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EDOARDO VESENTINI (*)

Small Eigenvalues of the Laplace-Beltrami Operator (**)

ABSTRACT. — The Friedrichs self-adjoint extension of the Laplace-Beltrami operator acting on exterior forms defined on a complete Riemannian manifold is considered. It is shown that, if the curvature operator defined by Weitzenböck satisfies suitable conditions, the part of the spectrum of the Friedrichs extension nearing the origin of the real line consists of isolated eigenvalues with finite multiplicities.

Sui piccoli autovalori dell'operatore di Laplace-Beltrami

RIASSUNTO. — Data una varietà Riemanniana completa, si considera l'estensione di Friedrichs dell'operatore di Laplace-Beltrami definito sulle forme esterne. Si dimostra che, se l'operatore di Weitzenböck soddisfa condizioni opportune, la parte dello spettro dell'estensione di Friedrichs contenuta in un intorno dell'origine della retta reale consta di autovalori isolati aventi molteplicità finita.

The spectrum of the Laplace-Beltrami operator on a manifold has been the object of intensive investigations in recent years. If the manifold is compact, the values of the resolvent function of the operator acting not only on functions but also on differential forms are compact operators. That implies the well-known fact whereby the spectrum of the Laplace-Beltrami operator consists entirely of eigenvalues with finite multiplicity. If the manifold is not compact, the situation is completely different, as the classical case of the Laplace operator acting on square integrable functions on R^n already shows. This paper investigates the spectrum of the Laplace-Beltrami operator on connected orientable Riemannian complete manifolds for which the curvature operator defined by the Weitzenböck operator acting on differential forms of a given degree is strongly positive outside a compact set. For example, the Weitzenböck operator acting on linear scalar-valued differential forms is expressed by the Ricci curvature and the above condition says that the Ricci curvature form is strongly positive outside a compact set. For complex manifolds that condition is reminiscent of the case in which

(*) Indirizzo dell'Autore: Scuola Normale Superiore, Piazza dei Cavalieri, I-56100 Pisa.

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an exhaustion function is strongly plurisubharmonic outside a compact set. Also in view of applications to the latter case, in this paper differential forms with values in a vector bundle will be systematically considered. The main result states that, near the origin of the real line, the behaviour of the spectrum of the Laplace-Beltrami operator on the complete Riemannian manifolds considered here is somewhat the same as for compact manifolds: there exists a neighbourhood of 0 in the positive half-line such that the part of the spectrum contained in that neighbourhood is a discrete set of eigenvalues whose eigenspaces have finite dimension.

1. - Let M be a smooth, connected, orientable Riemannian manifold of dimension n. Let $E \to M$ be a smooth Riemannian vector bundle over M, i.e. a vector bundle of finite rank with base M, endowed with a smooth positive definite scalar product $\langle\,|\,\rangle$ along the fibers and with a linear connection ∇ such that $X\langle e_1 | e_2 \rangle_x = \langle \nabla_X e_1 | e_2 \rangle_x + \langle e_1 | \nabla_X e_2 \rangle_x$ for all $x \in M$, all smooth vector fields X on M and all smooth sections e_1, e_2 of the vector bundle (cf. *e.g.* [4] for the definitions). Denoting by $T(M)$ and $T(M)^\star$ the tangent aand cotangent bundles of M, the same symbol $\langle\,|\,\rangle$ will be used throughout the following to indicate the scalar product defined, on any subbundle of the vector bundle

$$E \otimes T(M)^{\otimes p} \otimes T(M)^{\star \otimes q} \to M \tag{1}$$

(for $p, q = 0, 1, 2, \ldots$), by the Riemannian metric on M and by the scalar product on $E \to M$. Similarly, the same symbol ∇ will stand for the covariant derivative defined on the bundle (1) by the covariant derivative on E and by the Levi-Civita connection on the Riemannian manifold M. For any $x \in M$, the function $\xi \mapsto |\xi|_x := (\langle \xi | \xi \rangle_x)^{1/2}$ defines a norm on the fiber of the bundle (1) over the point x. Let dm be the volume element of the Riemannian metric of M. For any fixed $r \geqslant 1$ the function $\|\|$ defined by the integral

$$\|\varphi\|^r = \int_M |\varphi|_x^r \, dm(x) \tag{2}$$

is a norm on the Banach space of all sections φ of the bundle (1) for which the function $x \mapsto |\varphi|_x$ is measurable and the integral on the right hand side of (2) is convergent. If $r = 2$, the Banach space is a Hilbert space.

Let $\Lambda^p(E)$ be the space of smooth p-forms on M with values in the vector bundle E, and let $\mathcal{D}^p(E)$ be the subspace of $\Lambda^p(E)$ consisting of those forms which have compact support. For $r = 2$, consider the pre-Hilbert space defined by the integral (2) acting on all $\varphi \in \mathcal{D}^p(E)$, and let $L^p(E)$ be its completion. The norm on $L^p(E)$ defined by (2) comes from the scalar product $(\,|\,)$ expressed by

$$(\varphi | \psi) = \int_M \langle \varphi | \psi \rangle_x \, dm(x)$$

for φ, ψ in $L^p(M)$.

2. - Let $d\colon \Lambda^p(E) \to \Lambda^{p+1}(E)$ be the exterior differential operator defined in terms of the covariant derivative on E by the formula

$$(d\varphi)(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X_i} \varphi(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \varphi([X_i X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).$$

If $\star\colon \Lambda^p(E) \xrightarrow{\sim} \Lambda^{n-p}(E)$ denotes the Hodge star operator, the operator $\delta\colon \Lambda^p(E) \to \Lambda^{p-1}(E)$ expressed by $\delta = (-1)^p \star^{-1} d\star$ is the formal adjoint of d, in the sense that

$$(d\varphi \mid \psi) = (\varphi \mid \delta\psi) \tag{3}$$

whenever at least one of the two forms $\varphi \in \Lambda^p(E)$, $\psi \in \Lambda^{p+1}(E)$ has compact support. The Laplace-Beltrami operator $d\delta + \delta d\colon \Lambda^p(E) \to \Lambda^p(E)$ is a linear elliptic differential operator with smooth coefficients, and will be denoted by Δ or, when necessary, by Δ^p to indicate its dependence on p. By (3)

$$(\Delta\varphi \mid \psi) = (d\varphi \mid d\psi) + (\delta\varphi \mid \delta\psi) = (\varphi \mid \Delta\psi)$$

for all φ, $\psi \in \mathcal{D}^p(E)$. Since furthermore

$$(\Delta\varphi \mid \varphi) = \|d\varphi\|^2 + \|\delta\varphi\|^2 \geqslant 0 \tag{4}$$

for all $\varphi \in \mathcal{D}^p(E)$, Δ is a positive symmetric operator with domain $D(\Delta) = \mathcal{D}^p(E)$, and as such it has a self-adjoint extension. Its Friedrichs extension is defined as follows [10, 317-318]. Consider the positive-definite scalar product defined in $\mathcal{D}^p(E)$ by

$$\varphi, \psi \mapsto (\Delta\varphi \mid \psi) + (\varphi \mid \psi),$$

and let $|||\ |||$ be the associated norm. Since $\|\varphi\| \leqslant |||\varphi|||$ for all $\varphi \in \mathcal{D}^p(E)$, and since two Cauchy sequences for the norm $|||\ |||$ are equivalent if, and only if, they are equivalent for the norm $\|\ \|$, the identity map of $\mathcal{D}^p(E)$ extends uniquely to a continuous injective map of the completion of $\mathcal{D}^p(E)$ for the norm $|||\ |||$ into $L^p(E)$. Let $\mathcal{W}^p(E)$ be the image of this map. Since $\mathcal{D}^p(E) \subset \mathcal{W}^p(E)$, $\mathcal{W}^p(E)$ is dense in $L^p(E)$. The following proposition was established in [9, 18-24] in the complex-analytic case, for the operator $\bar{\partial}$ and its formal adjoint. However, the proof carries over with only notational changes to the case of E-valued p-forms.

Proposition 2.1: *If the Riemannian metric of M is complete, $\mathcal{W}^p(E)$ consists of those $\varphi \in L^p(E)$ whose differentials $d\varphi$ and $\delta\varphi$ in the weak sense are contained in $L^{p+1}(E)$ and in $L^{p-1}(E)$ respectively.*

In other words, there exist $\varphi' \in L^{p+1}(E)$ and $\varphi'' \in L^{p-1}(E)$ such that

$$(\varphi' \mid \tau) = (\varphi \mid \delta\tau), \quad (\varphi'' \mid \chi) = (\varphi \mid d\chi)$$

for all $\tau \in \mathcal{D}^{p+1}(E)$, $\chi \in \mathcal{D}^{p-1}(E)$. Let $\Delta^\star$ be the adjoint operator of Δ, let $D(\Delta^\star)$ be its domain and let

$$\mathcal{V}^p(E) = \mathcal{W}^p(E) \cap D(\Delta^\star).$$

Since Δ is symmetric, $\mathcal{V}^p(E)$ is dense in $L^p(E)$. The Friedrichs self-adjoint extension of Δ is the restriction of $\Delta^\star$ to $\mathcal{V}^p(E)$. It will be denoted by $\boldsymbol{\Delta}$ or, when necessary, by $\boldsymbol{\Delta}^p$ to underline its dependence on p. A simple density argument shows that, it φ and ψ are elements of $D(\boldsymbol{\Delta}) = \mathcal{V}^p(E)$, then

$$(\boldsymbol{\Delta}\varphi \,|\, \psi) = (d\varphi \,|\, d\psi) + (\delta\varphi \,|\, \delta\psi). \tag{5}$$

The positive square root $\boldsymbol{\Delta}^{1/2}$ of $\boldsymbol{\Delta}$ is a self-ajoint operator whose domain turns out to be $\mathcal{W}^p(E)$ (1).

3. - The following «Stampacchia inequality» was established in the complex analytic case in [2] and, for scalar-valued differential forms on a differentiable manifold, in [9]. Also the proof of this inequality carries over to the case of E-valued p-forms with only notational changes.

Proposition 3.1: *If the Riemannian metric of M is complete, for every $\chi \in \Lambda^p(E) \cap L^p(E)$ for which $\Delta\varphi \in L^p(E)$ the following inequality holds*

$$\|d\varphi\|^2 + \|\delta\varphi\|^2 \leq t\|\varphi\|^2 + \frac{1}{t}\|\Delta\varphi\|^2$$

for all $t > 0$.

For $\varphi \in \Lambda^p(E)$, $\nabla\varphi$ is a smooth section of the vector bundle

$$E \otimes \Lambda^p T(M)^\star \otimes T(M)^\star \to M.$$

Rellich's theorem (cf. e.g. [1, 144-148; 6]) implies

Proposition 3.2: *a) Let V be a bounded relatively compact open set in M with a smooth boundary. Let $\{\varphi_\nu\}$ be a sequence of E-valued smooth p-forms on V such*

(1) Proof: If $\varphi \in \mathcal{W}^p(E)$, there is a sequence $\{\varphi_\nu\}$ in $\mathcal{D}^p(E)$ converging to φ for the norm $|||\ |||$. Since $\lim_{\mu,\nu\to+\infty} (\Delta(\varphi_\nu - \varphi_\mu) \,|\, \varphi_\nu - \varphi_\mu) = 0$, $\{\boldsymbol{\Delta}^{1/2}\varphi_\nu\}$ is a Cauchy sequence in $L^p(E)$ for the norm $\|\ \|$. Because $\boldsymbol{\Delta}^{1/2}$ is closed, then $\varphi \in D(\boldsymbol{\Delta}^{1/2})$ (and $\boldsymbol{\Delta}^{1/2}\varphi = \psi$).

Viceversa, let $\varphi \in D(\boldsymbol{\Delta}^{1/2})$. Since $D(\boldsymbol{\Delta})$ is a core of $\boldsymbol{\Delta}^{1/2}$, there is a sequence $\{\varphi_\nu\}$ in $D(\boldsymbol{\Delta})$ such that

$$\lim_{\nu\to+\infty} \varphi_\nu = \varphi, \qquad \lim_{\nu\to+\infty} \boldsymbol{\Delta}^{1/2}\varphi_\nu = \boldsymbol{\Delta}^{1/2}\varphi$$

for the norm $\|\ \|$. That implies that $\{\varphi_\nu\}$ is a Cauchy sequence for the norm $|||\ |||$, and therefore $\varphi \in \mathcal{W}^p(E)$.

that

$$\sup_{\nu} \{\|\varphi_\nu\|_V^2 + \|\nabla\varphi_\nu\|_V^2\} < \infty .$$

There exists a subsequence $\{\varphi_{\nu_i}\}$ *such that*

$$\lim_{i,j\to+\infty} \|\varphi_{\nu_j} - \varphi_{\nu_i}\|_V = 0 . \tag{6}$$

b) Let $\{\varphi_\nu\}$ *be a sequence of forms* $\varphi_\nu \in \Lambda^p(E) \cap L^p(E)$ *such that*

$$\sup_{\nu} (\{\|\varphi_\nu\|^2 + \|\nabla\varphi_\nu\|^2) < \infty .$$

There exists a subsequence $\{\varphi_{\nu_i}\}$ *such that*

$$\lim_{i,j\to+\infty} \|\varphi_{\nu_i} - \varphi_{\nu_j}\|_K = 0$$

for every compact set $K \subset M$.

Expressing the operators d and δ in terms of the covariant derivative ∇, the Laplace-Beltrami operator is expressed by the Weitzenböck formula

$$\Delta\varphi = -\operatorname{tr}\nabla\nabla\varphi + \kappa\varphi , \tag{7}$$

where $\kappa = 0$ if $p = 0$, while, if $p \geqslant 1$, for every $x \in M, \kappa_x$ is an endomorphism of the fiber of $E \otimes \Lambda^p T(M)^\star \to M$ over x, which is symmetric with respect to the scalar product $\langle\,|\,\rangle_x$ and is expressed in terms of the curvatures of the Riemannian metrics on M and along the fibers of E (2). For any $\varphi \in \Lambda^p(E)$ the function $x \mapsto \langle \kappa\varphi \,|\, \varphi\rangle_x$ is smooth on M. It follows from (4) that

$$\|\nabla\psi\|^2 + (\kappa\psi\,|\,\psi) = (\Delta\psi\,|\,\psi) = \|d\psi\|^2 + \|\delta\psi\|^2 \tag{8}$$

for all $\psi \in \mathcal{D}^p(E)$.

(2) If $ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ is the representation of the Riemannian metric of M with respect to the local coordinates $x^1, \ldots, x^n$ on an open set on which E is trivial, and if $g^{\alpha\beta}$ is defined by the equation $\sum_\gamma g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$, the two terms appearing on the right hand side of (7) are expressed respectively by:

$$(\operatorname{tr}\nabla\nabla\varphi)^b_{\gamma_1\ldots\gamma_p} = \sum_{\alpha,\beta} g^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi^b_{\gamma_1\ldots\gamma_p} ,$$

where $\nabla_\alpha = \nabla_{\partial/\partial x^\alpha}$ and $h = 1, \ldots, \operatorname{rank} E$;

$$(\kappa\varphi)^b_{\gamma_1\ldots\gamma_p} = \sum_k \sum_j \sum_\alpha C^{b\ \alpha}_{k\ \gamma_j} \varphi^k_{\gamma_1\ldots(\alpha)_j\ldots\gamma_p} + \sum_j \sum_\alpha \mathcal{R}^\alpha{}_{\gamma_j} \varphi^b_{\gamma_1\ldots(\alpha)_j\ldots\gamma_p} - \sum_j \sum_{l\neq j} \sum_{\alpha,\beta} \mathcal{R}^{\alpha\ \beta}_{\gamma_j\ \gamma_l} \varphi^b_{\alpha\gamma_1\ldots(\beta)_l\ldots\hat{\gamma}_j\ldots\gamma_p} ,$$

where $\mathcal{R}^\alpha{}_{\gamma_j}$, $\mathcal{R}^{\alpha\ \beta}_{\gamma_j\ \gamma_l}$, $C^{b\ \alpha}_{k\ \gamma_j}$ are the local components of the Ricci tensor, the Riemann tensor and of the curvature tensor of the metric in E. Cf. e.g. [9] for further details.

4. - Since Δ is a positive self-adjoint operator, its spectrum $\sigma(\Delta)$ is a non-empty subset of the positive half-line: $\sigma(\Delta) \subset \boldsymbol{R}_+$. The purpose of this paper is to investigate the structure of $\sigma(\Delta)$ near 0 for a class of connected, orientable, complete Riemannian manifolds.

For any given $p = 1, \ldots, n$ suppose that the following condition is satisfied:

i_p) there exists a compact set $K \subset M$ and a positive constant c such that

$$\langle \kappa\varphi | \varphi \rangle_x \geqslant c |\varphi|_x^2$$

for all $x \in M \setminus K$ and all $\varphi \in \Lambda^p(E)$.

Note that K may be empty.

If M is compact, choosing $K = M$, condition i_p) is satisfied by the empty set.

Since K is compact, there is a constant $c_1 \geqslant 0$ such that

$$(\kappa\psi | \psi)_K \geqslant -c_1 \|\psi\|_K^2$$

for all $\chi \in \mathcal{D}^p(E)$. (8) implies then that

$$\|\nabla\psi\|^2 + c\|\psi\|^2 \leqslant (c + c_1)\|\psi\|_K^2 + (\Delta\psi | \psi) \qquad \text{for all } \psi \in \mathcal{D}^p(E). \tag{9}$$

By consequence, if $\{\psi_\nu\} \subset \mathcal{D}^p(E)$ is a Cauchy sequence in $\mathcal{V}^p(E)$ for the norm $|||\ |||$ then $\{\nabla\psi_\nu\}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}$ of those smooth sections of the vector bundle $E \otimes T(M)^{\star \otimes p+1}$ for which the integral (2), with $r = 2$, is finite. Furthermore, Cauchy sequences for the norm $|||\ |||$ which are equivalent for this latter norm yield equivalent Cauchy sequences in $\mathcal{H}$. Hence for any $\psi \in \mathcal{V}^p(E)$ there exists $\nabla\psi \in \mathcal{H}$, and (9) implies that

$$\|\nabla\psi\|^2 + c\|\psi\|^2 \leqslant (c + c_1)\|\psi\|_K^2 + (\Delta\psi | \psi) \qquad \text{for all } \psi \in D(\Delta). \tag{10}$$

If M is compact, choosing $K = M$ and $c = 0$, (10) yields

$$\|\nabla\psi\|^2 \leqslant c_1 \|\psi\|^2 + (\Delta\psi | \psi) \qquad \text{for all } \psi \in D(\Delta). \tag{11}$$

Under the hypothesis that M be compact, let $\{\varphi_\nu\}$ be a sequence in $L^p(E)$ with $\|\varphi_\nu\| \leqslant 1$ for all $\nu = 1, 2, \ldots$, and let $b \in \boldsymbol{R}$ be such that $b > c_1 + 1/2$. Since $b > 0$, $-b$ is contained in the resolvent set of Δ. Thus there is a unique $\psi_\nu \in \mathcal{V}^p(E)$ such that

$$(\Delta + bI)\psi_\nu = \varphi_\nu \qquad (\nu = 1, 2, \ldots).$$

Hence (11) reads

$$\|\nabla\psi_\nu\|^2 - c_1\|\psi_\nu\|^2 \leqslant (-b\psi_\nu + \varphi_\nu | \psi_\nu),$$

whence

$$\|\nabla\psi_\nu\|^2 + (b - c_1)\|\psi_\nu\|^2 \leqslant (\varphi_\nu | \psi_\nu) \leqslant \frac{1}{2} + \frac{1}{2}\|\psi_\nu\|^2,$$

i.e.

$$\|\nabla\psi_\nu\|^2 + \left(b - c_1 - \frac{1}{2}\right)\|\psi_\nu\|^2 \leqslant \frac{1}{2} \qquad \text{for } \nu = 1, 2, \ldots.$$

By Rellich's theorem, there is a subsequence of $\{\psi_\nu\}$ which is a Cauchy sequence in $L^p(E)$. That proves that the bounded operator $(\Delta + bI)^{-1}$ is compact, and therefore [5, 187] that $(\zeta I - \Delta)^{-1}$ is compact for every ζ in the resolvent set of Δ. Thus $\sigma((\zeta I - \Delta)^{-1})$ consists of a sequence of poles of the resolvent function of $(\zeta I - \Delta)^{-1}$ converging to 0 and whose spectral projectors have a finite dimensional range. By the spectral mapping theorem $\sigma(\Delta)$ consists of a sequence of poles of the resolvent function of Δ, contained in R_+ and diverging to $+\infty$. If $\zeta_0 \in C \setminus \sigma(\Delta)$ and if $t \geqslant 0$ is contained in $\sigma(\Delta)$, the spectral projector Q associated to the eigenvalue $1/(\zeta_0 - t)$ of $(\zeta_0 I - \Delta)^{-1}$ is expressed by the integral

$$Q = \frac{1}{2\pi i}\int_\Gamma (\zeta I - (\zeta_0 I - \Delta)^{-1})^{-1}\, d\zeta$$

extended to a small circle Γ centered in $1/(\zeta_0 - t)$ and oriented counterclockwise. A direct computation and Cauchy's integral theorem yields

$$Q = \frac{1}{2\pi i}\int_{\Gamma'} (\zeta I - \Delta)^{-1}\, d\zeta,$$

where Γ' is the image of Γ by the map $\zeta \mapsto \zeta_0 - 1/\zeta$. Hence Q is the spectral projector associated to the eigenvalue t of Δ (and therefore does not depend on the choice of $\zeta_0 \in C \setminus \sigma(\Delta)$). Since $(\zeta_0 I - \Delta)^{-1}$ is invertible, the space spanned by the ranges of the spectral projectors is dense in $L^p(E)$. In conclusion, the following known theorem holds:

THEOREM 4.1: *If the orientable, connected Riemannian manifold M is compact, the spectrum of Δ consists, for each $p = 0, 1, \ldots, n = \dim M$, of a sequence diverging to $+\infty$ of non-negative eigenvalues of Δ. The ranges of the corresponding spectral projectors are finite-dimensional and span a dense linear space in $L^p(E)$.*

5. - In the following, $p\sigma(\Delta)$ will denote the point-spectrum of Δ and $c\sigma(\Delta)$ the continuous spectrum, i.e. the set of points $\zeta \in \sigma(\Delta)$ such that $\zeta I - \Delta$ is injective with a dense range but $(\zeta I - \Delta)\mathcal{V}^p(E) \neq L^p(E)$; $r(\Delta)$ will stand for the resolvent set of Δ.

LEMMA 5.1: *If the orientable, connected Riemannian manifold M is complete and if, for a given p, $1 \leqslant p \leqslant n$, condition i_p) is satisfied, then $[0, c) \cap c\sigma(\Delta^p) = \emptyset$.*

PROOF: Let $a \in c\sigma(\Delta)$ be such that $0 \leqslant a < c$. There exists a sequence $\{\varphi_\nu\}$ in $\mathcal{V}^p(E)$ such that $\|\varphi_\nu\| = 1$ for all $\nu = 1, 2, \ldots$, and that

$$\lim_{\nu \to +\infty} \|(\Delta - aI)\varphi_\nu\| = 0. \tag{12}$$

It follows from (10) that

$$\|\nabla\varphi_\nu\|^2 + c - a \leqslant (c + c_1)\|\varphi_\nu\|_K^2 + \|(\Delta - aI)\,\varphi_\nu\| \qquad (\nu = 1, 2, \ldots). \tag{13}$$

By Rellich's theorem and the weak sequential compactness of the closed unit ball of the Hilbert space $L^p(E)$, there exists a subsequence $\{\varphi_{\nu_i}\}$ of $\{\varphi_\nu\}$ and a form $\varphi \in L^p(E)$ such that

$$\lim_{i \to +\infty} \|\varphi - \varphi_{\nu_i}\|_\Omega = 0 \tag{14}$$

for every compact set $\Omega \subset M$. If $\tau \in \mathcal{D}^p(E)$, (12) yields

$$(\varphi \,|\, (\Delta - aI)\,\tau) = \lim_{i \to +\infty} (\varphi_{\nu_i} \,|\, (\Delta - aI)\,\tau) = \lim_{i \to +\infty} ((\Delta - aI)\,\varphi_{\nu_i} \,|\, \tau) = 0, \tag{15}$$

whence

$$\Delta\varphi = a\varphi$$

in the weak sense. By regularization it can be assumed that $\varphi \in \Lambda^p(E) \cap L^p(E)$ and that the above equation holds pointwise. The Stampacchia inequality yields then

$$\|d\varphi\|^2 + \|\delta\varphi\|^2 \leqslant (1 + a^2)\|\varphi\|^2,$$

and by Proposition 3.1, $\varphi \in \mathcal{W}^p(E)$. Since the linear form $\tau \mapsto (\Delta\tau \,|\, \varphi)$ is continuous on $\mathcal{D}^p(E)$, then $\varphi \in D(\Delta^\star)$. Hence $\varphi \in D(\Delta)$, and (15) yields

$$\Delta\varphi = a\varphi. \tag{16}$$

Since $a \in c\sigma(\Delta)$, then $\varphi = 0$. By consequence (14) yields

$$\lim_{i \to +\infty} \|\varphi_{\nu_i}\|_K = 0,$$

so that (12) and (13) imply $c - a = 0$ contrary to the hypothesis $a < c$. QED

Since Δ is self-adjoint, then $\sigma(\Delta) = p\sigma(\Delta) \cup c\sigma(\Delta)$, and Lemma 5.1 implies then

Corollary 5.2: *If the orientable, connected Riemannian manifold M is complete and if condition i_p) is satisfied for a given p $(1 \leqslant p \leqslant n)$, then*

$$[0, c) \cap \sigma(\Delta) \subset p\sigma(\Delta).$$

Remark: Under the hypotheses of Corollary 5.2, if $K = \emptyset$, the numerical range of Δ is contained in the half line $[c, +\infty)$. Hence, also $\sigma(\Delta) \subset [c, +\infty)$.

Theorem 5.3: *If the orientable, connected Riemannian manifold M is complete and if condition i_p) is satisfied, then* $\dim \mathcal{H}_a^p(E) < \infty$ *for all* $a \in [0, c]$.

PROOF: If $\dim \mathcal{H}_a^p(E) = \infty$ for some $a \in [0, c]$, given a dense sequence $\{x_\nu\}$ in M $(\nu = 1, 2, \ldots)$, there is a sequence $\{\varphi_\nu\}$ in $\mathcal{H}_a^p(E)$ such that

$$\varphi_\nu(x_j) = 0 \qquad \text{whenever } j \leqslant \nu, \tag{17}$$

$$\|\varphi_\nu\| = 1 \qquad \text{for all } \nu. \tag{18}$$

Let V be a relatively compact open set in M, with a smooth boundary such that $K \subset V$ (if $K \neq \emptyset$, or any relatively compact open set in M, with a smooth boundary, if $K = \emptyset$). By (10)

$$\|\nabla\varphi_\nu\|^2 + c - a \leqslant (c + c_1)\|\varphi_\nu\|_K^2 \qquad \text{for all } \nu.$$

If $\varphi_\nu \equiv 0$ on V, then $\|\varphi_\nu\|_K = 0$, and therefore $\nabla\varphi_\nu = 0$, i.e. the function $M \ni x \mapsto |\varphi_\nu|_x$ is constant. Thus, by (17), $\varphi_\nu \equiv 0$ on M, contradicting (18). Hence $\|\varphi_\nu\|_V > 0$ for all ν. Re-norming φ_ν in such a way that

$$\|\varphi_\nu\|_V = 1 \qquad \text{for all } \nu, \tag{19}$$

(10) yields

$$\|\nabla\varphi_\nu\|^2 + (c - a)\|\varphi_\nu\|^2 \leqslant (c + c_1)\|\varphi_\nu\|_K^2 \leqslant (c + c_1).$$

By Proposition 3.2*a*), there is a subsequence $\{\varphi_{\nu_i}\}$ for which (6) holds. Hence there exists a square-summable E-valued, p-form φ on V such that

$$\lim_{i \to +\infty} \|\varphi - \varphi_{\nu_i}\|_V = 0.$$

Since φ_{ν_i} is a zero of the elliptic operator $\Delta - aI$, then φ_{ν_i} can be assumed to be smooth on V, and the sequence $\{\varphi_{\nu_i}\}$ converges to φ uniformly on all compact sets of V. By (17), the fact that the sequence $\{x_j\}$ is dense, implies that $\varphi \equiv 0$ on V. Thus $\lim_{i \to +\infty} \|\varphi_{\nu_i}\|_V = 0$, contradicting (19). QED

In particular, if $a = 0$, the space $\mathcal{H}_0^p(E)$ of square-integrable E-valued harmonic p-forms has finite dimension.

This latter statement holds under weaker conditions and also for $p = 0$. In fact, choosing $c = a = 0$, the above argument yields the following theorem (which was established in [9, 62-63] for scalar-valued differential forms).

THEOREM 5.4: *If the orientable, connected Riemannian manifold M is complete and if, for a given $p = 0, 1, \ldots, n$, $\langle K\varphi | \varphi \rangle_x \geqslant 0$ for all $x \in M \setminus K$ and for all $\varphi \in \Lambda^p(E)$, then* $\dim \mathcal{H}_0^p < \infty$.

Except for the latter statement, the case $p = 0$, in which condition i_p) is void, has been left aside so far. Since, for $p = 0$, $d\Delta = d\delta d = \Delta d$, if $\varphi \in \mathcal{V}^0(E)$ is such that $\Delta\varphi = a\varphi$ for some $a \geqslant 0$, then—as is easily checked—$d\varphi \in \mathcal{V}^1(E)$ and $\Delta d\varphi = ad\varphi$. Thus d defines a linear map $\mathcal{H}_a^0(E) \to \mathcal{H}_a^1(E)$. If $a > 0$, this map is injective because, if the image of $\varphi \in \mathcal{H}_a^0(E)$ vanishes, then $\Delta\varphi = 0$ and therefore

(being $a > 0$), $\varphi = 0$. Hence

$$\dim \mathcal{H}_a^0(E) \leqslant \dim \mathcal{H}_a^1(E)$$

for all $a > 0$. That implies

PROPOSITION 5.5: *If M is an orientable, connected, complete Riemannian manifold and if condition i_1) is satisfied, then $\dim \mathcal{H}_a^0 < \infty$ whenever $0 \leqslant a < c$. If, for $a > 0$, $\dim \mathcal{H}_a^0 > 0$, then a is an eigenvalue of Δ^1.*

6. - It will be shown now that, if condition i_p) is satisfied, every eigenvalue of Δ contained in $[0, c)$ is an isolated point of $\sigma(\Delta)$.

Indeed, let $\{a_\nu\}$ be a sequence of distinct eigenvalues of Δ, contained in $[0, c)$ and converging to a point $a \in [0, c)$. For every $\nu = 1, 2, \ldots$ choose $\varphi_\nu \in \mathcal{H}_{a_\nu}^p$ with $\|\varphi_\nu\| = 1$. By (10)

$$\|\nabla \varphi_\nu\|^2 \leqslant c_1 + a_\nu ,$$

and Rellich's theorem and the weak compactness of the closed unit ball of $L^p(E)$ imply that there exist $\varphi \in L^p(E)$ and a subsequence $\{\varphi_{\nu_i}\}$ of the sequence $\{\varphi_\nu\}$ such that (14) holds for every compact set $\Omega \subset M$. By consequence, for all $\tau \in \mathcal{D}^p(E)$,

$$(\varphi \,|\, (\Delta - aI)\tau) = \lim_{i \to +\infty} (\varphi_{\nu_i} \,|\, (\Delta - aI)\tau) =$$

$$= \lim_{i \to +\infty} ((\varphi_{\nu_i} \,|\, (\Delta - a_{\nu_i} I)\tau) + (a_{\nu_i} - a)(\varphi_{\nu_i} \,|\, \tau)) = \lim_{i \to +\infty} (a_{\nu_i} - a)(\varphi_{\nu_i} \,|\, \tau) = 0 .$$

As before one shows that $\varphi \in D(\Delta)$ and (16) holds, i.e. φ is either 0 or an eigenvector of Δ corresponding to the eigenvalue a. The inequality (10) and the fact that $\varphi \perp \varphi_{\nu_i}$, $\|\varphi_{\nu_i}\| = 1$ for all i, yield

$$\|\nabla(\varphi - \varphi_{\nu_i})\|^2 + (c - a)\|\varphi\|^2 + c - a_{\nu_i} \leqslant (c + c_1)\|\varphi - \varphi_{\nu_i}\|_K^2 ,$$

whence

$$0 \leqslant c - a_{\nu_i} \leqslant (c + c_1)\|\varphi - \varphi_{\nu_i}\|_K^2 .$$

Thus, by (14) with $\Omega = K$, $a = \lim_{i \to +\infty} a_{\nu_i} = c$, contradicting the hypothesis $a < c$.

In conclusion, the following theorem collects the main results established so far.

THEOREM 6.1: *If the connected, orientable Riemannian manifold M is complete and if, for a given $p = 1, \ldots, n = \dim M$, condition i_p) is satisfied, then the intersection $[0, c) \cap \sigma(\Delta^p)$ is either a finite (possibly empty) set or a sequence converging to c of eigenvalues of Δ^p whose corresponding eigenspaces have finite dimension.*

7. - The following considerations yield a description of the (finite set or the) sequence of eigenvalues of Δ in $[0, c)$.

If $\varphi \in \mathcal{H}_0^p(E)$, Stampacchia's inequality yields

$$\|d\varphi\|^2 + \|\delta\varphi\|^2 \leqslant t\|\varphi\|^2$$

for all $t > 0$, implying that $d\varphi = 0$, $\delta\varphi = 0$. Hence, if S^p and S_0^p are the orthogonal complements of $\mathcal{H}_0^p(E)$ in $L^p(E)$ and in $\mathcal{W}^p(E)$ then

$$S_0^p = S^p \cap \mathcal{W}^p(E).$$

Consider the numerical range of the restriction of Δ to $D(\Delta) \cap S^p$, and let a_1 be its greatest lower bound. It turns out that $a_1 > 0$, i.e. the following proposition holds.

Proposition 7.1: *If the connected, orientable Riemannian manifold M is complete, and if condition i_p) is satisfied for some $p = 1, \ldots, n$, then*

$$a_1 := \inf\{(\Delta\varphi|\varphi):\ \varphi \in D(\Delta) \cap S^p,\ \|\varphi\| = 1\} > 0.$$

Proof: If the proposition is false, there is a sequence $\{\varphi_\nu\}$ in $D(\Delta) \cap S^p$ such that $\|\varphi_\nu\| = 1$ and

$$(\Delta\varphi_\nu|\varphi_\nu) \leqslant \frac{1}{\nu} \qquad \text{for } \nu = 1, 2, \ldots. \tag{20}$$

The inequality (10) reads then

$$\|\nabla\varphi_\nu\|^2 + c - \frac{1}{\nu} \leqslant (c + c_1)\|\varphi_\nu\|_K^2 \qquad \text{for } (\nu = 1, 2, \ldots).$$

Rellich's theorem and the weak sequential compactness of the closed unit ball of the Hilbert space $L^p(E)$ imply that there exist $\varphi \in L^p(E)$ and a subsequence $\{\varphi_{\nu_i}\}$ of $\{\varphi_\nu\}$ such that (14) holds for every compact set $\Omega \subset M$; in particular

$$\lim_{i \to +\infty} \|\varphi - \varphi_{\nu_i}\|_K = 0. \tag{21}$$

If $\tau \in \mathcal{D}^p(E)$, (20), and (14) imply that

$$|(\varphi|\Delta\tau) = \lim_{i \to +\infty} |(\varphi_{\nu_i}|\Delta\tau)| = \lim_{i \to +\infty} |(\Delta\varphi_{\nu_i}|\tau)| =$$

$$= \lim_{i \to +\infty} |(\Delta^{1/2}\varphi_{\nu_i}|\Delta^{1/2}\tau)| \leqslant \|\Delta^{1/2}\tau\| \lim_{i \to +\infty} \|\Delta^{1/2}\varphi_{\nu_i}\| \leqslant \|\Delta^{1/2}\tau\| \lim_{i \to +\infty} \frac{1}{\sqrt{\nu_i}} = 0.$$

Hence $\Delta\varphi = 0$ in the weak sense. By regularization, φ can be assumed to be $\varphi \in \Lambda^p(E) \cap L^p(E)$ and to satisfy $\Delta\varphi = 0$ pointwise. Stampacchia's inequality yields then $d\varphi = 0$, $\delta\varphi = 0$, so that $\varphi \in \mathcal{W}^p(E)$. Since furthermore $\varphi \in D(\Delta^\star)$ then $\varphi \in D(\Delta)$ and $\Delta\varphi = 0$, i.e. $\varphi \in \mathcal{H}_0^p$.

By (10), (20) and (21)

$$c\|\varphi-\varphi_{\nu_i}\|^2 \leqslant (c+c_1)\|\varphi-\varphi_{\nu_i}\|_K^2 + (\Delta(\varphi-\varphi_{\nu_i})|\varphi-\varphi_{\nu_i}) =$$

$$= (c+c_1)\|\varphi-\varphi_{\nu_i}\|_K^2 + (\Delta\varphi_{\nu_i}|\varphi_{\nu_i}) \leqslant (c+c_1)\|\varphi-\varphi_{\nu_i}\|_K^2 + \frac{1}{\nu_i} \to 0$$

as $i \to +\infty$, whence

$$\lim_{i\to+\infty} \|\varphi-\varphi_{\nu_i}\| = 0. \tag{22}$$

This implies that $\varphi \in \mathcal{S}^p \cap \mathcal{H}_0^p$ and therefore $\varphi = 0$, i.e. the sequence φ_{ν_i} tends to zero in $L^p(E)$ contradicting the hypothesis $\|\varphi_{\nu_i}\| = 1$ for all i. QED

As a consequence, the norms $\varphi \mapsto (\|\varphi\|^2 + \|d\varphi\|^2 + \|\delta\varphi\|^2)^{1/2}$ and $\varphi \mapsto (\|d\varphi\|^2 + \|\delta\varphi\|^2)^{1/2}$ are equivalent on $\mathcal{S}_0^p$.

Let P be the orthogonal projector in $L^p(E)$ whose range is $\mathcal{S}^p$. Since $(I-P)L^p(E) = \mathcal{H}_0^p \subset D(\Delta)$, then $PD(\Delta) = D(\Delta) \cap \mathcal{S}^p$. Furthermore $\Delta D(\Delta) \subset \mathcal{S}^p$ and if $P\varphi_1 = P\varphi_2$ for φ_1, φ_2 in $D(\Delta)$ then $\Delta\varphi_1 = \Delta\varphi_2$.

Proposition 7.2: *Under the hypotheses of Theorem 6.1, for every $\chi \in \mathcal{S}^p$ there exists a unique $\varphi \in D(\Delta) \cap \mathcal{S}^p$ such that*

$$\Delta\varphi = \chi. \tag{23}$$

Proof: For every $\psi \in \mathcal{S}_0^p$, Proposition 7.1. yields

$$|(\psi|\chi)|^2 = |(\psi|P\chi)|^2 = |(P\psi|\chi)|^2 \leqslant \|\chi\|^2\|P\psi\|^2 \leqslant$$

$$\leqslant a_1\|\chi\|^2(\|dP\psi\|^2 + \|\delta P\psi\|^2) = a_1\|\chi\|^2(\|d\psi\|^2 + \|\delta\psi\|^2),$$

because $d((I-P)\psi) = 0$, $\delta((I-P)\psi) = 0$. Consequently the linear form $\psi \mapsto (\psi|\chi)$ is continuous on $\mathcal{S}_0^p$. Thus there exists a unique $\varphi \in \mathcal{S}_0^p$ such that

$$(\psi|\chi) = (d\psi|d\varphi) + (\delta\psi|\delta\varphi)$$

for all $\psi \in \mathcal{S}_0^p$. If $\tau \in \mathcal{D}^p(E)$, $d((I-P)\tau) = 0$, $\delta((I-P)\tau) = 0$. Hence

$$(\tau|\chi) = (P\tau|\chi) = (dP\tau|d\varphi) + (\delta P\tau|\delta\varphi) = (d\tau|d\varphi) + (\delta\tau|\delta\varphi)$$

for all $\tau \in \mathcal{D}^p(E)$, i.e. $\chi = \Delta\varphi$ in the weak sense. Since the linear form $\tau \mapsto (\Delta\tau|\varphi)$ is continuous on $\mathcal{D}^p(E)$, then $\varphi \in D(\Delta^\star)$. Therefore $\varphi \in D(\Delta)$, and

$$(\chi - \Delta\varphi|\tau) = (\chi|\tau) - (\varphi|\Delta\tau) = 0$$

for all $\tau \in \mathcal{D}^p(E)$. That proves (23). QED

Since $D(\Delta)$ is dense in $L^p(E)$, $PD(\Delta)$ is dense in $\mathcal{S}^p$, or, in other words, $D(\Delta) \cap \mathcal{S}^p$ is dense in $\mathcal{S}^p$.

The operator Δ_1 in $\mathcal{S}^p$ with domain $D(\Delta_1) = D(\Delta) \cap \mathcal{S}^p$ obtained by restricting Δ to the latter space is closed and symmetric. Since by Proposition 7.2 the range of Δ_1 is the

entire space $\mathcal{S}^p$, Δ_1 is self-adjoint. By Proposition 7.1

$$\sigma(\Delta_1) \subset [a_1, +\infty).$$

Let $\zeta \in \mathbf{C}$. For $\varphi \in D(\Delta) \cap \mathcal{S}^p$ and $\psi \in \mathcal{H}_0^p$

$$(\zeta I - \Delta)(\varphi + \psi) = (\zeta I_{\mathcal{S}^p} - \Delta)\varphi + \zeta\psi,$$

implying that

$$\sigma(\Delta_1) = \sigma(\Delta) \cap \mathbf{R}_+^\star .$$

If $a > 0$ is an eigenvalue of Δ, with eigenspace $\mathcal{H}_a^p$, then $\mathcal{H}_a^p \perp \mathcal{H}_0^p$, and therefore $\mathcal{H}_a^p \subset \mathcal{S}^p$.

That yields another proof of the fact that 0 is either contained in the resolvent set of Δ or is an isolated point of $\sigma(\Delta)$.

Since the closure of the numerical range of the self-adjoint operator Δ_1 is the closed convex hull of $\sigma(\Delta_1)$ [8, 327-330], if $a_1 < c$, a_1 is an eigenvalue of Δ: the first eigenvalue on the right hand side of 0.

Proposition 7.3: *If the connected, orientable Riemannian manifold M is complete, if condition i_p) is satisfied for some $p = 1, \ldots, n$, and if $a_1 < c$, then*

$$a_2 := \inf\{(\Delta\varphi|\varphi)\colon\ \varphi \perp \mathcal{H}_0^p \oplus \mathcal{H}_{a_1}^p,\ \ \varphi \in D(\Delta),\ \ \|\varphi\| = 1\} > a_1 .$$

Proof: If the proposition is false there is a sequence $\{\varphi_\nu\}$ in $D(\Delta)$ such that $\varphi_\nu \perp \mathcal{H}_0^p \oplus \mathcal{H}_{a_1}^p$, $\|\varphi_\nu\| = 1$ and

$$(\Delta\varphi_\nu|\varphi_\nu) \leqslant a_1 + \frac{1}{\nu} \qquad \text{for } \nu = 1, 2, \ldots. \tag{24}$$

Hence, by (10),

$$\|\nabla\varphi_\nu\|^2 + c - a_1 - \frac{1}{\nu} \leqslant (c + c_1)\|\varphi_\nu\|_K^2 \qquad \text{for } (\nu = 1, 2, \ldots).$$

By Rellich's theorem and the weak sequential compactness of the closed unit ball of the Hilbert space $L^p(E)$ there exist $\varphi \in L^p(E)$ and a subsequence $\{\varphi_{\nu_i}\}$ of $\{\varphi_\nu\}$ such that (14) holds for every compact set $\Omega \subset M$; in particular, also (21) holds.

If $\tau \in \mathcal{D}^p(E)$, then by (14) and (24),

$$|(\varphi|(\Delta - a_1 I)\tau) = \lim_{i \to +\infty} |(\varphi_{\nu_i}|(\Delta - a_1 I)\tau)| =$$

$$= \lim_{i \to +\infty} |((\Delta_1 - a_1 I)\varphi_{\nu_i}|\tau)| = \lim_{i \to +\infty} |((\Delta_1 - a_1 I)^{1/2}\varphi_{\nu_i}|(\Delta_1 - a_1 I)^{1/2}\tau)| \leqslant$$

$$\leqslant \|(\Delta_1 - a_1 I)^{1/2}\tau\| \lim_{i \to +\infty} \|(\Delta_1 - a_1 I)^{1/2}\varphi_{\nu_i}\| \leqslant \|(\Delta_1 - a_1 I)^{1/2}\tau\| \lim_{i \to +\infty} \frac{1}{\sqrt{\nu_i}} = 0.$$

Hence $\Delta\varphi = a_1\varphi$ in the weak sense, and—by the same kind of argument used in the proof of Proposition 7.1—one shows that $\varphi \in \mathcal{H}_{a_1}^p$. On the other hand by (10), (21)

and (24)

$$(c-a_1)\|\varphi-\varphi_{\nu_i}\|^2 \leqslant (c+c_1)\|\varphi-\varphi_{\nu_i}\|_K^2 + ((\Delta - a_1 I)(\varphi-\varphi_{\nu_i})|\varphi-\varphi_{\nu_i}) =$$

$$= (c+c_1)\|\varphi-\varphi_{\nu_i}\|_K^2 + ((\Delta - a_1 I)\,\varphi_{\nu_i}|\varphi_{\nu_i}) \leqslant (c+c_1)\|\varphi-\varphi_{\nu_i}\|_K^2 + \frac{1}{\nu_i} \to 0$$

as $i \to +\infty$ so that (22) holds.

The fact that $\varphi_\nu \perp \mathcal{H}^p_{a_1}$ implies then that $\varphi \perp \mathcal{H}^p_{a_1}$ and therefore $\varphi = 0$. Hence $\{\varphi_{\nu_i}\}$ tends to zero in $L^p(E)$ contradicting the fact that $\|\varphi_{\nu_i}\| = 1$ for all $i = 1, 2, \ldots$. QED

As before, one shows that the intersection $D(\Delta) \cap (\mathcal{H}^p_0 \oplus \mathcal{H}^p_{a_1})^\perp$ is dense in $(\mathcal{H}^p_0 \oplus \mathcal{H}^p_{a_1})^\perp$ for the norm $\| \ \|$, and that the restriction of Δ to that intersection is a self-adjoint operator Δ_2 in $(\mathcal{H}^p_0 \oplus \mathcal{H}^p_{a_1})^\perp$ for which

$$\sigma(\Delta_2) \subset [a_2, +\infty)$$

and

$$\sigma(\Delta) \cap (a_1, +\infty) = \sigma(\Delta_2).$$

By the same kind of argument as before one shows also that, if $a_2 < c$, then a_2 is an eigenvalue of Δ: the first eigenvalue on the right hand side of a_1.

Iteration of the above procedure yields a description of $[0, c) \cap \sigma(\Delta)$.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York/San Francisco/London, 1975.

[2] A. Andreotti - E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math., n. 25 (1965), pp. 81-130; Erratum, *ibid.*, pp. 153-155.

[3] J. Eells - L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc., 10 (1978), pp. 1-68.

[4] J. Eells - L. Lemaire, *Selected topics in harmonic maps*, Confer. Board Math. Sci. Amer. Math. Soc., n. 50 (1980), pp. 1-85.

[5] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin/Heidelberg/New York, 1966.

[6] R. Narasimhan, *Lectures on Topics in Analysis*, Tata Institute of Fundamental Research, Bombay, 1965.

[7] F. Ricci, *L^2-vector bundle valued forms and the Laplace-Beltrami operator*, Rend. Sem. Mat. Univ. Padova, 76 (1986).

[8] M. H. Stone, *Linear Transformations in Hilbert Space*, Amer. Math. Soc. Colloquium Publications, Vol. 15, Providence, R.I., 1932.

[9] E. Vesentini, *Lectures on Levi Convexity of Complex Manifolds and Cohomology Vanishing Theorems*, Tata Institute of Fundamental Research, Bombay, 1966.

[10] K. Yosida, *Functional Analysis*, II Ed., Springer-Verlag, Berlin/Heidelberg/New York, 1968.

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