.12) $$\|v - pv\|_{BH(\Omega)} \leq S|D^2 v|_T \quad \forall v \in BH(\Omega).$$

PROOF: Take $v_h \in C^\infty(\Omega)$, such that, referring to Remark 2.11,

$$d_1(v_h, v) \to 0, \quad \int_\Omega v_h\,dx = 0,$$

then the well known Poincaré inequality in $W^{1,1}(\Omega)$ gives the existence of a constant $C = C(\Omega)$ such that

$$\|v_h\|_{L^1(\Omega)} \leq C\|Dv_h\|_{L^1(\Omega)} = C|Dv_h|_T \quad \forall h \in N$$

and since $|Dv_h|_T \to |Dv|_T$, we get (2.11).

Then, by choosing $v_h \in C^\infty(\Omega)$ such that $d_2(v_h, v) \to 0$ and by using (2.10), (2.11), we get

$$\begin{aligned}
\|v_h - pv_h\|_{BH(\Omega)} = \|v_h - pv_h\|_{W^{2,1}(\Omega)} = {}& \\
&= \|v_h - pv_h\|_{L^1(\Omega)} + \|D(v_h - pv_h)\|_{L^1(\Omega, R^m)} + \|D^2 v_h\|_{L^1(\Omega, M_{m\times m})} \leq \\
&\leq (1 + C)\|D(v_h - pv_h)\|_{L^1(\Omega, R^m)} + \|D^2 v_h\|_{L^1(\Omega, M_{m\times m})} \leq \\
&\leq (1 + \sqrt{m}\,C(1 + C))\|D^2 v_h\|_{L^1(\Omega, M_{m\times m})} = \\
&= S|D^2 v_h|_T .
\end{aligned}$$

Taking the limit as $h \to +\infty$, inequality (2.12) follows. ■

By summarizing Theorems 2.4, 2.6, 2.7 and 2.12 and with a simple calculation in the case $m = 1$, we get

THEOREM 2.13: For any set $\Omega \subset R^m$ with property $\mathcal{R}$, there is a constant $\vartheta(\Omega) > 0$, such that for every $v \in BH(\Omega)$

$$\|v - pv\|_{L^{m/(m-2)}(\Omega)} \leq \vartheta(\Omega)|D^2 v|_T \quad \text{if } m > 2,$$
$$\|v - pv\|_{L^\infty(\Omega)} \leq \vartheta(\Omega)|D^2 v|_T \quad \text{if } m = 1, 2.$$

More precisely $\vartheta(\Omega) = (1/4)MS$ if $m = 2$ and $\vartheta(\Omega) = (1/2)|\Omega|$ if $m = 1$.

REMARK 2.14: We remark that if σ is a continuous seminorm on $BH(\Omega)$ and a norm on $\mathcal{P}_1$, then $\sigma(v) + |D^2 v|_T$ is a norm on $BH(\Omega)$ equivalent to $\|\cdot\|_{BH(\Omega)}$. We may set for instance

$$\sigma(v) = \left|\int_\Omega v\,dx\right| + \sum_{i=1}^m \left|\int_\Omega D_i v\,dx\right|.$$

In the next theorem we point out the structure of the singular part of the hessian matrix of a function $v \in SBH(\Omega)$. On this subject we refer also to [AG] and [AL].

THEOREM 2.15: Let $\Omega \subset R^m$ be an open set and $v \in SBH(\Omega)$. Then

1) $(D^2 v)^s = [Dv] \otimes n\, d\mathcal{H}^{m-1} \llcorner S_{Dv} = \left[\dfrac{\partial v}{\partial n}\right] n \otimes n\, d\mathcal{H}^{m-1} \llcorner S_{Dv}$,

2) $|(D^2 v)^s|_T = \displaystyle\int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^{m-1} = \int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial n}\right]\right| d\mathcal{H}^{m-1}$,

3) $|(D^2 v)^s|_T = |\Delta^s v|_T$,

where $\partial v/\partial n = n \cdot Dv$, $(D^2 v)^s$ and $\Delta^s v$ denote respectively the singular part of the distributional hessian and laplacian of v with respect to $\mathcal{L}^m$.

PROOF: By the definition of $SBH(\Omega)$ we have $Dv \in SBV(\Omega, R^m)$ so that the first equalities in 1) and in 2) immediately follow. By Lemma 2.10 we get

$$[Dv] = \left[\frac{\partial v}{\partial n}\right] n.$$

Since the singular part of the hessian matrix is rank one and symmetric, again by Lemma 2.10 we get

$$|(D^2 v)^s|_T = \int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial n}\right]\right| d\mathcal{H}^{m-1} = \int_{S_{Dv}} \left|\sum_{i=1}^m [D_i v] n_i\right| d\mathcal{H}^{m-1} = |\Delta^s v|_T,$$

and the proof is achieved. ■

We state two basic results on SBV functions, that will be applied to the gradients of a minimizing sequence for the functional (1.3). The first is a compactness property and the other one is related to semicontinuity.

THEOREM 2.16: Let $\Omega \subset \mathbf{R}^m$ be an open set with property $\mathcal{R}$. Let $\phi : [0, +\infty[\to$ $\to [0, +\infty]$ be a convex, non decreasing function satisfying the condition

$$\lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty, \tag{2.13}$$

let a, b be strictly positive constants and let $\{z_h\}_{h \in \mathbf{N}}$ be a sequence of functions in $SBV(\Omega, \mathbf{R}^k)$ such that

$$\int_\Omega z_h dx = 0 \quad \forall h \in \mathbf{N},$$

$$\sup_{h \in \mathbf{N}} \left\{ \int_\Omega \phi(|\nabla z_h|)\, dx + \int_{S_{z_h}} (a + b|z_h^+ - z_h^-|)\, d\mathcal{H}^{m-1} \right\} < +\infty.$$

Then there is a function $z \in SBV(\Omega, \mathbf{R}^k)$ and a subsequence $\{z_{h_i}\}_{i \in \mathbf{N}}$ such that

1) $z_{h_i} \to z$ strongly in $L^1(\Omega, \mathbf{R}^k)$;

2) $\nabla z_{h_i} \rightharpoonup \nabla z$ weakly in $L^1(\Omega, M_{k,m})$;

3) $Dz_{h_i} - \nabla z_{h_i} dx = (z_{h_i}^+ - z_{h_i}^-) \otimes n\, d\mathcal{H}^{m-1} \llcorner S_{z_{h_i}}$ converges weakly* in $\mathcal{M}(\Omega, M_{k,m})$ to $Dz - \nabla z\, dx = (z^+ - z^-) \otimes n\, d\mathcal{H}^{m-1} \llcorner S_z$;

4) $\int_\Omega z\, dx = 0$.

PROOF: We can assume $\phi(s) \geq cs - d$ for every $s \in [0, +\infty[$, so that there exists a constant C' such that

$$|Dz_h|_T \leq C'$$

and, by Theorem 2.12,

$$\|z_h\|_{L^1} \leq C'.$$

Hence there exists a subsequence, still denoted by z_h, and a function $z \in BV(\Omega, \mathbf{R}^k)$ such that $z_h \to z$ in $L^1(\Omega, \mathbf{R}^k)$. Let $\alpha > 0$ and denote by z_h^α the vector valued function whose components are $(z_h^\alpha)_j = ((z_h)_j \wedge \alpha) \vee (-\alpha)$ $(j = 1, \dots, k)$. Then for every α we have $z_h^\alpha \to z^\alpha$ in $L^1(\Omega, \mathbf{R}^k)$ and

$$\sup_h \left\{ \|z_h^\alpha\|_{L^\infty} + \int_\Omega |Dz_h^\alpha| \right\} < +\infty,$$

so by Theorem 2.1 of [AM2] there exists a subsequence such that $z_{h_i}^\alpha \to z^\alpha \in$ $\in SBV(\Omega, \mathbf{R}^k)$ in $L^1(\Omega, \mathbf{R}^k)$. Since $z \in BV(\Omega, \mathbf{R}^k)$ and $z^\alpha \in SBV(\Omega, \mathbf{R}^k)$ for every $\alpha > 0$, then we obtain $z \in SBV(\Omega, \mathbf{R}^k)$.

The second assertion is proved in [AM2], Theorem 2.2. The third assertion follows by difference and the fourth one is trivial. ■

THEOREM 2.17: Let $\{z_h\}_{h\in N}$ be a sequence in $SBV(\Omega, R^k)$. Assume that $\{z_h\}_{h\in N}$ converges in measure to z and that $\{\nabla z_h\}_{h\in N}$ is weakly compact in $L^1(A, M_{k,m})$ for every open set $A \subset\subset \Omega$. Moreover let $\theta: [0, +\infty[\to [1, +\infty[$ be a concave, non decreasing function. Then

$$\int_{S_z} \theta(|z^+ - z^-|)\, d\mathcal{H}^{m-1} \le \liminf_{h\to+\infty} \int_{S_{z_h}} \theta(|z_h^+ - z_h^-|)\, d\mathcal{H}^{m-1}.$$

PROOF: See Theorem 3.7 in [AM2]. ■

3. - The Neumann problem

Let us denote by $\Omega \subset R^m$ an open set with property $\mathcal{R}$ (see Definition 2.5). We study the following variational problem

(3.1) minimize the functional $\mathcal{F} = E - L$ over $SBH(\Omega)$,

where (taking into account Remark 2.3)

$$E(v) = \int_\Omega Q(\nabla^2 v)\, dx + \int_{S_{\nabla v}} \left(1 + \left|\left[\frac{\partial v}{\partial n}\right]\right|\right) d\mathcal{H}^{m-1}, \tag{3.2}$$

$Q: M_{m,m} \to [0, +\infty)$ is a given positive definite quadratic form such that

$$\exists \alpha > 0: Q(\xi) \ge \alpha|\xi|^2 \quad \forall \xi \in M_{m,m}, \tag{3.3}$$

and L is the prescribed transverse load satisfying

$$\begin{cases} (L, v) = \int_\Omega gv\, dx + \int_{\Gamma\cap\Omega} hv\, d\mathcal{H}^{m-1} + \int_{\partial\Omega} lv\, d\mathcal{H}^{m-1}, \\ \Gamma \text{ is a } C^2 \text{ hypersurface in } R^m, \quad g \in L^q(\Omega),\ h \in L^r(\Gamma),\ l \in L^s(\partial\Omega) \\ \text{with } q > \frac{m}{2},\ r, s > m-1 \quad (r, s \ge 1 \text{ if } m = 2). \end{cases} \tag{3.4}$$

First we consider the case $m = 2$, so that Ω may be regarded as the natural state of an unloaded *elastic-plastic plate*. We remark that for the load L we have

$$|L|_T = \|g\|_{L^1(\Omega)} + \|h\|_{L^1(\Gamma\cap\Omega)} + \|l\|_{L^1(\partial\Omega)}.$$

THEOREM 3.1: Assume $m = 2$, the property $\mathcal{R}$, (3.2), (3.3), (3.4) and

(3.5) $\langle L, v\rangle = 0$ for every affine displacement v (compatibility condition),

(3.6) $$|L|_T < 4\frac{\alpha \wedge 1}{MS} \quad \text{(safe load condition)},$$

where M, S are defined in Theorems 2.7 and 2.12. Then the problem (3.1) has a solution. Moreover (3.5) is a necessary condition in order to have $\inf \mathcal{F} > -\infty$.

PROOF: The necessity of (3.5) follows from the simple remark that

$$E(v) = \langle L, v \rangle \quad \text{for any } v \text{ affine}. \tag{3.7}$$

Let us show now the sufficiency of (3.2)-(3.6). Referring to (2.8) and Theorem 2.13, for every $v \in SBH(\Omega)$ we get

$$\langle L, v \rangle = \langle L, v - pv \rangle \leqslant |L|_T \|v - pv\|_{L^\infty(\Omega)} \leqslant \delta |L|_T |D^2 v|_T; \tag{3.8}$$

by Hölder inequality and by the inequality $\beta^2 \geqslant \alpha\beta - \alpha^2/4 \ \forall \alpha, \beta \in R$, we get

$$\int_\Omega |\nabla^2 v|^2 \, dx \geqslant |\Omega|^{-1} \left(\int_\Omega |\nabla^2 v| \, dx \right)^2 \geqslant \int_\Omega |\nabla^2 v| \, dx - \frac{|\Omega|}{4}. \tag{3.9}$$

By Remark 2.3(3) and by (3.2), (3.3), (3.8) and (3.9), for every $v \in SBH(\Omega)$ we have

$$\mathcal{F}(v) \geqslant a \int_\Omega |\nabla^2 v| \, dx + \int_{S_{Dv}} |[Dv]| \, d\mathcal{H}^1 - \langle L, v \rangle - a \frac{|\Omega|}{4} \geqslant \tag{3.10}$$

$$\geqslant ((a \wedge 1) - \delta |L|_T) |D^2 v|_T - a \frac{|\Omega|}{4}.$$

Then, by (3.6) the functional is bounded from below. If $\{v_h\}_{h \in N}$ is a minimizing sequence for $\mathcal{F}$, there is $C' > 0$ such that

$$|D^2 v_h|_T \leqslant C'. \tag{3.11}$$

Set

$$u_h = v_h - pv_h,$$

thanks to (3.5) $\{u_h\}_{h \in N}$ is a minimizing sequence too, and by (2.12) and (3.11) we get the existence of C'' such that

$$\|u_h\|_{BH(\Omega)} \leqslant C''. \tag{3.12}$$

Then by Theorem 2.4, up to subsequences, there is $u \in W^{1,1}(\Omega)$ such that

$$u_h \to u \quad \text{in } W^{1,1}(\Omega). \tag{3.13}$$

By applying Theorem 2.16 to $z_h = Du_h$, we get

$$u \in SBH(\Omega). \tag{3.14}$$

By (3.3) we may assume

$$\nabla^2 u_h \rightharpoonup \nabla^2 u \quad \text{weakly in } L^2(\Omega).$$

By lower semicontinuity of positive definite quadratic forms

$$\int_\Omega Q(\nabla^2 u)\,dx \le \liminf_h \int_\Omega Q(\nabla^2 u_h)\,dx. \tag{3.15}$$

By applying Theorem 2.17 with $z_h = Du_h$ and $\theta(s) = 1 + |s|$, due to la Vallée Poussin criterion and Theorem 2.15, we get

$$\int_{S_{Du}} \left(1 + \left|\left[\frac{\partial u}{\partial n}\right]\right|\right) d\mathcal{H}^1 \le \liminf_h \int_{S_{Du_h}} \left(1 + \left|\left[\frac{\partial u_h}{\partial n}\right]\right|\right) d\mathcal{H}^1. \tag{3.16}$$

By Theorems 2.4 and 2.9, and by the Rellich Theorem for $W^{1,1}(\partial\Omega)$

(3.17) $\qquad L$ is linear and w^*-$BH(\Omega)$ continuous;

hence, by summarizing (3.14)-(3.17),

$$\mathcal{F}(u) \le \liminf_h \mathcal{F}(u_h) = \lim_h \mathcal{F}(v_h) = \inf \mathcal{F},$$

so that u minimizes $\mathcal{F}$ over $SBH(\Omega)$. ∎

Hypothesis (3.6) is a safe load condition which prevents plastic collapse of the free plate when submitted to a balanced system of applied forces.

Remark 3.2: Theorem 3.1 still holds modifying L by adding in (3.4) any linear functional $\mathcal{L}$ that is w^*-$BH(\Omega)$ continuous on the sublevels of E, i.e.

$$\sup_h E(v_h) < +\infty, \qquad v_h \xrightarrow{*} v \Rightarrow \lim_h \langle \mathcal{L}, v_h\rangle = \langle \mathcal{L}, v\rangle.$$

One interesting example is

$$\mathcal{L} = \sum_{i=1}^{+\infty} c_i \delta_{x_i} \quad \text{where} \quad \sum_{i=1}^{+\infty} |c_i| < +\infty, \; x_i \in \Omega, \tag{3.18}$$

as shown by the following statement.

Lemma 3.3: Let $m = 2$ and let Ω, E be as like as in Theorem 3.1. Assume

$$v_h \in SBH(\Omega) \ \forall h \in N, \quad \sup_h E(v_h) < +\infty, \quad v_h \to v \text{ in } L^1(\Omega).$$

Then $\lim_h v_h(x) = v(x)$ for every $x \in \Omega$. Moreover, if (3.18) holds then $\lim_h \langle \mathcal{L}, v_h\rangle = \langle \mathcal{L}, v\rangle$.

Proof: By the assumption we have

$$\sup_h |D^2 v_h|_T < +\infty,$$

hence, by Theorem 2.12,

$$\sup_h \|v_h - pv_h\|_{L^\infty(\Omega)} < +\infty ,$$

and by Theorems 2.4 and 2.6 we conclude that $(v_h - pv_h)$ converges strongly in $W^{1,1}(\Omega)$ and is bounded in L^∞. The boundedness of (v_h) in $L^1(\Omega)$ and the structure of pv_h entail that also (v_h) converges to v in $W^{1,1}(\Omega)$ and is bounded in $L^\infty(\Omega)$. By Theorem 2.16 we have also $v \in SBH(\Omega)$. Following the argument of Theorem 1.3 of [DE2], fix $x_0 \in \Omega$ and $\varphi \in C_0^\infty(R^2)$ with $\varphi(0) = 1$, $\varphi \geq 0$, φ decreasing with respect to $|x|$. Let $\varphi_h(x) = \varphi(h(x - x_0))$ for every $h \in N$. Set $w_h = |v_h - v|$. For h big enough, $\varphi_h w_h$ has compact support in Ω, hence by Theorem 2.6 we have:

$$w_h(x_0) = \varphi_h(x_0) w_h(x_0) \leq |D^2(\varphi_h w_h)|_T \leq$$

$$\leq \int_\Omega \varphi_h |D^2 w_h| + 2 \int_\Omega |D\varphi_h| \, |Dw_h| \, dx + \int_\Omega |D^2 \varphi_h| \, w_h \, dx .$$

Since both w_h and φ_h converge to 0 strongly in $W^{1,1}(\Omega)$ and $D^2 \varphi_h$ is bounded in L^1, then for any $i \in N$ we have:

$$\limsup_h w_h(x_0) \leq \limsup_h \int_\Omega \varphi_h |D^2 w_h| \leq \limsup_h \int_\Omega \varphi_i |D^2 w_h| = \int_\Omega \varphi_i \, d\mu ,$$

where, by Theorem 2.16, μ is a non atomic measure. By the arbitrariness of i we get $\limsup_h w_h \leq \mu(\{x_0\}) = 0$, so that we have proved $\lim_h v_h(x_0) = v(x_0)$.

The last statement of the theorem follows by the boundedness of $\{v_h\}_{h \in N}$ in $L^\infty(\Omega)$ and by the dominated convergence theorem in the space of absolutely convergent series. ■

Arguing as like as in the proof of Theorem 3.1 we can show the following statement for an *elastic-plastic rod*. It is worth remarking that $\mathcal{H}^0$ is the counting measure.

THEOREM 3.4: Assume $m = 1$, Ω is a bounded open interval in $\boldsymbol{R}$ (the undeformed state of the rod), and the deformation energy E satisfies (3.2), (3.3). The transverse load L is a measure with bounded total variation in Ω satisfying (3.5) and

$$|L|_T < 2 \frac{a \wedge 1}{|\Omega|} .$$

Then the problem (3.1) has a solution. Moreover (3.5) is a necessary condition in order to have $\inf \mathcal{F} > -\infty$.

We conclude this section by considering the case $\Omega \subset \boldsymbol{R}^m$ with $m > 2$.

THEOREM 3.5: Let $m > 2$ and $\Omega \subset R^m$ with the property $\mathcal{R}$. Assume (3.2), (3.3), (3.4) and (3.5). Then there exists $\xi = \xi(\Omega, \Gamma, q, r, s, a) > 0$ such that

$$\|L\|_* \stackrel{\text{def}}{=} \|g\|_{L^q(\Omega)} + \|b\|_{L^r(\Gamma)} + \|l\|_{L^s(\partial\Omega)} < \xi, \tag{3.19}$$

entails that the problem (3.1) has a solution. Condition (3.5) is necessary in order to have $\inf \mathcal{F} > -\infty$.

PROOF: The argument is the same as in the proof of Theorem 3.1, except when showing inequality (3.8) and the w^*-continuity of L. By Theorems 2.4 and 2.9 we have for every $v \in SBH(\Omega)$

$$\begin{aligned}(3.20)\quad \langle L, v\rangle = \langle L, v - pv\rangle \leq \\ \leq \|g\|_{L^q(\Omega)} \|v - pv\|_{L^{q/(q-1)}(\Omega)} + \|b\|_{L^r(\Gamma)} \|v - pv\|_{L^{r/(r-1)}(\Gamma)} + \\ + \|l\|_{L^s(\partial\Omega)} \|v - pv\|_{L^{s/(s-1)}(\partial\Omega)} \leq \xi' \|L\|_* |D^2 v|_T,\end{aligned}$$

where ξ' follows by Theorems 2.13 and 2.9. Choosing $\xi = (a \wedge 1)/\xi'$, we obtain a compact minimizing sequence as in Theorem 3.1. Moreover L is w^*-$BH(\Omega)$ continuous due to the summability of g, b, l and to the embeddings in Theorems 2.4, 2.9. ∎

REMARK 3.6: Analogously to the framework of various models for image segmentation (see [BZ]), it is interesting to consider a different «perturbation term» added to the functional (3.2), namely one can consider (for any m) the functional

$$\int_\Omega Q(\nabla^2 v)\, dx + \int_{S_{Dv}} \left(1 + \left|\left[\frac{\partial v}{\partial n}\right]\right|\right) d\mathcal{H}^{m-1} + \int_\Omega |v - g|^2 dx. \tag{3.21}$$

We emphasize that the functional (3.21), under the only assumptions (3.3) and $g \in L^2(\Omega)$, with Ω satisfying property $\mathcal{R}$, has a minimizer in $SBH(\Omega)$ by the same argument of the proof of Theorem 3.5.

4. - THE DIRICHLET PROBLEM

Let $\Omega \subset R^m$ be an open set, $w: R^m \to R$ and let L_0 be the prescribed load. We study the variational problem

$$\text{minimize the functional } \mathcal{F}_0 = E_0 - L_0 \quad \text{over } \mathcal{C}_0, \tag{4.1}$$

where $\mathcal{C}_0 = \{v \in SBH(R^m);\ v = w \text{ in } R^m \setminus \overline{\Omega}\}$,

$$E_0(v) = \int_{R^m} Q(\nabla^2 v)\, dx + \int_{S_{Dv}} \left(1 + \left|\left[\frac{\partial v}{\partial n}\right]\right|\right) d\mathcal{H}^{m-1}. \tag{4.2}$$

Q is a quadratic form with property (3.3), and the data w, L_0 satisfy

$$\text{(4.3)} \qquad \begin{cases} w \in W^{2,2}(\mathbf{R}^m), \quad \operatorname{spt} w \text{ compact}, \\ \langle L_0, v\rangle = \int\limits_{\Omega} gv\,dx + \int\limits_{\Gamma\cap\Omega} hv\,d\mathcal{H}^{m-1}, \\ \Gamma \text{ is a } C^2 \text{ hypersurface in } \mathbf{R}^m, \quad g \in L^q(\Omega), h \in L^r(\Gamma) \\ \text{with } q > \frac{m}{2}, r > m-1 \quad (r \geq 1 \text{ if } m = 2). \end{cases}$$

As usual in non reflexive problems, we prescribe the Dirichlet datum by imposing coincidence with a suitable function outside $\overline{\Omega}$. As a consequence the Dirichlet condition on the gradient is relaxed since an admissible displacement $v \in \mathcal{C}\!l_0$ may have a discontinuity set for the gradient even on the boundary of Ω; in such case the amount of the penalization is (as follows by using Lemma 2.10)

$$\int\limits_{S_{Dv}\cap\partial\Omega} \left(1 + \left|\left(\frac{\partial w}{\partial N}\right)^{+} - \left(\frac{\partial v}{\partial N}\right)^{-}\right|\right) d\mathcal{H}^{m-1},$$

where $+, -$ denote respectively the outer and inner trace.

At first we set $m = 2$; in this case problem (4.1) may be regarded as a weak formulation of the *clamped elastic-plastic plate*.

THEOREM 4.1: Assume $m = 2$ and Ω is any open set. Assume also (3.3), (4.2), (4.3) and

$$\text{(4.4)} \qquad |L_0|_T < 4(a \wedge 1).$$

Then the problem (4.1) has a solution.

PROOF: Referring to Theorem 2.6 we have

$$\text{(4.5)} \qquad \langle L_0, v\rangle \leq |L_0|_T \|v\|_{L^*} \leq \frac{1}{4}|L_0|_T |D^2 v|_{T(\mathbf{R}^2)} \quad \forall v \in \mathcal{C}\!l_0.$$

Assume $\operatorname{spt} w \subset B_r$. Then, arguing as in (3.9), (3.10) and replacing Ω by B_r, we obtain

$$\mathcal{F}_0(v) \geq a\int\limits_{\mathbf{R}^2} |\nabla^2 v|\,dx + \int\limits_{S_{Dv}} |[Dv]|\,d\mathcal{H}^1 - \langle L_0, v\rangle - a\frac{|B_r|}{4} \geq$$

$$\geq \left((a \wedge 1) - \frac{1}{4}|L_0|_T\right)|D^2 v|_{T(\mathbf{R}^2)} - a\frac{|B_r|}{4} \quad \forall v \in \mathcal{C}\!l_0,$$

hence by assumption (4.4) the functional $\mathcal{F}_0$ is bounded from below and coercive in $SBH(\mathbf{R}^2)$. If $\{u_h\}_{h\in\mathbf{N}}$ is a minimizing sequence for $\mathcal{F}_0$, then by (2.6) and Remark 2.14 we get (3.12) and the conclusion follows as in Theorem 3.1. ■

By considering the case $m = 1$, we can show the following statement for an *elastic-plastic clamped rod.*

THEOREM 4.2: Let $m = 1$, M is any open interval in R, $w \in W^{2,2}(R)$, spt w compact and the deformation energy satisfies (3.3), (4.2). The transverse load L_0 is a measure with bounded total variation in Ω satisfying

$$|L_0|_T < 2(a \wedge 1).$$

Then the problem (4.1) has a solution.

We conclude this section by considering the case $\Omega \subset R^m$ with $m > 2$.

THEOREM 4.3: Let $m > 2$ and $\Omega \subset R^m$ is any open set. Assume (3.3), (4.2), (4.3). Then there exists $\tau = \tau(\Omega, \Gamma, q, r, a) > 0$ such that

$$\|L_0\|_* \stackrel{\text{def}}{=} \|g\|_{L^q(\Omega)} + \|b\|_{L^r(\Gamma)} < \tau,$$

entails that the problem (4.1) has a solution.

PROOF: The argument is the same as in the proof of Theorem 4.1, except when showing inequality (4.5). By Theorems 2.4 and 2.9 we have

$$\langle L_0, v\rangle \leq \|g\|_{L^{m/2}(\Omega)}\|v\|_{L^{m/(m-2)}(\Omega)} + \|b\|_{L^{m-1}(\Gamma)}\|v\|_{L^{(m-1)/(m-2)}(\Gamma)} \leq \tau'\|L_0\|_* \, |D^2 v|_{T(R^m)},$$

where τ' follows by Theorems 2.9, 2.12, and 2.13. Hence we obtain a compact minimizing sequence and the conclusion follows as in Theorem 3.5. ■

REMARK 4.4: We enphasize that no regularity of $\partial\Omega$ is required in this section. The fact is that property $\mathcal{R}$, assumed elsewhere, is used only to extend outside Ω the competing functions.

All the statements of this section still hold when the datum w satisfies

$$w \in SBH(R^m), \quad \text{spt}\, w \text{ compact}, \quad E_0(w) < +\infty.$$

5. - THE OBSTACLE PROBLEM

In this section we study the existence of equilibrium for the *elastic-perfectly plastic plate in presence of a flat rigid obstacle* and subject to prescribed dead loads. We still assume that the natural state of the unloaded plate is an open bounded subset Ω of R^2 with property $\mathcal{R}$ (see Definition 2.5).

We introduce the following weak formulation of the obstacle problem in any dimension $m \geq 1$

$$(5.1) \qquad \begin{cases} \text{minimize the following functional over } SBH(\Omega), \\ \mathcal{G}(v) = \begin{cases} E(v) - \langle L, v \rangle & \text{if } v \geq 0 \text{ on } U, \\ +\infty & \text{elsewhere}, \end{cases} \end{cases}$$

where E, L are given by (3.2)-(3.4), and $U \subset \boldsymbol{R}^m$ is a given closed set (assuming U closed is not restrictive if $m \leq 2$ due to the continuity of the functions in $SBH(\Omega)$). The inequality $v \geq 0$ has the usual sense in the case $m = 1, 2$, otherwise (referring to sections 4 and 5 of [BBGT]) it has to be understood in the sense quasi everywhere with respect to q-capacity, where $q = m/(m-1)$ and, for any set $T \subset \boldsymbol{R}^m$,

$$\mathrm{cap}_q(T) = \inf \{ \|v\|_{W^{1,q}(\boldsymbol{R}^m)},\ v \text{ l.s.c. and } v(x) \geq 1\ \forall x \in T \}.$$

Notice that the functional $\mathcal{G}$ shows a lack of coerciveness due to the fact that no Dirichlet type condition is imposed: say, the plate is free at the boundary, and fulfils a unilateral constraint on the unknown contact set.

Definition 5.1: Whenever L has nonvanishing resultant (*i.e.* $\langle L, 1 \rangle \neq 0$), we can define the center of mass c of the given system of forces by

$$(5.2) \qquad c = (c_1, \ldots, c_m), \quad c_j = \frac{\langle L, x_j \rangle}{\langle L, 1 \rangle} \quad (j = 1, \ldots, m),$$

where $(x_1, \ldots, x_m)$ is any orthonormal coordinate system in $\boldsymbol{R}^m$.

We can now state the main results of this section whose proofs are postponed.

Theorem 5.2: (Necessary conditions for any m.) Assume there is a solution to problem (5.1). Then

$$(5.3) \qquad \langle L, 1 \rangle \leq 0,$$

and

$$(5.4) \qquad \text{if } \langle L, 1 \rangle = 0, \quad \text{then } \langle L, x_j \rangle = 0 \quad (j = 1, \ldots, m),$$

$$(5.5) \qquad \text{if } \langle L, 1 \rangle < 0, \quad \text{then } c \in \mathrm{co}\, U,$$

where $\mathrm{co}\, U$ is the closed convex hull of U.

By reinforcing the necessary conditions we obtain the existence.

Theorem 5.3: Let $m = 2$, and $\Omega \subset \boldsymbol{R}^2$ with the property $\mathcal{R}$. Assume (3.2), (3.3), (3.4) and

$$(5.6) \qquad \langle L, 1 \rangle < 0,$$

$$(5.7) \qquad c \in \mathring{U}.$$

Then there is $\eta = \eta(\Omega, U, a) > 0$ such that the following smallness condition

$$|L|_T < \eta, \tag{5.8}$$

entails the existence of a solution for problem (5.1).

Remark 5.4: The assumption (5.7) may be substituted by $c \in (\mathrm{co}\, U)^{\circ}$, as it will be clear from the proof. Hypothesis (5.8) is a geometric safe load condition: it entails that the obstacle reaction can balance the load avoiding plastic collapse of the plate. The strict inequality entails compactness of the minimizing sequences.

If $(\Omega \cap \mathrm{co}\, U) \subsetneq \Omega$ then inequality (5.8) is stricter than (3.6) in the sense that, if the constant M, S (defined in Theorems 2.7 and 2.12) are optimal, then η is smaller than $4(a \wedge 1)/MS$, as one can see by the proof of Lemma 5.13.

Remark 5.5: Theorem 5.3 also holds modifying the functional L as like as in Remark 3.2.

Remark 5.6: For any m, if $\langle L, 1 \rangle = 0$ the problem (5.1) has solution if and only if $\langle L, x_j \rangle = 0$ for any $j = 1, \ldots, m$ (the only if part follows from (5.4)).

Theorem 5.7: Let $m = 1$, Ω is any open interval in R and the deformation energy E satisfies (3.2), (3.3). Assume that the transverse load L is a measure with bounded total variation in Ω satisfying (5.6). Then there is $\mu = \mu(\Omega, U, a) > 0$ such that

$$|L|_T < \mu,$$

entails that the problem (5.1) has a solution if and only if $c \in \mathrm{co}\, U$.

Theorem 5.8: Let $m > 2$ and $\Omega \subset R^m$ with the property $\mathcal{R}$. Let $U \subset R^m$ be a closed set. Assume (3.2), (3.3), (3.4), (5.6) and (5.7). Then there exists $\zeta = \zeta(\Omega, U, \Gamma, q, r, s, a) > 0$ such that

$$\|L\|_* \stackrel{\text{def}}{=} \|g\|_{L^q(\Omega)} + \|h\|_{L^r(\Gamma)} + \|l\|_{L^s(\partial\Omega)} < \zeta,$$

entails that the problem (5.1) has a solution.

Finally we give a statement which includes the case of *thin obstacles*, where the notation ri denotes the relative interior of a set, say the set of the internal points in the topology of the affine convex hull, and U_{es} denotes the essential part of the set U with respect to the capacity (see Definition 4.3 of [BBGT])

$$U_{es} = \cap \{C : C \text{ closed}, \mathrm{cap}_e(U \setminus C) = 0\}.$$

Theorem 5.9: Assume $\langle L, y \rangle = 0$ for every $y \perp \mathrm{span}\{U - c\}$. Then Theorems 5.3 and 5.7 still hold when (5.7) is substituted by

$$c \in \mathrm{ri}(\mathrm{co}\, U). \tag{5.7)'$$

Theorem 5.8 still holds when substituting (5.7) by

$$c \in \operatorname{ri}(\operatorname{co} U_{\infty}). \tag{5.7)'$$

We recall one definition and two statements of the abstract theory of minimization of noncoercive and nonconvex functionals (see [BBGT], and Definition 2.1, Proposition 2.3, Theorem 2.3 of [BT]).

DEFINITION 5.10: Given $G: V \to (-\infty, +\infty]$ where V is the dual of a separable Banach space, the *sequential recession functional* of G is

$$G_{\infty}(v) = \inf\left\{\liminf_{h} \frac{G(t_h v_h)}{t_h} : v_h \rightharpoonup v \ w^*\text{-}V, t_h \to +\infty\right\}.$$

THEOREM 5.11: If G is bounded from below in V, then

$$G_{\infty}(v) \geq 0 \quad \forall v \in V. \tag{5.9}$$

THEOREM 5.12: Assume V is the dual of a separable Banach space, and assume $G: V \to (-\infty, +\infty]$ is a proper w^*-sequentially lower semicontinuous functional such that

$$G_{\infty}(v) \geq 0 \quad \forall v \in V, \tag{5.10}$$

$$\{t_h \to +\infty, v_h \xrightarrow{w^*} v, G(t_h v_h) \text{ bounded}\} \Rightarrow v_h \to v \text{ strongly in } V, \tag{5.11}$$

$$G(v - z) = G(v) \quad \forall z \in \ker G_{\infty}, \ \forall v \in V. \tag{5.12}$$

Then G achieves a finite minimum.

We can now prove the existence of weak solutions for the obstacle problem.

LEMMA 5.13: For any $m \geq 1$, assume $B = B_r(c) \subset U$ for some $r > 0$. Define the map $p_1: BH(\Omega) \to \mathscr{P}_1$ in this way

$$(p_1 v)(x) = v_B + (\nabla v)_B \cdot (x - c), \tag{5.13}$$

where

$$v_B = |B|^{-1} \int_B v \, dx, \quad (\nabla v)_B = |B|^{-1} \int_B \nabla v \, dx.$$

Then

$$p_1 v = v \quad \forall v \in \mathscr{P}_1,$$

$$v_B \geq 0 \quad \forall v \in SBH(\Omega) \quad \text{with } v \geq 0 \text{ in } U, \tag{5.14}$$

$$(5.15) \qquad \exists C = C(\Omega, U): \begin{cases} \|v - p_1 v\|_{L^\infty(\Omega)} \le C|D^2 v|_T & \text{if } m = 1, 2, \\ \|v - p_1 v\|_{L^{m/(m-2)}(\Omega)} \le C|D^2 v|_T & \text{if } m > 2. \end{cases}$$

We notice that the smaller is r the bigger is C, since, referring to the notations of Theorem 2.13, $C(\Omega, B_r(c)) \ge \delta(B_r(c))$.

Proof: The first part of the statement is trivial. Let us show the existence of C. It is enough showing the existence of C' such that

$$(5.16) \qquad \|v - p_1 v\|_{L^1(B_r)} \le C' |D^2 v|_{T(B_r)}$$

and the inequality follows, by contradiction, in a standard way. Inequality (5.16) is a consequence of Theorem 2.12. ∎

Proof of Theorem 5.2: We apply Theorem 5.11. By explicit computation of the recession functional $\mathfrak{F}_\infty$ we get

$$(5.17) \qquad \mathfrak{F}_\infty(v) = \begin{cases} \displaystyle\int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^1 - \langle L, v\rangle & \text{if } \nabla^2 v \equiv 0 \text{ and } v \ge 0 \text{ in } U, \\ +\infty & \text{elsewhere}. \end{cases}$$

Assume problem (5.1) has a solution, then $\mathfrak{F}(v)$ is bounded from below in $SBH(\Omega)$. Hence, by substituting $v \equiv 1$ in (5.9) we get $\mathfrak{F}_\infty(1) = -\langle L, 1\rangle \ge 0$, say (5.3).

Assume $\langle L, 1\rangle = 0$; then there are $k_j^+, k_j^- \in R$ such that each one of the functions $v_j^+(x) = k_j^+ + x_j$, $v_j^-(x) = k_j^- - x_j$, is nonnegative on U; again by substitution in (5.9) we get $\langle L, \pm x_j\rangle \le 0$, for $j = 1, \dots, m$, that is (5.4).

Assume $\langle L, 1\rangle < 0$; then c is well defined; since any displacement $v \in \mathcal{P}_1$ can be written as $v(x) = v(c) + \lambda\cdot(x - c)$ for suitable $\lambda \in R^2$, we get by (5.9)

$$\langle L, 1\rangle v(c) = \langle L, v\rangle \le 0 \quad \forall v \in \mathcal{P}_1 \text{ s.t. } v \ge 0 \text{ in } U.$$

This means $v(c) \ge 0$ $\forall v$ affine linear and nonnegative on U and Hahn-Banach Theorem entails $c \in \operatorname{co} U$. ∎

Proof of Theorem 5.3: We apply Theorem 5.12. First of all the closedness of the unilateral constraint and Theorem 2.13 imply $\mathfrak{F}$ is sequentially w^*-lower semicontinuous. From the definition of c we get $\langle L, x_j\rangle = \langle L, c_j\rangle$ for all j, hence by (5.6), (5.14) and (5.15) we get

$$\langle L, p_1 v\rangle = \langle L, v_B\rangle + \langle L, (\nabla v)_B\cdot(x - c)\rangle = \langle L, v_B\rangle \le 0 \quad \forall v \text{ with } v \ge 0 \text{ on } U,$$

$$\langle L, v\rangle \le \langle L, v - p_1 v\rangle \le |L|_T \|v - p_1 v\|_{L^\infty(\Omega)} \le C|L|_T |D^2 v|_T.$$

Arguing as in the proof of Theorem 3.1 we get for a suitable constant C'' the inequality

$$(5.18) \qquad \mathfrak{F}(v) \ge (a \wedge 1 - C''|L|_T)|D^2 v|_T - a\frac{|\Omega|}{4} \quad \forall v \in SBH(\Omega): v \ge 0 \text{ on } U.$$

By comparison, taking into account (5.8) with $r_5 = (a \wedge 1)/C^*$ and (5.17), we get

$$\begin{cases} \mathcal{G}_\infty(v) \geqslant 0 & \forall v \in SBH(\Omega), \\ \mathcal{G}_\infty(v) = 0 & \text{iff } D^2 v \equiv 0, v \geqslant 0 \text{ on } U \text{ and } \langle L, v \rangle = 0. \end{cases} \tag{5.19}$$

Then (5.9) is satisfied.

Assume now

$$t_k \to +\infty, \quad v_k \xrightarrow{w^*} v, \quad \mathcal{G}(t_k v_k) \text{ bounded},$$

then, by (5.18) we get

$$t_k (a \wedge 1 - C^* |L|_T) |D^2 v_k|_T \leqslant C'',$$

so that

$$|D^2 v_k|_T \to 0.$$

By the compact embedding $SBH(\Omega) \subset W^{1,1}(\Omega)$ (see Theorem 2.4),

$$v_k \to v \quad \text{strongly in } SBH(\Omega)$$

and the compactness property (5.11) is fulfilled.

It is left to show the compatibility (5.12). But (5.17) and (5.19) imply

$$\ker \mathcal{G}_\infty = \{v \in \mathcal{P}_1 : v \geqslant 0 \text{ on } U, \langle L, v \rangle = 0\}$$

and, since $\langle L, v \rangle = v(c) \langle L, 1 \rangle$ $\forall v \in \mathcal{P}_1$, by (5.6) we get

$$\ker \mathcal{G}_\infty = \{v \in \mathcal{P}_1 : v \geqslant 0 \text{ on } U, v(c) = 0\}.$$

Finally assumption (5.7) gives

$$\ker \mathcal{G}_\infty = \{0\},$$

and then (5.12) becomes trivially satisfied. ■

Proof of Theorem 5.7: The only if part follows from (5.5). The if part, if $c \in (\mathrm{co}\, U)^\circ$, follows exactly in the same way of Theorem 5.3; if $c \in \partial(\mathrm{co}\, U)$ we take the solution u_c of the same problem with a reduced obstacle $\{c\}$ instead of U and then, since the right and left derivative of u_c exist finite everywhere, one gets that $u = u_c + \langle \lambda, x - c \rangle \geqslant 0$ on U, for suitable λ, is a solution of problem (5.1). ■

Proof of Theorem 5.8: The only differences with the proof of Theorem 5.3 are: the closedness of the admissible set $\{v \in SBH(\Omega) : \tilde{v} \geqslant 0 \text{ on } U\}$, which follows by Proposition 4.8 of [BBGT], and the estimate (5.18) which is substituted by

$$\mathcal{G}(v) \geqslant (a \wedge 1 - C^* \|L\|_*) |D^2 v|_T - a \frac{|\Omega|}{4} \quad \forall v \in SBH(\Omega). \tag{5.20}$$

Estimate (5.20) can be obtained as like as in the proof of Theorem 3.5. ■

PROOF OF THEOREM 5.9: The argument is exactly the same as in the corresponding proofs. The only differences are: Lemma 5.13 and verification of necessary conditions (5.11) and of compatibility condition (5.12). The modified Lemma 5.13 consists in averaging over $\tilde{B}$, which is a disk of the same dimension of the affine convex hull of the obstacle and is contained in $\mathrm{co}(U_{\infty})$, and setting

$$(\tilde{p}_1 v)(x) = v_{\tilde{B}} + (\nabla v)_{\tilde{B}} \cdot (x - c);$$

then (5.14) becomes $v_{\tilde{B}} \geq 0 \ \forall v \in SBH(\Omega): v \geq 0$ in U and in (5.15) $v - p_1 v$ may be substituted by $v - \tilde{p}_1 v - p_2 v$ where $(p_2 v)(x) = (\nabla v)_{\Omega} \cdot (\pi x - c)$ and π is the orthogonal projection on $(\mathrm{span}\,\{U - c\})^{\perp}$, if this is non void.

We have

$$\ker \mathcal{G}_\infty = \{v \in \mathcal{P}_1 : v \geq 0 \text{ on } U, \langle L, v\rangle = 0\} =$$
$$= \{v \in \mathcal{P}_1 : v \geq 0 \text{ on } U, \nabla v \perp \{U - c\}\},$$

which now may be non trivial. The necessary condition follows from $\langle L, y\rangle = 0$.

About the compatibility we have

$$z - k \in \mathrm{Dom}\,\mathcal{G} \quad \forall z \in \mathrm{Dom}\,\mathcal{G}, \forall k \in \ker \mathcal{G}_\infty,$$

and $\mathcal{G}$ is invariant by adding or subtracting any $k \in \ker \mathcal{G}_\infty$ since $\langle L, x_j - c_j\rangle = 0$. ■

6. - NECESSARY CONDITIONS

Let $\Omega \subset R^2$ be an open set; in this section we specialize the quadratic form Q in order to deal with the elastic perfectly-plastic plate submitted to a transverse load and we consider the local functional (referring to Remark 2.3)

$$\mathcal{F}(v, A) = \int_A d((1 - \nu)|\nabla^2 v|^2 + \nu(\Delta^e v)^2)\,dx + \tag{6.1}$$
$$+ \mathcal{H}^1(S_{Dv} \cap A) + \int_{S_{Dv} \cap A} |[Dv]|\,d\mathcal{H}^1 - \int_A gv\,dx,$$

for every open set $A \subseteq \Omega$, $v \in SBH(A)$ and

$$0 \leq \nu < 1, \quad d > 0, \quad g \in L^q(\Omega) \text{ with } q \geq 1. \tag{6.2}$$

In the following we use the notation $D_{ij} v = D_i D_j v$ and, for any unit vector e, we use also $\partial v / \partial e = e \cdot Dv$.

DEFINITION 6.1: We say that u is a local minimizer of the functional $\mathcal{F}(\cdot, \Omega)$ if

$$u \in SBH(A), \quad \mathcal{F}(u, A) < +\infty$$

and

$$\mathcal{F}(u, A) \le \mathcal{F}(u + \varphi, A),$$

for every open subset $A \subset\subset \Omega$ and for every $\varphi \in SBH(\Omega)$ with compact support in A.

REMARK 6.2: If $\int_\Omega g\,dx = 0$, then u is a local minimizer of $\mathcal{F}(\cdot, \Omega)$ if and only if $u + c$ is a local minimizer of $\mathcal{F}(\cdot, \Omega)$ for any $c \in \mathbf{R}$.

If $\int_\Omega xg\,dx = 0$, then u is a local minimizer of $\mathcal{F}(\cdot, \Omega)$ if and only if $u + a \cdot x$ is a local minimizer of $\mathcal{F}(\cdot, \Omega)$ for any $a \in \mathbf{R}^2$.

Next we want to evaluate the first variation in certain directions of the energy functional (6.1) around a local minimizer. First we recall a Green formula. Here and in the following we assume the functions u, v regular enough to have all the traces that are needed. Let $0 \le \nu < 1$ and consider the following decomposition of the biharmonic operator:

$$\Delta^2 = (1 - \nu)(D_1^4 + D_2^4 + 2D_{12}^2) + \nu(D_1^4 + D_2^4 + 2D_{11}D_{22}).$$

Let A be an open subset of Ω. For every $v, \varphi \in W^{2,2}(A)$ we set

$$a_A(v, \varphi) = \int_A (1 - \nu)(D^2 v):(D^2 \varphi)\,dx + \int_A \nu \Delta v\, \Delta\varphi\,dx.$$

Since $0 \le \nu < 1$, the form a is bilinear, symmetric and positive definite on $W^{2,2}(A)$. If A is C^2 uniformly regular and if we denote by $\mathbf{n}$ the outward unit normal to ∂A and by $\mathbf{t}$ the unit tangent vector which orients ∂A counter clockwise, then we obtain for every $\varphi \in W^{2,2}(A)$ and for every $v \in W^{2,2}(A) \cap \{v; \Delta^2 v \in L^2(A)\}$ the following Green formula (see [LI], pp. 75-76):

$$(6.3)\quad a_A(v, \varphi) = \int_A (\Delta^2 v)\varphi\,dx + \int_{\partial A}\left((1 - \nu)S(v) - \frac{\partial}{\partial \mathbf{n}}(\Delta v)\right)\varphi\,d\mathcal{H}^1 + \\ + \int_{\partial A}((1 - \nu)T(v) + \nu\Delta v)\frac{\partial \varphi}{\partial \mathbf{n}}\,d\mathcal{H}^1,$$

where

$$S(v) = -\frac{\partial}{\partial \mathbf{t}}(\mathbf{t} \cdot D^2 v\,\mathbf{n}),$$

$$T(v) = \mathbf{n} \cdot D^2 v\,\mathbf{n}.$$

For instance, in a flat portion of ∂A parallel to the x_1 axis, if $\nu = 0$, one gets the identities:

$$S(v) = -\frac{\partial}{\partial n}\left(\frac{\partial^2 v}{\partial t^2}\right) = -D_{112}v, \qquad T(v) = \frac{\partial^2 v}{\partial n^2} = D_{22}v.$$

THEOREM 6.3: Let u be a local minimizer of $\mathcal{F}(\cdot, \Omega)$. Then

$$\Delta^2 u = \frac{1}{2d} g \qquad \text{in } \Omega \setminus \overline{S_{Du}}.$$

PROOF: For every open set $A \subset\subset \Omega \setminus \overline{S_{Du}}$, for every $\varepsilon \in R$ and for every $\varphi \in C_0^\infty(A)$ we have

$$0 \leq \mathcal{F}(u + \varepsilon\varphi, A) - \mathcal{F}(u, A) = \varepsilon\left(2d a_A(u, \varphi) - \int_A g\varphi\, dx\right) + o(\varepsilon),$$

hence

$$a_A(u, \varphi) = \frac{1}{2d}\int_A g\varphi\, dx,$$

for every $\varphi \in C_0^\infty(A)$. The thesis follows integrating by parts. ∎

THEOREM 6.4: Let $g \in L^q(\Omega)$ with $q \geq 2$, and let u be a local minimizer of $\mathcal{F}(\cdot, \Omega)$. Assume $B \subset \Omega$ is an open ball such that $S_{Du} \cap B$ is the graph of a C^2 function. Denote by B^+, B^- the two connected components of $B \setminus S_{Du}$. Let $n = (n_1, n_2)$ be the unit normal vector to S_{Du} pointing toward B^+, and let $t = (t_1, t_2)$ be the unit tangent vector to S_{Du} defined by $t_1 = n_2$, $t_2 = -n_1$. Then the following relationships hold on $S_{Du} \cap B$

$$\text{(6.4)} \qquad \begin{cases} \left[(1-\nu)\dfrac{\partial}{\partial t}(t \cdot \nabla^2 u\, n) + \dfrac{\partial}{\partial n}\Delta^a u\right] = 0, \\[2ex] (1-\nu)(n \cdot \nabla^2 u\, n)^+ + \nu(\Delta^a u)^+ = \dfrac{1}{2d}\operatorname{sign}\left[\dfrac{\partial u}{\partial n}\right], \\[2ex] (1-\nu)(n \cdot \nabla^2 u\, n)^- + \nu(\Delta^a u)^- = \dfrac{1}{2d}\operatorname{sign}\left[\dfrac{\partial u}{\partial n}\right], \end{cases}$$

where for a function w we denote by w^+, w^- the traces from B^+, B^- and we set $[w] = w^+ - w^-$.

PROOF: Let $\varphi \in C^\infty(\Omega)$ be a function such that $\operatorname{spt}\varphi \subset B$, $\varphi|_{\overline{B^+}} \in C^2(\overline{B^+})$ and $\varphi|_{\overline{B^-}} \in C^2(\overline{B^-})$. Then $\varphi \in SBH(B)$ and for every $\varepsilon \in R$ we have

$$S_{D(u+\varepsilon\varphi)} \subset S_{Du};$$

moreover, since for every $a, b \in R$ with $|b| < |a|$ it holds $|a + b| - |a| = b \operatorname{sign} a$, taking into account that the possible dependence on t is even, we have by (6.3)

$$0 \leqslant \mathcal{F}(u + \varepsilon\varphi, B) - \mathcal{F}(u, B) \leqslant 2d\varepsilon\left(a_{B^+}(u, \varphi) + a_{B^-}(u, \varphi) - \frac{1}{2d}\int_B g\varphi\, dx\right) +$$

$$+ \int_{S_{Du}\cap B}\left(\left|\left[\frac{\partial(u + \varepsilon\varphi)}{\partial n}\right]\right| - \left|\left[\frac{\partial u}{\partial n}\right]\right|\right) d\mathcal{H}^1 + o(\varepsilon) =$$

$$= 2d\varepsilon\left(\int_{B^+\cup B^-}\varphi\Delta^2 u\, dx - \frac{1}{2d}\int_B g\varphi\, dx - \int_{S_{Du}}\left[(1-\nu)Su - \frac{\partial}{\partial n}\Delta^t u\right]\varphi\, d\mathcal{H}^1 -\right.$$

$$\left. - \int_{S_{Du}}\left[((1-\nu)Tu + \nu\Delta^t u)\frac{\partial\varphi}{\partial n}\right] d\mathcal{H}^1 + \frac{1}{2d}\int_{S_{Du}}\left[\frac{\partial\varphi}{\partial n}\right]\operatorname{sign}\left[\frac{\partial u}{\partial n}\right] d\mathcal{H}^1\right) + o(\varepsilon).$$

Taking into account Theorem 6.3 and by the arbitrariness of φ and of the two traces of $\partial\varphi/\partial n$ on the two sides of S_{Du} we obtain the thesis. ■

Remark 6.5: By Theorem 6.4 it follows that

$$\left[(1-\nu)\frac{\partial^2 u}{\partial n^2} + \nu\Delta^t u\right] = 0 \quad \text{on } S_{Du}\cap B.$$

More explicitly, on a flat portion Σ of S_{Du}, the first condition in (6.4) and the previous one become

$$\begin{cases} \left[(1-\nu)\dfrac{\partial}{\partial n}\left(\dfrac{\partial^2 u}{\partial t^2}\right) + \dfrac{\partial^3 u}{\partial n^3} + \dfrac{\partial}{\partial n}\left(\dfrac{\partial^2 u}{\partial t^2}\right)\right] = 0 & \text{on } \Sigma, \\ \left[\dfrac{\partial^2 u}{\partial n^2} + \nu\dfrac{\partial^2 u}{\partial t^2}\right] = 0 & \text{on } \Sigma. \end{cases}$$

Now we want to compute the first variation of the functional (6.1) with respect to some directions which are different from those considered in Theorems 6.3 and 6.4.

Theorem 6.6: Let $g \in C^1(\Omega)$ and let u, B, n, t be as in Theorem 6.4. Then for every $\eta \in C_0^2(B, \mathbf{R}^2)$ the following equation holds:

$$\int_\Omega d\{(1-\nu)|\nabla^2 u|^2 + \nu(\Delta^t u)^2\}\operatorname{div}\eta\, dx -$$

$$-2d\int_\Omega\{(1-\nu)\nabla^2 u : (2\nabla^2 u\, D\eta + Du\, D^2\eta) + \nu\Delta^t u(Du\cdot\Delta\eta + 2\nabla^2 u : D\eta)\}\, dx +$$

$$+\int\limits_{S_{Du}}\left\{\left(1+\left|\left[\frac{\partial u}{\partial n}\right]\right|\right)t\cdot D\eta\, t-\left|\left[\frac{\partial u}{\partial n}\right]\right|(nD\eta)\cdot n\right\}d\mathcal{H}^1=$$

$$=\int\limits_{\Omega}(Dg\cdot\eta+g\operatorname{div}\eta)\,u\,dx\,.$$

Proof: Let $\eta\in C_0^2(B,R^2)$ and let $\varepsilon\in R$ small enough, so that the map $\tau_\varepsilon(x)=$ $=x+\varepsilon\eta(x)$ is a diffeomorphism of B onto itself. Set $u_\varepsilon(\tau_\varepsilon(x))=u(x)$. By using

$$D\tau_\varepsilon(x)=I+\varepsilon D\eta(x)\quad\text{ and }\quad(D\tau_\varepsilon(x))^{-1}=I-\varepsilon D\eta(x)+o(\varepsilon)\,,$$

where $o(\varepsilon)$ is an infinitesimal of order greater than ε uniformly in x, we compute

$$\begin{aligned}|(\nabla^2u_\varepsilon)\circ\tau_\varepsilon|^2&=|\nabla^2u(D\tau_\varepsilon)^{-2}-Du(D\tau_\varepsilon)^{-1}D^2\tau_\varepsilon(D\tau_\varepsilon)^{-2}|^2=\\&=|\nabla^2u(I-2\varepsilon D\eta+o(\varepsilon))-\varepsilon Du\,D^2\eta+o(\varepsilon)|^2=\\&=|\nabla^2u|^2-2\varepsilon\nabla^2u:(2\nabla^2u\,D\eta+Du\,D^2\eta)+o(\varepsilon)\,.\end{aligned}$$

Similarly we obtain

$$|(\Delta^e u_\varepsilon)\circ\tau_\varepsilon|^2=|\Delta^e u|^2-2\varepsilon\Delta^e u(2\nabla^2u:D\eta+Du\cdot\Delta\eta)+o(\varepsilon)\,.$$

Taking into account that

$$\det(I+\varepsilon D\eta)=1+\varepsilon\operatorname{div}\eta+o(\varepsilon)\,,$$

and by using the change of variables $y=\tau_\varepsilon(x)$, we get

$$\begin{aligned}&(6.5)\qquad 0\leqslant\mathcal{F}(u_\varepsilon,B)-\mathcal{F}(u,B)=\\&=\int\limits_B d((1-\nu)|\nabla^2u_\varepsilon(y)|^2+\nu|\Delta^e u_\varepsilon(y)|^2)\,dy-\int\limits_B d((1-\nu)|\nabla^2u(x)|^2-\nu|\Delta^e u(x)|^2)\,dx+\\&+\int\limits_{S_{Du_\varepsilon}\cap B}(1+|[Du_\varepsilon(y)]|)\,d\mathcal{H}^1(y)-\int\limits_{S_{Du}\cap B}(1+|[Du(x)]|)\,d\mathcal{H}^1(x)-\\&-\int\limits_B g(y)u_\varepsilon(y)\,dy+\int\limits_B g(x)u(x)\,dx=\varepsilon d\int\limits_B((1-\nu)|\nabla^2u|^2+\nu(\Delta^e u)^2)\operatorname{div}\eta\,dx-\\&-2\varepsilon d\int\limits_B((1-\nu)\nabla^2u:(2\nabla^2u\,D\eta+Du\,D^2\eta)+\nu\Delta^e u(2\nabla^2u:D\eta+Du\cdot\Delta\eta))\,dx+\\&+\int\limits_{S_{Du_\varepsilon}\cap B}(1+|[Du_\varepsilon(y)]|)\,d\mathcal{H}^1(y)-\int\limits_{S_{Du}\cap B}(1+|[Du(x)]|)\,d\mathcal{H}^1(x)-\\&\qquad\qquad-\int\limits_B((g\circ\tau_\varepsilon)(1+\varepsilon\operatorname{div}\eta)-g)\,u\,dx+o(\varepsilon)\,.\end{aligned}$$

Let γ be a parametrization of $S_{Du} \cap B$ by arclength defined on $[0, L]$ and $\gamma^\varepsilon(s) = \gamma(s) + \varepsilon r_\varepsilon(\gamma(s))$; then $t(\gamma(s)) = \gamma'(s)$ and

$$\mathcal{H}^1(S_{Du_\varepsilon} \cap B) = \int_0^L |(\gamma^\varepsilon)'| \, ds =$$

$$= \int_0^L (|\gamma_1' + \varepsilon D_1 r_{\varepsilon 1} \gamma_1' + \varepsilon D_2 r_{\varepsilon 1} \gamma_2'|^2 + |\gamma_2' + \varepsilon D_1 r_{\varepsilon 2} \gamma_1' + \varepsilon D_2 r_{\varepsilon 2} \gamma_2'|^2)^{1/2} ds =$$

$$= \int_0^L (1 + \varepsilon \gamma' D r_\varepsilon \gamma') \, ds + o(\varepsilon),$$

hence

$$\mathcal{H}^1(S_{Du_\varepsilon} \cap B) - \mathcal{H}^1(S_{Du} \cap B) = \varepsilon \int_0^L \gamma'(s) \cdot D r_\varepsilon(\gamma(s)) \gamma'(s) \, ds + o(\varepsilon) = \tag{6.6}$$

$$= \varepsilon \int_{S_{Du}} t \cdot D r_\varepsilon t \, d\mathcal{H}^1 + o(\varepsilon).$$

Since

$$(Du_\varepsilon) \circ \tau_\varepsilon = Du (D\tau_\varepsilon)^{-1} = Du(I - \varepsilon D r_\varepsilon + o(\varepsilon))$$

then

$$\int_{S_{Du_\varepsilon} \cap B} |[Du_\varepsilon(y)]| \, d\mathcal{H}^1(y) - \int_{S_{Du} \cap B} |[Du(x)]| \, d\mathcal{H}^1(x) = \tag{6.7}$$

$$= \int_{S_{Du_\varepsilon} \cap B} (|[Du - \varepsilon Du D r_\varepsilon + o(\varepsilon)]| (1 + \varepsilon t \cdot D r_\varepsilon t + o(\varepsilon)) - |[Du]|) \, d\mathcal{H}^1 =$$

$$= \varepsilon \int_{S_{Du} \cap B} (|[Du]| t \cdot D r_\varepsilon t - |[Du]| (n D r_\varepsilon) \cdot n) \, d\mathcal{H}^1 + o(\varepsilon) =$$

$$= \varepsilon \int_{S_{Du}} \left(\left| \left[\frac{\partial u}{\partial n} \right] \right| t \cdot D r_\varepsilon t - \left| \left[\frac{\partial u}{\partial n} \right] \right| (n D r_\varepsilon) \cdot n \right) d\mathcal{H}^1 + o(\varepsilon).$$

Moreover we have

$$\int_B ((g \circ \tau_\varepsilon)(1 + \varepsilon \operatorname{div} r_\varepsilon) - g) u \, dx = \varepsilon \int_B (Dg \cdot r_\varepsilon + g \operatorname{div} r_\varepsilon) u \, dx + o(\varepsilon). \tag{6.8}$$

From (6.5)-(6.8) the thesis follows. ■

REMARK 6.7: We notice that since $t = (n_2, -n_1)$, then

$$t \cdot D\eta\, t = \sum_{i=1}^{2} \delta_i \tau_{\eta i},$$

where $\delta_i = D_i - n_i \sum_{k=1}^{2} n_k D_k$ are the tangential derivatives on S_{Dw}. Let $\nu \in C^1(B, R^2)$ be a vector field such that $\nu(x) = n(x)$ for every $x \in S_{Dw} \cap B$, let $\varphi \in C_0^\infty(B)$ and set $\eta = \varphi\nu$. Then, arguing as in [GI], p. 120, we obtain

$$\int_{S_{Dw}} t \cdot D\eta t \, d\mathcal{H}^1 = \int_{S_{Dw}} H\varphi \, d\mathcal{H}^1,$$

where H is the curvature of the curve S_{Dw}.

REMARK 6.8: It is clear from the proof that Theorem 6.6 holds even if $S_{Dw} \cap B$ is a finite union of C^2 curves possibly extinguishing in B.

7. - LOCAL MINIMIZERS IN R^m

In this section we show a Caccioppoli type inequality as a preliminary step to a further study of regularity, and we apply it to show that local minimizers in R^m, with finite energy and bounded singular set, are affine. For the sake of simplicity, we assume in the following

(7.1) $$\mathcal{F}(v, A) = \int_A Q(\nabla^2 v)\, dx + \mathcal{H}^{m-1}(S_{Dv} \cap A) + \int_{S_{Dv} \cap A} |[Dv]| \, d\mathcal{H}^{m-1}$$

where $A \subseteq R^m$ is an open set, $v \in SBH(A)$ and Q is specialized as follows

$$Q(\nabla^2 v) = (1 - \nu)|\nabla^2 v|^2 + \nu(\Delta' v)^2.$$

PROPOSITION 7.1: Let $x \in R$ and $y \in R^{m-1}$. For any $c \neq 0$, the finite dihedron $u(x, y) = c|x|$ is not a local minimizer for $\mathcal{F}(\cdot, R^m)$.

PROOF: Without loss of generality we assume $c > 0$. Define u_0 as follows

$$u_0(x, y) = \begin{cases} \dfrac{x^2}{2\alpha} + \dfrac{\alpha}{2} & \text{if } |x| < \alpha,\ |y| < \beta, \\ (\beta + 1 - |y|)\dfrac{x^2}{2\alpha} + (|y| + 1 - \beta)\dfrac{\alpha}{2} & \text{if } |x| < \alpha, \beta \leq |y| < \beta + 1, \\ (\beta + 2 - |y|)\alpha & \text{if } |x| \leq (\beta + 2 - |y|)\alpha \\ & \text{and } \beta + 1 \leq |y| \leq \beta + 2, \\ |x| & \text{elsewhere}. \end{cases}$$

If we choose $\alpha > 2c^2/(2c + 1)$ and some β, depending on α and c, large enough we have, for any $r > \alpha + \beta + 2$,

$$\mathcal{F}(cu_0, B_r) < \mathcal{F}(u, B_r),$$

which contradicts the minimality of u. ■

The following proposition gives a Caccioppoli inequality for a local minimizer of $\mathcal{F}$.

PROPOSITION 7.2: If u is a local minimizer of $\mathcal{F}(\cdot, A)$ then for every $\rho > 0$ such that $B_{2\rho} \subset A$ and for every $\lambda \in \mathbf{R}^m$, $\xi \in \mathbf{R}$ we have

$$\int_{B_\rho} Q(\nabla^2 u)\,dx \leq \frac{c}{\rho^2} \int_{B_{2\rho}\setminus B_\rho} |Du - \lambda|^2\,dx + \frac{c}{\rho^4} \int_{B_{2\rho}\setminus B_\rho} |u - \xi - \lambda\cdot x|^2\,dx,$$

where c is a constant independent of u and ρ.

PROOF: By Remark 6.2 we can assume $\lambda = 0$, $\xi = 0$. Let $\varphi \in C_0^\infty(B_{2\rho})$ such that

$$0 \leq \varphi \leq 1, \qquad \varphi \equiv 1 \text{ in } B_\rho, \qquad |D\varphi| \leq \frac{c'}{\rho}, \qquad |D^2\varphi| \leq \frac{c'}{\rho^2}.$$

For $|\varepsilon| < 1$ set $u_\varepsilon = u + \varepsilon\varphi^4 u$, then

$$\begin{aligned} &Du_\varepsilon = (1 + \varepsilon\varphi^4)\,Du + 4\varepsilon u\varphi^3 D\varphi, \\ &S_{Du_\varepsilon} = S_{Du}, \quad [Du_\varepsilon] = (1 + \varepsilon\varphi^4)[Du]. \end{aligned}$$

Now we have

$$|\nabla^2 u_\varepsilon|^2 = |\nabla^2 u|^2 +$$

$$+ 2\varepsilon(\varphi^4|\nabla^2 u|^2 + 8\varphi^3 Du\,\nabla^2 u D\varphi + 12\varphi^2 u\,D\varphi\nabla^2 u\,D\varphi + 4\varphi^3 u\nabla^2 u : D^2\varphi) + o(\varepsilon),$$

$$(\Delta^e u_\varepsilon)^2 = (\Delta^e u)^2 +$$

$$+ 2\varepsilon(\varphi^4(\Delta^e u)^2 + 8\varphi^3 Du\,\Delta^e u\,D\varphi + 12\varphi^2 u|D\varphi|^2\Delta^e u + 4\varphi^3 u\Delta^e u\Delta\varphi) + o(\varepsilon),$$

and

$$\int_{S_{Du_\varepsilon}} |[Du_\varepsilon]|\,d\mathcal{H}^{m-1} = \int_{S_{Du}} |[Du]|\,d\mathcal{H}^{m-1} + \varepsilon\int_{S_{Du}} \varphi^4|[Du]|\,d\mathcal{H}^{m-1}.$$

By the minimality of u we get

$$(1-\nu)\int_{B_{2\rho}}(\varphi^4|\nabla^2 u|^2 + 8\varphi^3 Du\,\nabla^2 u\,D\varphi + 12\varphi^2 u\,D\varphi\,\nabla^2 u\,D\varphi + 4\varphi^3 u\,\nabla^2 u : D^2\varphi)\,dx +$$

$$+\nu\int_{B_{2\rho}}(\varphi^4(\Delta u)^2 + 8\varphi^3 Du\,\Delta u\,D\varphi + 12\varphi^2 u|D\varphi|^2\Delta u + 4\varphi^3 u\,\Delta u\,\Delta\varphi)\,dx +$$

$$+\int_{S_{Du}\cap B_{2\rho}}\frac{1}{2}\varphi^4|[Du]|\,d\mathcal{H}^{m-1} = 0.$$

By using the above equation and Hölder inequality we obtain

$$\int_{B_{2\rho}}\varphi^4 Q(\nabla^2 u)\,dx \le \int_{B_{2\rho}}\varphi^4 Q(\nabla^2 u)\,dx + \int_{S_{Du}\cap B_{2\rho}}\frac{1}{2}\varphi^4|[Du]|\,d\mathcal{H}^{m-1} \le$$

$$\le c^*\int_{B_{2\rho}}(|\varphi^2\nabla^2 u|\,|\varphi\,Du\,D\varphi| + |\varphi^2\nabla^2 u|\,|u|\,|D\varphi|^2 + |\varphi^2\nabla^2 u|\,|\varphi u\,D^2\varphi|)\,dx \le$$

$$\le c^{**}\left(\int_{B_{2\rho}}\varphi^4|\nabla^2 u|^2 dx\right)^{1/2}\left\{\frac{1}{\rho}\left(\int_{B_{2\rho}\setminus B_\rho}|Du|^2 dx\right)^{1/2} + \frac{1}{\rho^2}\left(\int_{B_{2\rho}\setminus B_\rho}u^2 dx\right)^{1/2}\right\},$$

hence by Young inequality we have

$$\int_{B_\rho}Q(\nabla^2 u)\,dx \le \int_{B_{2\rho}}\varphi^4 Q(\nabla^2 u)\,dx \le \frac{2c^{**}}{\rho^2}\int_{B_{2\rho}\setminus B_\rho}|Du|^2 dx + \frac{2c^{**}}{\rho^4}\int_{B_{2\rho}\setminus B_\rho}u^2 dx,$$

and the inequality is proved. ∎

We give now an application of the above inequality.

Proposition 7.3: If u is a local minimizer of $\mathcal{F}(\cdot, R^m)$ such that S_{Du} is bounded and

$$\int_{R^m}Q(\nabla^2 u)\,dx < +\infty, \tag{7.2}$$

then u is affine.

Proof: Assume $S_{Du}\subset B_{\rho_0}$ and $\rho > \rho_0$. By using Proposition 7.2 with

$$\lambda = |B_{2\rho}\setminus B_\rho|^{-1}\int_{B_{2\rho}\setminus B_\rho}Du\,dx, \qquad \xi = |B_{2\rho}\setminus B_\rho|^{-1}\int_{B_{2\rho}\setminus B_\rho}u\,dx,$$

and applying Poincaré inequality to $u \in W^{2,2}(B_{2\rho} \setminus B_\rho)$, we get

$$\int_{B_\rho} Q(\nabla^2 u)\,dx \leq \frac{c}{\rho^2} \int_{B_{2\rho}\setminus B_\rho} |Du - \lambda|^2\,dx + \frac{c}{\rho^4} \int_{B_{2\rho}\setminus B_\rho} |u - \xi - \lambda \cdot x|^2\,dx \leq$$

$$\leq c' \int_{B_{2\rho}\setminus B_\rho} |\nabla^2 u|^2\,dx \leq \frac{c'}{1-\nu} \int_{B_{2\rho}\setminus B_\rho} Q(\nabla^2 u)\,dx.$$

Setting $c'' = c'/(1-\nu)$ and filling the hole we obtain

$$(1 + c'') \int_{B_\rho} Q(\nabla^2 u)\,dx \leq c'' \int_{B_{2\rho}} Q(\nabla^2 u)\,dx,$$

so that

$$\int_{B_\rho} Q(\nabla^2 u)\,dx \leq \theta \int_{B_{2\rho}} Q(\nabla^2 u)\,dx \quad \text{with } \theta = \frac{c''}{1 + c''} < 1,$$

and, for every $k \in \boldsymbol{N}$,

$$\int_{B_\rho} Q(\nabla^2 u)\,dx \leq \theta^k \int_{B_{2^k\rho}} Q(\nabla^2 u)\,dx.$$

By the assumption (7.2) and the arbitrariness of k we conclude that

$$\int_{B_\rho} Q(\nabla^2 u)\,dx = 0.$$

By the arbitrariness of ρ, u is affine in $\boldsymbol{R}^m \setminus B_{\rho_0}$ and then, by the minimality, u is affine in $\boldsymbol{R}^m$. ■

Remark 7.4: It is clear from the proof that Propositions 7.1, 7.2, 7.3 hold true for any Q satisfying (3.3).

Remark 7.5: The Caccioppoli type inequality and Proposition 7.3 hold true even for local minimizers for $\mathcal{F} - L$, where L satisfies (3.4).

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