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On the Motion of a Fluid in a Basin with Slipping Conditions on the Side (**) (***)

SUMMARY. — The motion in a basin of a viscous incompressible fluid is studied under the assumption that the velocity of the fluid does not necessarily vanish on the side of the basin. The model consists of variational inequalities obtained from the Navier-Stokes equations by imposing on the functions intervening in the problem some limitations of obvious physical significance, in order to obtain a «physically consistent» model.

Sul moto di un fluido in un bacino con condizioni di scivolamento sulla parete

RIASSUNTO. — Si studia il moto di un fluido viscoso incomprimibile in un bacino, nell'ipotesi che il fluido stesso non abbia necessariamente velocità nulla sulle pareti laterali dello stesso. Il modello consta di disequazioni variazionali, ottenute dalle equazioni di Navier-Stokes, imponendo alle grandezze considerate limitazioni di ovvio significato fisico, in modo che il modello stesso risulti «fisicamente consistente».

1. - THE PROBLEM AND THE MODEL

It is well known that the mathematical model of the motion of a viscous incompressible fluid of unit density and viscosity μ in a given domain Ω, with boundary Γ, is represented by Navier-Stokes equations

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \mu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \operatorname{div} \boldsymbol{u} = 0 . \end{cases} \tag{1.1}$$

It is possible to associate to (1.1) some different conditions, taking into account either different behaviour on Γ (boundary conditions) or different initial conditions, or even more the fact that Navier-Stokes equations are non relativistic (consistency conditions).

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With respect to boundary conditions we can suppose that Γ is completely constituted by a material surface which is a known function of the time t ([1,2]), or a constant (se for ex. references in [3]); finally we can suppose that a part of Γ is a free surface ([3]).

In every case we can have different boundary conditions on the part of Γ which is a material fixed surface, in order to represent different behaviour of the fluid.

In [3] the motion of a viscous incompressible fluid is studied in a basin, assuming that the velocity is zero on the bottom and side.

Conforming to this paper, we can observe that, because of the non slipping condition, the free surface cannot move at the side of the basin. It is then clear that such a condition cannot give a good model of the motion of the fluid, if we are interested in studying the solution near to the intersection of the free surface with a part of Γ as would be the case of the side of the basin, or if a solid were partially immersed in the fluid. This problem will, in fact, be the object of a future research.

In order to eliminate this difficulty we can use different boundary conditions on the bottom and side: we will precisely consider a «non slipping» condition on the bottom and a suitable «slipping» condition on the side of tha basin.

Let us precisely introduce the following notations:

A = open bounded set of (x,y) plane, with boundary ∂A, satisfying locally a Lipschitz condition;

$\tilde{\partial} A$ = the set of the singular points of ∂A;

$Q_\varphi = \{(x,y) \in A,\ -1 < z < \varphi(x,y,t),\ 0 < t < T\}$;

$\Omega_{\varphi,t} = \{(x,y) \in A,\ -1 < z < \varphi(x,y,t)\}$;

$\Lambda_{\bar{t}} = \{(x,y) \in A,\ 0 < t < \bar{t}\}$;

$\Gamma_{1,\varphi,t} = \partial\Omega_{\varphi,t} \cap \{z = \varphi(x,y,t)\}$;

$\Gamma_2 = \partial\Omega_{\varphi,t} \cap \{z = -1\}$;

$\Gamma_{3,\varphi,t} = \{(x,y) \in \partial A - \tilde{\partial} A,\ -1 < z < \varphi(x,y,t)\}$.

In what follows (cfr. [3]), $z = \varphi(x,y,t)$ will represent the equation of the free surface at the time t, $\Omega_{\varphi,t}$ the domain in which the motion takes place (which obviously depends on t), $\Gamma_{1,\varphi,t}$ the free surface, Γ_2 the bottom of the basin, $\Gamma_{3,\varphi,t}$ the side of the basin depending obviously on t, because we have not supposed $\varphi(x,y,t) = \text{cost}$, for $(x,y) \in \partial A$ (as in [3]).

Denoting by $\boldsymbol{u}$ the velocity of the fluid, by p the pressure, by μ the viscosity, and by f the external force, the motion in Q_φ of the fluid (assumed to be incompressible and of unit density) is governed by the classical Navier-Stokes equations (1.1) to which we add the following boundary conditions.

$$\boldsymbol{u}(x,y,z,t) = 0 \quad \text{when } (x,y,z) \in \Gamma_2,\ 0 < t < T, \tag{1.2}$$

$$\frac{\partial\varphi}{\partial t} + u_1(x,y,\varphi(x,y,t),t)\frac{\partial\varphi}{\partial x} + u_2(x,y,\varphi(x,y,t),t)\frac{\partial\varphi}{\partial y} + \\ - u_3(x,y,\varphi(x,y,t),t) = 0, \quad \text{when } (x,y) \in A,\ 0 < t < T, \tag{1.3}$$

(1.4) $p(x,y,\varphi(x,y,t),t)=0\,,\qquad$ when $(x,y)\in A,\ 0<t<T\,,$

(1.5) $\dfrac{\partial \boldsymbol{u}}{\partial \nu}(x,y,\varphi(x,y,t),t)=0\,,\qquad$ when $(x,y)\in A,\ 0<t<T\,,$

(1.6) $\boldsymbol{u}(x,y,z,t)\cdot\boldsymbol{n}=0,\qquad$ when $(x,y,z)\in\Gamma_{3,\varphi,t},\ 0<t<T\,,$

(1.7) $\left(\dfrac{\partial \boldsymbol{u}(x,y,z,t)}{\partial n}\right)_\pi=\rho(x,y,z,t)\,\boldsymbol{u}(x,y,z,t)\quad$ when $(x,y,z)\in\Gamma_{3,\varphi,t},\ 0<t<T\,,$

where u_i are the components of $\boldsymbol{u}$, $\boldsymbol{\nu}$ is the normal to $\Gamma_{1,\varphi,t}$, $\boldsymbol{n}$ is the normal to $\Gamma_{3,\varphi,t}$, $(\boldsymbol{v})_\pi$ denotes the orthogonal projection of the vector $\boldsymbol{v}$ on the π plane tangential to $\Gamma_{3,\varphi,t}$ and $\rho(x,y,z)$ is a known function on $\Gamma_{3,\varphi,t}$.

We add also the initial conditions

(1.8) $\varphi(x,y,0)=\bar{\varphi}(x,y)\qquad$ when $(x,y)\in A\,,$

(1.9) $\boldsymbol{u}(x,y,z,0)=\bar{\boldsymbol{u}}(x,y,z)\qquad$ when $(x,y,z)\in Q_{\bar{\varphi},0}\,.$

The physical meaning of conditions (1.2)-(1.5) and (1.8), (1.9) is well known.

Conditions (1.6) and (1.7), which are the suitable conditions mentioned above, are valid on the side only of the basin: the first one imposes that the fluid cannot leave the side of the basin, the second one means that the fluid can slip on the same side, in such a way that its velocity is directed as the tangential component of the velocity in an internal very close point. In other words, there is a sort of friction that does not change the direction of the velocity.

Infact, if we indicate by $(\boldsymbol{v})_n$ the projection of a generic vector $\boldsymbol{v}$ on the n axis, by n the abscissa on the n axis (with the origin in $P\in\Gamma_{3,\varphi,t}$) by $Q(n)$ a point on n axis, by $\boldsymbol{\omega}(n)$ a suitable infinitesimal vector of order higher than n, if $\boldsymbol{u}$ is smooth enough we have successively (by (1.6) and projecting on n)

$$\boldsymbol{u}(Q)-\boldsymbol{u}(P)=n\,\frac{\partial \boldsymbol{u}(P)}{\partial n}+\boldsymbol{\omega}=n\left\{\left(\frac{\partial \boldsymbol{u}(P)}{\partial n}\right)_n+\left(\frac{\partial \boldsymbol{u}(P)}{\partial n}\right)_\pi\right\}+\boldsymbol{\omega}\,,$$

$$(\boldsymbol{u}(Q))_\pi-\boldsymbol{u}(P)=n\left(\frac{\partial \boldsymbol{u}(P)}{\partial n}\right)_\pi+\boldsymbol{\omega}$$

and precisely, by (1.7)

$$(\boldsymbol{u}(Q))_\pi=(1+n\rho(P))\,\boldsymbol{u}(P)+(\boldsymbol{\omega})_\pi\,.$$

Regarding the initial conditions (1.8), (1.9) we obviously suppose that $\bar{\varphi}(x,y)>-1$, and $\bar{\boldsymbol{u}}(x,y)$ satisfies (1.2) and (1.5)-(1.7).

Let us finally observe that (1.3) makes sense if $\varphi(x,y,t)$ is regular enough only, and if it maintains a positive distance from the bottom.

Denoting by M_1, M_2, σ some appropriate positive constants, we shall therefore as-

sume that

$$(1.10)\qquad \begin{cases} |\varphi_{xx}| \leq M_1, & |\varphi_{xt}| \leq M_1, & |\varphi_{tt}| \leq M_1, \\ |\varphi_{yy}| \leq M_1, & |\varphi_{yt}| \leq M_1, & \varphi \geq -1+\sigma; \end{cases}$$

moreover we assume that the following consistency condition holds

$$(1.11)\qquad |u(x,y,z,t)| \leq M_2 .$$

2. - Weak formulation of the constitutive equations

Let h be a function $\in C^1(Q_\varphi)$, with

$$(2.1)\qquad \begin{cases} \operatorname{div} h = 0, & h = 0 \quad \text{on } \Gamma_2 \times (0,T), \\ h \cdot n = 0 & \text{on } \Gamma_{3,\varphi,t}. \end{cases}$$

Multiplying the first of (1.1) by h and integrating over Q_φ, applying Green's formulas and bearing in mind (1.4) and (1.5), we obtain

$$(2.2)\qquad \int_{Q_\varphi} \left\{ \frac{\partial u}{\partial t} \cdot h + \mu \left(\frac{\partial u}{\partial x} \cdot \frac{\partial h}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial h}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial h}{\partial z} \right) + (u \cdot \nabla) u \cdot h - f \cdot h \right\} dQ - \\ - \mu \int_0^T dt \int_{\Gamma_{3,\varphi,t}} \frac{\partial u}{\partial n} \cdot h \, d\Gamma = 0 .$$

Resolving both h and $\partial u / \partial n$ into the components on the n axes and π plane and bearing in mind (1.7), (2.1) we have

$$\int_{\Gamma_{3,\varphi,t}} \left(\frac{\partial u}{\partial n} \cdot h \right) d\Gamma = \int_{\Gamma_{3,\varphi,t}} \left\{ \left(\frac{\partial u}{\partial n} \right)_n + \left(\frac{\partial u}{\partial n} \right)_\pi \right\} \cdot (h)_\pi = \int_{\Gamma_{3,\varphi,t}} \left(\frac{\partial u}{\partial n} \right)_\pi \cdot (h)_\pi \, d\Gamma - \int_{\Gamma_{3,\varphi,t}} \rho u \cdot h \, d\Gamma ,$$

and (2.2) can be written

$$(2.3)\qquad \int_{Q_\varphi} \left\{ \frac{\partial u}{\partial t} \cdot h + \mu \left(\frac{\partial u}{\partial x} \cdot \frac{\partial h}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial h}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial h}{\partial z} \right) + (u \cdot \nabla) u \cdot h - f \cdot h \right\} - \\ - \mu \int_{\Gamma_{3,\varphi,t}} \rho u \cdot h \, d\Gamma = 0 .$$

If we introduce the following new set of variables

$$(2.4)\qquad \xi = x, \quad \eta = y, \quad \zeta = \frac{z - \varphi(x,y,t)}{1 + \varphi(x,y,t)}, \quad \tau = t$$

(whose inverse is

$$x=\xi, \quad y=\eta, \quad z=[1+\varphi(\xi,\eta,\tau)]\,\zeta+\varphi(\xi,\eta,\tau), \quad t=\tau),$$

the sets defined in § 1 are trasformed in the following way:

$$A\to A^*=A, \quad \Lambda_t\to\Lambda_t^*=\Lambda_t\,(\Lambda\to\Lambda^*=\Lambda), \quad \tilde{\partial}A\to\tilde{\partial}A^*=\tilde{\partial}A,$$

$$Q_\varphi\to Q^*=\{(\xi,\eta)\in A,\ -1<\zeta<0,\ 0<\tau<T\},$$

$$\Omega_{\varphi,t}\to\Omega^*=\{(\xi,\eta)\in A,\ -1<\zeta<0\},$$

$$\Gamma_{1,\varphi,t}\to\Gamma_1^*=\{\partial\Omega^*\cap(\zeta=0)\}, \quad \Gamma_2\to\Gamma_2^*=\Gamma_2,$$

$$\Gamma_{3,\varphi,t}\to\Gamma_3^*=\{(\xi,\eta)\in\partial A-\tilde{\partial}A,\ -1<\zeta<0\}.$$

Moreover, indicating by v any vector function, we obviously use the following notations

$$v^*(\xi,\eta,\zeta,\tau)=v(\xi,\eta,(1+\varphi(\xi,\eta,\tau))\,\zeta+\varphi(\xi,\eta,\tau),\tau),$$

(that is $v^*=\{v_1^*,v_2^*,v_3^*\}$,

$$v_i^*(\xi,\eta,\zeta,\tau)=v_i(\xi,\eta,(1+\varphi)\,\zeta+\varphi,\tau) \qquad (i=1,2,3)),$$

and therefore

$$\begin{aligned} v(x,y,z,t)&=v^*\left(x,y,\frac{z-\varphi(x,y,t)}{1+\varphi(x,y,t)},t\right),\\ v_i(x,y,z,t)&=v_i^*\left(x,y,\frac{z-\varphi(x,y,t)}{1+\varphi(x,y,t)},t\right), \qquad (i=1,2,3). \end{aligned} \tag{2.5}$$

By (2.4) and (2.5) it follows

$$\zeta=\frac{z-\varphi}{1+\varphi}=\frac{z+1}{1+\varphi}-1,$$

$$\frac{\partial v}{\partial x}=\frac{\partial v^*}{\partial x}+\frac{\partial v^*}{\partial z}\,\frac{-\varphi_x(1+\varphi)-\varphi_x(z-\varphi)}{(1+\varphi)^2}=\frac{\partial v^*}{\partial\xi}-\frac{(1+z)\,\varphi_x}{(1+\varphi)^2}\,\frac{\partial v^*}{\partial\zeta}$$

and finally

$$\frac{\partial v}{\partial x}=\frac{\partial v^*}{\partial\xi}-\frac{\zeta+1}{1+\varphi}\,\varphi_\xi\,\frac{\partial v^*}{\partial\zeta}. \tag{2.6}$$

In the same way we have

$$\frac{\partial v}{\partial y}=\frac{\partial v^*}{\partial\eta}-\frac{\zeta+1}{1+\varphi}\,\varphi_\eta\,\frac{\partial v^*}{\partial\zeta}, \tag{2.7}$$

$$\frac{\partial v}{\partial t} = \frac{\partial v^*}{\partial \tau} - \frac{\zeta+1}{1+\varphi}\varphi_\tau \frac{\partial v^*}{\partial \zeta}, \tag{2.8}$$

$$\frac{\partial v}{\partial z} = \frac{\partial v^*}{\partial \zeta}\left(\frac{1}{1+\varphi}\right). \tag{2.9}$$

We shall transform (2.3) firstly by observing that it is

$$\frac{\partial \boldsymbol{u}}{\partial t}\cdot \boldsymbol{h} = \left(\frac{\partial \boldsymbol{u}^*}{\partial \tau} - \frac{\zeta+1}{1+\varphi}\varphi_\tau \frac{\partial \boldsymbol{u}^*}{\partial \zeta}\right)\cdot \boldsymbol{h}^*, \tag{2.10}$$

$$\frac{\partial \boldsymbol{u}}{\partial x}\cdot\frac{\partial \boldsymbol{h}}{\partial x} = \frac{\partial \boldsymbol{u}^*}{\partial \xi}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \xi} - \frac{\zeta+1}{1+\varphi}\varphi_\xi\left(\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \xi} + \frac{\partial \boldsymbol{u}^*}{\partial \xi}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \zeta}\right) + \frac{(\zeta+1)^2}{(1+\varphi)^2}\varphi_\xi^2 \frac{\partial \boldsymbol{u}^*}{\partial \zeta}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \zeta}, \tag{2.11a}$$

$$\frac{\partial \boldsymbol{u}}{\partial y}\cdot\frac{\partial \boldsymbol{h}}{\partial y} = \frac{\partial \boldsymbol{u}^*}{\partial \eta}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \eta} - \frac{\zeta+1}{1+\varphi}\varphi_\eta\left(\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \eta} + \frac{\partial \boldsymbol{u}^*}{\partial \eta}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \zeta}\right) + \frac{(\zeta+1)^2}{(1+\varphi)^2}\varphi_\eta^2 \frac{\partial \boldsymbol{u}^*}{\partial \zeta}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \eta}, \tag{2.11b}$$

$$\frac{\partial \boldsymbol{u}}{\partial z}\cdot\frac{\partial \boldsymbol{h}}{\partial z} = \frac{1}{(1+\varphi)^2}\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\cdot\frac{\partial \boldsymbol{h}^*}{\partial \zeta}, \tag{2.11c}$$

$$(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = u_1^*\frac{\partial \boldsymbol{u}}{\partial x} + u_2^*\frac{\partial \boldsymbol{u}}{\partial y} + u_3^*\frac{\partial \boldsymbol{u}}{\partial z} = u_1^*\left(\frac{\partial \boldsymbol{u}^*}{\partial \xi} - \frac{\zeta+1}{1+\varphi}\varphi_\xi\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\right) + u_2^*\left(\frac{\partial \boldsymbol{u}^*}{\partial \eta} - \frac{\zeta+1}{1+\varphi}\varphi_\eta\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\right) + u_3^*\left(\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\frac{1}{1+\varphi}\right),$$

and finally

$$\begin{aligned}(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} &= u_1^*\frac{\partial \boldsymbol{u}}{\partial \xi} + u_2^*\frac{\partial \boldsymbol{u}}{\partial \eta} + u_3^*\frac{\partial \boldsymbol{u}}{\partial \zeta} + \\ &\quad - \frac{1}{1+\varphi}\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\{(\zeta+1)(u_1^*\varphi_\xi + u_2^*\varphi_\eta) + u_3^*\varphi\} = \\ &= (\boldsymbol{u}^*\cdot\nabla^*) - \frac{1}{1+\varphi}\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\{(\zeta+1)(u_1^*\varphi_\xi + u_2^*\varphi_\eta) + u_3^*\varphi\},\end{aligned} \tag{2.11d}$$

where ∇^* is the operator $\left\{\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right\}$.

Introducing now the vector

$$\boldsymbol{w}_\varphi = (\zeta+1)\,\varphi_\xi\,\boldsymbol{i} + (\zeta+1)\,\varphi_\eta\,\boldsymbol{j} + \varphi\boldsymbol{k} = (\zeta+1)\,\mathrm{grad}\,\varphi + \varphi\boldsymbol{k}, \tag{2.12}$$

we have

$$(2.13)\qquad (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = (\boldsymbol{u}^*\cdot\nabla^*)\,\boldsymbol{u}^* + \frac{1}{1+\varphi}(\boldsymbol{u}^*\cdot\boldsymbol{w}_\varphi)\frac{\partial \boldsymbol{u}^*}{\partial\zeta}.$$

If we indicate by

$$(2.14)\qquad x=\alpha(s),\quad y=\beta(s),\quad z=-1,\quad (0\leqslant s\leqslant 1,\ (\alpha,\beta)\in\partial A),$$

a regular arc of ∂A (oriented positively when A is kept on the left), by (2.4) it follows that

$$(2.15)\qquad d\Gamma_{3,\varphi} = ds\,dz = ds(1+\varphi)\,d\zeta = \{1+\varphi(\alpha(s),\beta(s),\tau)\}\,d\Gamma_3^*;$$

moreover

$$(2.16)\qquad \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} = \begin{vmatrix} 1 & 0 & (\zeta+1)\,\varphi_\xi \\ 0 & 1 & (\zeta+1)\,\varphi_\eta \\ 0 & 0 & 1+\varphi \end{vmatrix} = 1+\varphi.$$

By applying finally (2.10)-(2.16) to (2.3) we obtain

$$(2.17)\qquad \int_0^T \{(\boldsymbol{u}^{*\prime}(\tau),\boldsymbol{h}^*(\tau))_{L^2_\varphi(\Omega^*)} + \mu a_\varphi(\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau)) +$$

$$+ b_\varphi(\boldsymbol{u}^*(\tau),\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau)) + (\boldsymbol{f}^*(\tau),\boldsymbol{h}^*(\tau))_{L^2_\varphi(\Omega^*)} - \mu(\rho_\varphi^*(\tau)\,\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau))_{L^2(\Gamma_3^*)}\}\,d\tau = 0,$$

where

$$(\boldsymbol{u}^*,\boldsymbol{v}^*)_{L^2_\varphi(\Omega^*)} = \int_{\Omega^*} \boldsymbol{u}^*\cdot\boldsymbol{v}^*(\varphi+1)\,d\xi\,d\eta\,d\zeta,$$

$$(2.18)\qquad a_\varphi(\boldsymbol{u}^*,\boldsymbol{v}^*) = \int_{\Omega^*}\Big\{(\boldsymbol{u}_\xi^*\cdot\boldsymbol{v}_\xi^* + \boldsymbol{u}_\eta^*\cdot\boldsymbol{v}_\eta^*) + \frac{1}{(1+\varphi)^2}[1+(\zeta+1)^2(\varphi_\xi^2+\varphi_\eta^2)\,\boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}_\zeta^*] -$$

$$-\frac{\zeta+1}{\varphi+1}[(\boldsymbol{u}_\xi^*\cdot\boldsymbol{v}_\zeta^* + \boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}_\xi^*)\,\varphi_\xi + (\boldsymbol{u}_\eta^*\cdot\boldsymbol{v}_\zeta^* + \boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}_\eta^*)\,\varphi_\eta] +$$

$$+\frac{\zeta+1}{\mu(1+\varphi)}\varphi_\tau\,\boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}^*\Big\}(\varphi+1)\,d\xi\,d\eta\,d\zeta;$$

$$(2.19)\qquad b_\varphi(\boldsymbol{u}^*,\boldsymbol{v}^*,\boldsymbol{h}^*) = \int_{\Omega^*}\Big\{(\boldsymbol{u}^*\cdot\nabla^*)\,\boldsymbol{v}^* + \frac{1}{1+\varphi}(\boldsymbol{u}^*\cdot\boldsymbol{w}_\varphi)\frac{\partial\boldsymbol{v}^*}{\partial\zeta}\Big\}\cdot\boldsymbol{h}^*(1+\varphi)\,d\xi\,d\eta\,d\zeta,$$

$$\rho_\varphi^* = (1+\varphi)\,\rho^*.$$

Conditions (1.2)-(1.7) change by (2.4) as follows

$$(1.2)\rightarrow \boldsymbol{u}^*(\xi,\eta,-1,\tau)=0,\quad 0<\tau<T;$$

$$(1.3) \rightarrow \varphi_\tau + u_1^*(\xi,\eta,0,\tau)\,\varphi_\xi + u_2^*(\xi,\eta,0,\tau)\,\varphi_\eta - u_3^*(\xi,\eta,0,\tau) = 0, \quad (\xi,\eta)\in A, \quad 0<\tau<T;$$

$$(1.4) \rightarrow p(\xi,\eta,0,\tau) = 0, \quad (\xi,\eta)\in A, \quad 0<\tau<T.$$

In order to transform (1.5) we observe that the normal $\boldsymbol{\nu}$ to the surface $z=\varphi(x,y,z)$ is

$$\boldsymbol{\nu} = (\varphi_x \boldsymbol{i} + \varphi_y \boldsymbol{j} - \boldsymbol{k})(1+\varphi_x^2+\varphi_y^2)^{-1/2}$$

then, by (2.6) and (2.7),

$$\frac{\partial \boldsymbol{u}}{\partial \nu} = \left\{ \left[\frac{\partial \boldsymbol{u}^*}{\partial \xi} - \frac{\zeta+1}{\varphi+1}\varphi_\xi \frac{\partial \boldsymbol{u}^*}{\partial \zeta} \right] \varphi_\xi + \right.$$

$$\left. + \left[\frac{\partial \boldsymbol{u}^*}{\partial \eta} - \frac{\zeta+1}{\varphi+1}\varphi_\eta \frac{\partial \boldsymbol{u}^*}{\partial \zeta} \right] \varphi_\eta - \frac{1}{\varphi+1}\frac{\partial \boldsymbol{u}}{\partial \zeta} \right\} \{1+\varphi_\xi^2+\varphi_\eta^2\}^{-1/2};$$

imposing now $\zeta=0$ and introducing the vector

$$\boldsymbol{v}_\varphi = \left\{ \varphi_\xi, \varphi_\eta, -\frac{1+\varphi_\xi^2+\varphi_\eta^2}{1+\varphi} \right\},$$

we have

$$(1.5) \rightarrow (\boldsymbol{v}_\varphi \cdot \nabla^*)\,\boldsymbol{u}^*(\xi,\eta,0,\tau) = 0, \qquad (\xi,\eta)\in A, \qquad 0<\tau<T.$$

Regarding (1.6) we have that the vector

$$\boldsymbol{n} = \{\beta'(s), -\alpha(s), 0\} \tag{2.20}$$

is normal to $\Gamma_{3,\varphi,t}$ at the point $P(\alpha(s),\beta(s),z)$, $\forall z\in(-1,\varphi(\alpha(s),\beta(s)))$ and also to Γ_3^*, at the point

$$P^*(\alpha(s),\beta(s),\zeta), \qquad \forall \zeta = \frac{z-\varphi}{z+\varphi} \in (-1,0).$$

Then by (2.20) it follows

$$\boldsymbol{u}(\alpha(s),\beta(s),z,t)\cdot\boldsymbol{n} = u_1(\alpha,\beta,z,t)\,\beta' - u_2(\alpha,\beta,z,t)\,\alpha' =$$

$$= u_1^*(\alpha,\beta,\zeta,\tau)\,\beta' - u_2^*(\alpha,\beta,\zeta,\tau)\,\alpha' = 0$$

and finally

$$(1.6) \rightarrow \boldsymbol{u}^*(\alpha,\beta,\zeta,\tau)\cdot\boldsymbol{n} = 0, \qquad (-1<\zeta<0).$$

In order to transform (1.7) we shall first calculate $\partial \boldsymbol{u}/\partial n$. From (2.6), (2.7), (2.20)

it follows

$$\frac{\partial \boldsymbol{u}}{\partial n} = \frac{\partial \boldsymbol{u}}{\partial x}\beta' - \frac{\partial \boldsymbol{u}}{\partial y}\alpha' = \left\{\frac{\partial \boldsymbol{u}^*}{\partial \xi} - \frac{\zeta+1}{\varphi+1}\varphi_\xi \frac{\partial \boldsymbol{u}^*}{\partial \zeta}\right\}\beta' - \left\{\frac{\partial \boldsymbol{u}^*}{\partial \eta} - \frac{\zeta+1}{\varphi+1}\varphi_\eta \frac{\partial \boldsymbol{u}^*}{\partial \zeta}\right\}\alpha' =$$

$$= \frac{\partial \boldsymbol{u}^*}{\partial \xi}\beta' - \frac{\partial \boldsymbol{u}^*}{\partial \eta}\alpha' - \frac{\zeta+1}{\varphi+1}\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\{\varphi_\xi\beta' - \varphi_\eta\alpha'\} = \frac{\partial \boldsymbol{u}^*}{\partial n} - \frac{\zeta+1}{\varphi+1}\frac{\partial \boldsymbol{u}^*}{\partial \zeta}(\boldsymbol{n}\cdot \operatorname{grad}^* \psi).$$

Since (2.4) don't change neither $\boldsymbol{n}$ nor π, we have finally

$$(1.7) \rightarrow \left(\frac{\partial \boldsymbol{u}^*(\alpha,\beta,\zeta,\tau)}{\partial n} - \frac{\zeta+1}{\varphi+1}\frac{\partial \boldsymbol{u}^*(\alpha,\beta,\zeta,\tau)}{\partial \zeta}(\boldsymbol{n}\cdot \operatorname{grad}^* \varphi)\right)_\pi =$$

$$= \rho^*(\alpha,\beta,\zeta,\tau)\,\boldsymbol{u}^*(\alpha,\beta,\zeta,\tau), \qquad 0<\tau<T, \quad -1<\zeta<0.$$

Finally we shall calculate $\operatorname{div} \boldsymbol{u}$. From (2.6), (2.7), (2.9) it follows

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} =$$

$$= \frac{\partial u_1^*}{\partial \xi} - \frac{\zeta+1}{\varphi+1}\varphi_\xi \frac{\partial u_1^*}{\partial \zeta} + \frac{\partial u_2^*}{\partial \eta} - \frac{\zeta+1}{\varphi+1}\varphi_\eta \frac{\partial u_2^*}{\partial \zeta} + \frac{1}{\zeta+1}\frac{\partial u_3^*}{\partial \zeta} =$$

$$= \operatorname{div}^* \boldsymbol{u}^* - \frac{1}{\varphi+1}\left\{(\zeta+1)\left[\varphi_\xi \frac{\partial u_1^*}{\partial \zeta} + \varphi_\eta \frac{\partial u_2^*}{\partial \zeta}\right] + \varphi \frac{\partial u_3^*}{\partial \zeta}\right\}.$$

Setting now, for any $\boldsymbol{v}$,

$$\operatorname{div}_\varphi \boldsymbol{v}^* = \operatorname{div}^* \boldsymbol{v}^* - \frac{1}{\varphi+1}\left\{\frac{\partial \boldsymbol{v}^*}{\partial \zeta}\cdot \boldsymbol{w}_\varphi^*\right\} = \operatorname{div}^* \boldsymbol{v}^* - \frac{1}{\varphi+1}\frac{\partial \boldsymbol{v}^*}{\partial \zeta}\cdot\{(\zeta+1)\operatorname{grad}\varphi + \varphi \boldsymbol{k}\},$$

and applying (2.4) to the second of (1.1), by (2.12) we have

$$\operatorname{div} \boldsymbol{u} = \operatorname{div}^* \boldsymbol{u} - \frac{1}{1+\varphi}\left\{\frac{\partial \boldsymbol{u}^*}{\partial \zeta}\cdot \boldsymbol{w}_\varphi^*\right\} = \operatorname{div}_\varphi \boldsymbol{u}^* = 0.$$

It will be useful to change (2.17) observing that if $\boldsymbol{u}$ and $\boldsymbol{h}$ satisfy (1.2), (1.3), (1.6), the following relations hold:

$$\int_{\Omega_{\varphi,t}} (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot\boldsymbol{h}\,d\Omega = \int_{\Omega_{\varphi,t}} (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{h}\cdot\boldsymbol{u}\,d\Omega + \int_{\Gamma_{1,\varphi,t}} (\boldsymbol{u}\cdot\boldsymbol{\nu})(\boldsymbol{u}\cdot\boldsymbol{h})\,d\Gamma_1,$$

$$(\boldsymbol{u}\cdot\boldsymbol{\nu})_{\Gamma_{1,\varphi,t}} = (\{u_1\varphi_x + u_2\varphi_y - u_3\}\{1+\varphi_x^2+\varphi_y^2\}^{-1/2})_{\Gamma_{1,\varphi,t}} = -\frac{\partial\varphi}{\partial t}\{1+\varphi_x^2+\varphi_y^2\}^{-1/2},$$

$$d\Gamma_1 = (1+\varphi_x^2+\varphi_y^2)^{1/2}\,dA = (1+\varphi_\xi^2+\varphi_\eta^2)^{1/2}\,dA = (1+\varphi_\xi^2+\varphi_\eta^2)^{1/2}\,d\Gamma_1^*,$$

$$\int_{\Omega_{\varphi,t}} (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot\boldsymbol{h}\,d\Omega = -\int_{\Omega_{\varphi,t}} (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{h}\cdot\boldsymbol{u}\,d\Omega - \int_{\Gamma_{1,\varphi,t}} (1+\varphi_x^2+\varphi_y^2)^{-1/2}\,\varphi_t\,(\boldsymbol{u}\cdot\boldsymbol{h})\,d\Gamma_1\,,$$

and, by (2.19),

$$b_\varphi(\boldsymbol{u}^*,\boldsymbol{w}^*,\boldsymbol{h}^*) = -\,b_\varphi(\boldsymbol{u}^*,\boldsymbol{h}^*,\boldsymbol{w}^*) + (g_\varphi^*\,\boldsymbol{u}^*,\boldsymbol{h}^*)_{L^2(\Gamma_1^*)}\,, \tag{2.21}$$

(where $g_\varphi = -\,\varphi_\tau(\xi,\eta,\tau)$)

$$(\boldsymbol{u}^*,\boldsymbol{v}^*)_{L^2(\Gamma_1^*)} = \int_{\Gamma_1^*} (\boldsymbol{u}^*\cdot\boldsymbol{v}^*)\,d\Gamma_1^*\,.$$

Therefore we can change (2.17) into the following

$$\int_0^T \{(\boldsymbol{u}^{*\prime}(\tau),\boldsymbol{h}^*(\tau))_{L^2_\varphi(\Omega^*)} + \mu a((\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau)) + b_\varphi(\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau),\boldsymbol{u}^*(\tau)) + \tag{2.22}$$

$$+\,(\boldsymbol{f}^*(\tau),\boldsymbol{h}^*(\tau))_{L^2_\varphi(\Omega^*)} - \mu(\rho_\varphi^*(\tau)\,\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau))_{L^2(\Gamma_1^*)} - (g_\varphi(\tau)\,\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau))_{L^2(\Gamma_1^*)}\}\,d\tau = 0\,.$$

3. - Some functional spaces

In the following paragraphs we shall denote by x, y, z, t the new set of variables and drop the star symbol used in §2.

Since the fluid is incompressible, its volume $\int_A \varphi(x,y,t)\,dA$ is constant; moreover, setting $z=\zeta=-1$ at the bottom of the basin, we can suppose

$$\int_A \varphi(x,y,t)\,dA = 0\,, \qquad \forall t\in(0,T)\,.$$

In order to take in account this fact, we consider the space $\mathcal{N} = \left\{\psi\in C^\infty(A), \int_A \psi\,dA = 0\right\}$ and indicate by $\mathcal{N}_0$ the closure of $\mathcal{N}$ in $H^1(A)$.

We observe first that, directly from the definition of the space $\mathcal{N}_0$, we have

$$\int_A \psi\,dA = 0\,, \qquad \forall\psi\in\mathcal{N}_0\,. \tag{3.1}$$

Considering in fact the sequence $\{\varphi_n\}$, $\varphi_n\in\mathcal{N}$, with

$$\|\varphi_n-\psi\| \xrightarrow[H^1(A)]{} 0\,, \tag{3.2}$$

we have

$$\left|\int_A \psi\,dA\right| = \left|\int_A \psi\,dA - \int_A \varphi_n\,dA\right| = |(\psi-\varphi_n,1)_{L^2(A)}| \leq \sqrt{m(A)}\,\|\psi-\varphi_n\|_{L^2}$$

and (3.2) implies (3.1).

Now we can prove the following property: *setting obviously*

$$\|\psi\|^2_{H^1(A)} = \int_A (\psi^2 + \psi_x^2 + \psi_y^2)\, dA = \|\psi\|^2_{L^2} + \|\psi\|^2_{H_0^1(A)} \tag{3.3}$$

there exists a constant $C(A)$ *such that*

$$\|\psi\|_{H^1(A)} \leqslant C(A)\|\psi\|_{H_0^1(A)}, \qquad \forall \psi \in \mathcal{N}_0. \tag{3.4}$$

Let us prove (3.4) first for $\psi \in \mathcal{N}$, $\psi \not\equiv 0$: it is

$$\int_A \psi\, dA = 0, \qquad \psi \in C^\infty(A),$$

and moreover $\psi < 0$ if $(x,y) \in A^-$, $\psi > 0$ if $(x,y) \in A^+$; from (3.3) we have then

$$\|\psi\|_{H^1} \leqslant (\|\psi\|_{H_0^1} + \|\psi\|_{L^2}) \leqslant \|\psi\|_{H_0^1} + \left(\int_{A^+} \psi^2\, dA + \int_{A^-} \psi^2\, dA \right)^{1/2} \leqslant$$

$$\leqslant \|\psi\|_{H_0^1} + (\|\psi\|^2_{L^2(A^+)} + \|\psi\|^2_{L^2(A^-)})^{1/2}.$$

Since $\psi \in H_0^1(A^+) \cap H_0^1(A^-)$, there exists a constant C_{1A} such that

$$\|\psi\|^2_{L^2(A^+)} \leqslant C_{1A}\|\psi\|^2_{H_0^1(A^+)}; \qquad \|\psi\|^2_{L^2(A^-)} \leqslant C_{1A}\|\psi\|^2_{H_0^1(A^-)},$$

and then

$$\|\psi\|_{H^1} \leqslant \|\psi\|_{H_0^1} + C_{1A}(\|\psi\|^2_{H_0^1(A^+)} + \|\psi\|^2_{H_0^1(A^-)})^{1/2} \leqslant \tag{3.5}$$

$$\leqslant \|\psi\|_{H_0^1} + C_{1A}\|\psi\|_{H_0^1} = (1 + C_{1A})\|\psi\|_{H_0^1}.$$

Let us suppose now $\psi \in \mathcal{N}_0$ and a consider a sequence $\{\psi_n\}$ such that $\psi_n \in \mathcal{N}$, and

$$\lim_{n \to \infty} \|\psi_n - \psi\|_{H^1(A)} = 0.$$

By (3.5) we have

$$\|\psi\|_{H^1} \leqslant \|\psi_n - \psi\|_{H^1} + \|\psi_n\|_{H^1} \leqslant \|\psi_n - \psi\|_{H^1} + (1 + C_{1A})\|\psi_n\|_{H_0^1} \leqslant$$

$$\leqslant \|\psi_n - \psi\|_{H^1} + (1 + C_{1A})(\|\psi_n - \psi\|_{H_0^1} + \|\psi\|_{H_0^1})$$

and finally

$$\|\psi\|_{H^1} \leqslant (1 + C_{1A})\|\psi\|_{H_0^1} + (2 + C_{1A})\|\psi_n - \psi\|_{H^1}. \tag{3.6}$$

From this inequality it follows first that

$$\|\psi\|_{H_0^1} = 0 \Rightarrow \|\psi\|_{H^1} = 0,$$

then, if $\|\psi\|_{H_0^1} \neq 0$, there exists $\bar{n}$ such that $\|\psi_n - \psi\|_{H^1} < \|\psi\|_{H_0^1}$, $\forall n > \bar{n}$. Finally (3.4) follows from (3.6) setting $C(A) = 3 + 2C_{1A}$.

It will be useful to introduce also the following spaces.

Let us indicate by $\mathcal{M}$ the space of functions $v \in C^\infty(\overline{\Omega})$:

$$\begin{cases} v(x,y,-1)=0 & (x,y)\in A\,, \\ v(x,y,z)\cdot \boldsymbol{n}=0 & (x,y,z)\in \Gamma_3\,, \end{cases}$$

and by V^s the closure of $\mathcal{M}$ in $H_0^s(\Omega)$ (s integer ≥ 0).

Moreover, if $\psi(x,y)\in C^1(\overline{A})$, let us indicate by V_ψ^s the space of

$$v\in V^s:\ \mathrm{div}_\psi v=0\,.$$

Setting

$$(u,v)_{V_\psi^s}=(u,v)_{H^s(\Omega)}\,,$$

V_ψ^s are Hilbert spaces, the embedding of V_ψ^{s+1} in V_ψ^s being moreover completely continuous.

By the usual interpolation procedure, between Hilbert spaces, it is then possible to define V_ψ^σ $\forall$ real $\sigma\geq 0$ in such a way that

$$[V_\psi^\alpha, V_\psi^\beta]=V_\psi^{\alpha(1-\theta)+\beta\theta} \qquad \alpha,\beta\geq 0,\quad 0\leq\theta\leq 1\,, \tag{3.7}$$

and $V_\psi^{\sigma_1}$ is dense and has compact embedding in $V_\psi^{\sigma_2}$ if $\sigma_1>\sigma_2\geq 0$. Moreover, we shall denote by $V_\psi^{-\sigma}$ the dual space of V_ψ^σ, so that (3.7) holds $\forall$ real α,β.

In what follows, we shall have to consider the case in which the functions ψ and ρ depend also on the parameter t; we shall therefore denote by $V_{\psi,\bar t}^\sigma$ the space corresponding to the function $\psi(x,y,\bar t)$. For the sake of simplicity, anyhow, wherever there may not be any possibility of confusion, we shall set $V_\psi=V_{\psi,t}^1$, $V_\psi'=(V_{\psi,t}^1)'$ and denote, $\forall$ fixed t, the duality between V_ψ' and V_ψ by the notation $\langle\ ,\ \rangle$; finally, $\forall v\in V_\psi$ and $w\in V_\psi'$, we shall set $\langle w,v\rangle_\psi=\langle w,v(1+\psi)\rangle$.

4. - Associated variational inequality

As in [3], in order to take into account the consistency conditions (1.10) and (1.11) we shall denote by K_1, K_2 the following closed sets

$$K_1=\{v\in H^1(\Omega):\ |v|\leq M_2 \text{ a.e. in } \Omega\}\,,$$

$$K_2=\{\psi\in H^1(\Lambda):\ |\psi_{xx}|\leq M_1,\ |\psi_{yy}|\leq M_1,\ |\psi_{xy}|\leq M_1,$$

$$|\psi_{xt}|\leq M_1,\ |\psi_{yt}|\leq M_1,\ |\psi_{tt}|\leq M_1,\ \ \psi\geq -1+\sigma,\ \text{ a.e., in } \Lambda\}\,,$$

M_1, M_2, σ being the constants appearing in (1.10), (1.11), and we shall consider the following system constituted by variational inequality associated respectively to (2.22)

and (1.3):

$$(4.1)\qquad \frac{1}{2}\|\boldsymbol{u}(t)-\boldsymbol{h}(t)\|_{V_\varphi^0}-\frac{1}{2}\|\bar{\boldsymbol{u}}-\boldsymbol{h}(0)\|_{V_\varphi^0}=$$

$$=\int_0^t\Big\{\langle \boldsymbol{h}',\boldsymbol{u}-\boldsymbol{h}\rangle_\varphi+\mu a_\varphi(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})-b_\varphi(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h},\boldsymbol{u})+\frac{1}{2}(d_\varphi(\boldsymbol{u}-\boldsymbol{h}),\boldsymbol{u}-\boldsymbol{h})_{L^2_\varphi(\Omega)}+$$

$$+(g_\varphi\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})_{L^2(\Gamma_1)}-\mu(\rho_\varphi\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})_{L^2(\Gamma_3)}-\langle \boldsymbol{f},\boldsymbol{u}-\boldsymbol{h}\rangle_\varphi\Big\}\,d\eta\leq 0,$$

$$(4.2)\qquad \int_{\Lambda_t}\{\varphi_\eta+u_1(x,y,\eta)\,\varphi_x+u_2(x,y,\eta)\,\varphi_y-u_3(x,y,\eta)\}(\varphi-l)\,d\Lambda\leq 0,$$

where in (4.1) we have set

$$d_\varphi=\frac{\varphi'}{\varphi+1},$$

and in (4.2) we have indicated by $u_i(x,y,\eta)$ the trace of $u_i(x,y,z,\eta)$ on the surface $z=0$.

We shall then say that $\{\boldsymbol{u},\varphi\}$ is a solution in $[0,T]$ of (4.1), (4.2) satisfying the initial condition if

i) $\{\boldsymbol{u},\varphi\}\in L^2(0,T;V_\varphi\cap K_1)\times L^2(0,T;\mathcal{N}_0(A))\cap K_2$,

ii) $\varphi(0)=\bar{\varphi}$,

iii) (4.1) holds a.e. in $0\text{—}T$ and

$$\forall \boldsymbol{h}(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^1(0,T;V_\varphi'),$$

(4.2) holds $\forall t\in[0,T]$, $\forall l\in K_2$.

It is well known (cfr. [3]) that if $\{\boldsymbol{u},\varphi\}$ is solution of system (4.1), (4.2) it is also solution of the physical problem considered, provided the Navier-Stokes model is physically consistent.

5. - The existence theorem

According to what has been shown in § 4, the Navier-Stokes model is associated with the function $\Phi=\{\boldsymbol{u},\varphi\}$ which satisfies conditions i), ii), iii). In the following §§ we shall prove first existence uniqueness and continuous dependence Theorems for (4.1), whith fixed φ, an analogous Theorems for (4.2) with fixed $\boldsymbol{u}$.

Those Theorems allow to prove an existence Theorem given for (4.1), (4.2) in the sense in § 4.

Precisely we shall prove the following Theorems *a*) for (4.1) and Theorems *b*) for (4.2).

THEOREM a_1): *Assume that*

$$\varphi \in K_2; \qquad \bar{u} \in V^0_{\varphi,0} \cap K_1;$$

$$\rho \in L^\infty(\Gamma_3 \times (0,T)), \qquad f(t) \in L^2(0,T;V'_\varphi).$$

There exists then a unique function $u(t) \in L^2(0,T;V_\varphi \cap K_1) \cap H^s(0,T;V^{s-2}_\varphi)$ $(s<1/2)$ $(\Rightarrow \in H^\sigma(0,T;V^{1/2+\sigma}_\varphi), (\sigma<1/12))$, *which satisfies, a.e. in* $(0,T)$, *the inequality* (4.1), $\forall h(t) \in L^2(0,T;V_\varphi \cap K_1) \cap H^1(0,T;V'_\varphi)$.

THEOREM a_2): *Let* $\{\varphi_n\}$ *be a sequence of functions* $\in K_2$ *and such that* $\varphi_n \to \varphi$ *strongly in* $C^1(\bar{\Lambda})$ *and let* $\{u_n\}$ *be the sequence of the corresponding solutions of* (4.1) *whose existence and uniqueness has been proved in Theorem* a_1). *Then denoting by* u *the solution corresponding to* φ *(which obviously* $\in K_2$), *we have*

$$\lim_{n\to\infty} u_n(t) = u(t) \tag{5.1}$$

in the weak topology of $L^2(0,T;H^1(\Omega))$, *the weak* topology of* $L^\infty(Q)$ *and the strong topology of* $H^\sigma(0,T;H^{\sigma+1/2}(\Omega))$ $(\sigma<1/12)$ *and of* $L^4(Q)$.

THEOREM b_1): *Assume that* $\omega \in L^\infty(\Gamma_1 \times (0,T))$, $\bar{\varphi} \in \mathcal{N}_0(A)$,

$$|\bar{\varphi}_{xx}| \leq M_1, \qquad |\bar{\varphi}_{yy}| \leq M_1, \qquad \bar{\varphi} \geq -1+\sigma \ (\sigma>0).$$

There exists then a unique function $\varphi(t) \in L^2(0,T;\mathcal{N}_0(A)) \cap K_2$ *such that* $\varphi(0) = \bar{\varphi}$ *and satisfying,* $\forall l \in K_2$ *and* $t \in [0,T]$, *the inequality* (4.2).

THEOREM b_2): *Let* $\{u_n\}$ *be a sequence of functions* $\in L^\infty(\Gamma_1 \times (0,T))$ *such that* $u_n \to u$ *strongly in* $L^2(\Gamma_1 \times (0,T))$ *and let* $\{\varphi_n\}$ *be the sequence of the corresponding solutions of* (4.2), *whose existence and uniqueness has been proved in Theorem* b_1).

Then, denoting by φ *the solution corresponding to* u *it is*

$$\lim_{n\to\infty} \varphi_n = \varphi \tag{5.2}$$

in the strong topology of $C_1(\bar{\Lambda})$.

COROLLARY b_3: *Assume that* $u \in L^2(\Gamma_1 \times (0,T))$, $\bar{\varphi} \in \mathcal{N}_0(A)$,

$$|\bar{\varphi}_{xx}| \leq M_1, \qquad |\bar{\varphi}_{yy}| \leq M_1, \qquad \bar{\varphi} \geq -1+\sigma \ (\sigma>0).$$

There exists then a unique function $\varphi(t) \in L^2(0,T;\mathcal{N}_0(A)) \cap K_2$ *such that* $\varphi(0) = \bar{\varphi}$ *and satisfying,* $\forall l \in K_2$ *and* $t \in [0,T]$, *the inequality* (4.2).

From Theorems *a*) and *b*) we can deduce the following

THEOREM 1: *Assume that*

$$f(t) \in L^2(0,T;V'_\varphi), \qquad \rho \in L^\infty(\Gamma_3 \times (0,T)), \qquad \bar{\varphi} \in \mathcal{N}_0(A);$$

$$|\bar{\varphi}_{xx}| \leq M_1, \qquad |\bar{\varphi}_{yy}| \leq M_1, \qquad \bar{\varphi} \geq -1+\sigma \ (\sigma>0), \qquad \bar{u} \in K_1 \cap V^0_{\bar{\varphi}}.$$

There exists then $\Phi = \{u, \varphi\}$ *satisfying* i), ii), iii).

6. - Some auxiliary lemmas

We shall state the following Lemmas in before the proof of Theorems *a*). The proofs of Lemmas and theorems is quite analogous to the proofs in [3]. We shall any way give them for the reader's convenience.

Lemma 1: *Assume that*

$$\rho \in L^\infty(\Gamma_3 \times (0,T)), \qquad \varphi \in K_2, \qquad \bar{u} \in V^0_{\bar{\varphi},0},$$

$$f(t) \in L^2(0,T;V'_\varphi) \qquad (\Rightarrow g_\varphi \in L^\infty(\Gamma_1 \times (0,T))).$$

Then there exists a unique function

$$u(t) \in L^2(0,T;V_\varphi) \cap H^1(0;T;V'_\varphi) \cap C^0(0,T;L^2(\Omega))$$

such that $u(0) = \bar{u}$ *and*

$$(6.1) \qquad \int_0^t \{\langle u', h\rangle_\varphi + \mu a_\varphi(u,h) + (g_\varphi u, h)_{L^2(\Gamma_1)} - \mu(\rho_\varphi u, h)_{L^2(\Gamma_3)} - \langle f, h\rangle_\varphi\}\, d\Omega = 0$$

$\forall h(t) \in L^2(0,T;V_\varphi)$ *and* $\forall t \in [0,T]$.

Observe that, by a well known existence theorem for abstract differential equation (see, for instance, [4], ch. 3, Th. 1.1), we can prove that there exists $u(t) \in L^2(0,T;V_\varphi)$ satisfying the equation

$$(6.2) \qquad \int_0^T \{-(u,h')_{L^2(\Omega)} + \mu a_\varphi(u,h) - (d_\varphi u, h)_{L^2(\Omega)} - \mu(\rho_\varphi u, h)_{L^2(\Gamma_3)} + \\ + (g_\varphi u, h)_{L^2(\Gamma_1)} - \langle f, h\rangle_\varphi\}\, dt - (u, h(0))_{L^2(\Gamma_3)} = 0,$$

$\forall h(t) \in L^2(0,T;V_\varphi) \cap H^1(0,T;L^2(\Omega))$, $h(T) = 0$.

In fact, by (2.18) and classical embedding and trace theorems, there exist three positive constants, $\lambda, \alpha_1, \alpha_2$, such that

$$(6.3) \qquad \alpha_1 \|v\|^2_{V_\varphi} \geqslant \mu a_\varphi(v,v) - (d_\varphi v, v)_{L^2_\varphi(\Omega)} - (\rho_\varphi v, v)_{L^2(\Gamma_3)} + \\ + (g_\varphi v, v)_{L^2(\Gamma_1)} + \lambda \|v\|^2_{L^2_\varphi(\Omega)} \geqslant \alpha_2 \|v\|^2_{V_\varphi}, \qquad \forall v \in V_\varphi.$$

On the other hand, by (6.2)

$$(6.4) \qquad \int_0^T \langle u', h\rangle_\varphi\, dt = [(u,h)_{L^2_\varphi(\Omega)}]_0^T - \int_0^T \langle u, h\rangle_\varphi\, dt + \int_0^T (d_\varphi u, h)_{L^2_\varphi(\Omega)} = \\ = \int_0^T \{-\mu a_\varphi(u,h) + \mu(\rho_\varphi u, h)_{L^2(\Gamma_3)} - (g_\varphi u, h)_{L^2(\Gamma_1)} + \langle f, h\rangle_\varphi\}\, dt$$

$\forall h(t) \in L^2(0,T;V_\varphi) \cap H_0^1(0,T;L^2(\Omega))$ and

$$\left| \int_0^T \{-\mu a_\varphi(u,h) - (g_\varphi u,h)_{L^2(\Gamma_1)} + \mu(\rho_\varphi u,h)_{L^2(\Gamma_3)} + \langle f,h\rangle_\varphi\}\, dt \right| \leq c\|h\|_{L^2(0,T;V_\varphi)}$$

$\forall h \in L^2(0,T;V_\varphi)$.

Hence

$$\int_0^T \{-\mu a(u,h) - (g_\varphi u,h)_{L^2(\Gamma_1)} + \mu(\rho_\varphi u,h)_{L^2(\Gamma_3)} + \langle f,h\rangle_\varphi\} = \int_0^T \langle Bu,h\rangle_\varphi\, dt$$

with $Bu \in L^2(0,T;V_\varphi')$.

It follows therefore from (6.4) that $u' = Bu$ in $L^2(0,T;V_\varphi') \cap H^{-1}(0,T;L^2(\Omega))$ and consequently, $u(t) \in H^1(0,T;V_\varphi')$.

The property that $u(t) \in C^0(0,T;L^2(\Omega))$ can then be proved in a classical way (see, for instance [5]).

It is then obvious by (6.2), that $u(t)$ satisfies (6.1).

The uniqueness of the solution can be deduced directly from (6.1) setting $h = u_1 - u_2$, with u_1, u_2 solutions of (6.1)

Lemma 2: *Suppose that the assumptions made in Lemma* 1 *are verified and let* P *be the operator defined by*

$$Pv = v \quad \text{when } |v| \leq M_2, \qquad Pv = M_2 \frac{v}{|v|} \quad \text{when } |v| > M_2.$$

There exists then, $\forall \varepsilon > 0$, *a unique function*

$$u(t) \in L^2(0,T;V_\varphi) \cap H^1(0,T;V_\varphi') \cap C^0(0,T;L^2(\Omega))$$

such that $u(0) = \bar{u}$ *and*

$$(6.5) \qquad \int_0^t \Big\{ \langle u',h\rangle_\varphi + \mu a_\varphi(u,h) + (g_\varphi u,h)_{L^2(\Gamma_1)} + \\ - \mu(\rho_\varphi u,h)_{L^2(\Gamma_3)} + \frac{1}{\varepsilon}(u - Pu,h)_{L^2_\varphi(\Omega)} - \langle f,h\rangle_\varphi \Big\}\, d\eta,$$

$\forall h(t) \in L^2(0,T;V_\varphi)$ *and* $\forall t \in [0,T]$.

Consider, in fact, the transformation, from $H^{1/4}(0,T;V_{\varphi,t}^{1/4})$ to itself, $v \to u =$

$= S_{\varepsilon,\lambda}(v)$, where u is the solution given by Lemma 1, of the equation

$$(6.6)\qquad \int_0^t \Big\{ \langle u', h\rangle_\varphi + \mu a_\varphi(u,h) +$$

$$+ \lambda \Big[(g_\varphi u, h)_{L^2(\Gamma_1)} - \mu(\rho_\varphi u, h)_{L^2(\Gamma_3)} + \frac{1}{\varepsilon}(v - Pv, h)_{L^2_\varphi(\Omega)} - \langle f, h\rangle_\varphi \Big] \Big\} d\eta = 0,$$

with $u(0) = \lambda \bar{u}$, λ being a real parameter $\in [0,1]$.

Let u be an eventual solution of $u = S_{\varepsilon,\lambda}(u)$, *i.e.* such that

$$\int_0^T \Big\{ \langle u', h\rangle_\varphi + \mu a_\varphi(u,h) +$$

$$+ \lambda \Big[(g_\varphi u, h)_{L^2(\Gamma_1)} - \mu(\rho_\varphi u, h)_{L^2(\Gamma_3)} + \frac{1}{\varepsilon}(u - Pu, h)_{L^2_\varphi(\Omega)} - \langle f, h\rangle_\varphi \Big] \Big\} d\eta = 0,$$

with $u(0) = \lambda u$. Setting $h = u$, we obtain, bearing in mind (6.3),

$$(6.7)\qquad \|u\|_{C^0(0,T;L^2(\Omega)) \cap L^2(0,T;V'_\varphi) \cap H^1(0,T;V'_\varphi)} \leq N_1,$$

with N_1 independent of $\lambda \in [0,1]$.

Denoting, moreover, by $\{v_n\}$ a sequence such that $v_n \to v$ in the weak topology of $H^{1/4}(0,T;V^{1/4}_{\varphi,t})$, let $\{u_n\}$ be the sequence defined by $u_n = S_{\varepsilon,\lambda}(v)$. We have analogously to (6.7),

$$\|u_n\|_{C^0(0,T;L^2(\Omega)) \cap L^2(0,T;V_\varphi) \cap H^1(0,T;V'_\varphi)} \leq N_1, \qquad \text{(independent of } n)$$

in the weak topology of $L^2(0,T;V_\varphi) \cap H^1(0,T;V'_\varphi)$, the weak* topology of $L^\infty(0,T;L^2(\Omega))$ and, since the embedding of $L^2(0,T;V_\varphi) \cap H^1(0,T;V'_\varphi)$ into $H^{1/4}(0,T;V^{1/4}_{\varphi,t})$ is completely continuous, in the strong topology of $H^{1/4}(0,T;V^{1/4}_{\varphi,t})$.

Writing (6.6) for the functions u_n, v_n and letting $n \to \infty$, we then obtain that the limit functions u, v satisfy (6.6); by the uniqueness theorem proved in Lemma 1, the whole sequence $\{u_n\}$ must then necessarily converge to u. It follows that $S_{\varepsilon,\lambda}$ is, $\forall \varepsilon > 0$ and $\forall \lambda \in [0,1]$ completely continuous, while the eventual fixed points satisfy (6.7) and, obviously, $S_{\varepsilon,0}(v) = 0$.

By the Leray-Schauder fixed point theorem, there exists then $u^* = S_{\varepsilon,0}(u^*)$; the function u^* thus determined is obviously a solution of (6.5).

The uniqueness of this solution can be proved as in Lemma 1, observing that the operator $I - P$ is monotone.

LEMMA 3: *Let u_ε be the solution given by Lemma 2. Denoting by N_3, N_4 quantities independent of ε, we have, $\forall\varepsilon>0$, $s<1/2$*

$$\|u_\varepsilon\|_{L^2(0,T;V_\varphi)\cap L^\infty(0,T;L^2(\Omega))}\leq N_3,\qquad \|u_\varepsilon\|_{H^s(0,T;V_\varphi^{s-2})}\leq N_4. \tag{6.8}$$

Setting, in fact, in (6.5), $u=h=u_\varepsilon$, we obtain

$$\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega)}-\frac{1}{2}\|\bar{u}\|^2_{L^2(\Omega)}+$$

$$+\int_0^t\Big\{\mu a_\varphi(u_\varepsilon,u_\varepsilon)-(d_\varphi u_\varepsilon,u_\varepsilon)_{L^2_\varphi(\Omega)}-\mu(\rho_\varphi u_\varepsilon,u_\varepsilon)_{L^2(\Gamma_3)}+(g_\varphi u_\varepsilon,u_\varepsilon)_{L^2(\Gamma_1)}+$$

$$+\frac{1}{\varepsilon}(u_\varepsilon-Pu_\varepsilon,u_\varepsilon)_{L^2_\varphi(\Omega)}-\langle f,u_\varepsilon\rangle_\varphi\Big\}\,d\eta=0,$$

from which the first of (6.8) follows directly. Moreover,

$$\frac{1}{\varepsilon}\int_0^T(u_\varepsilon-Pu_\varepsilon,u_\varepsilon)_{L^2_\varphi(\Omega)}\,dt\leq N_5. \tag{6.9}$$

On the other hand, by (6.9) and the definition of P,

$$\frac{M_2}{\varepsilon}-\int_Q|u_\varepsilon-Pu_\varepsilon|\,dQ\leq\frac{1}{\varepsilon}\int_Q|u_\varepsilon-Pu_\varepsilon|\,|u_\varepsilon|\,dQ=\frac{1}{\varepsilon}\int_0^T(u_\varepsilon-Pu_\varepsilon,u_\varepsilon)_{L^2(\Omega)}\,dt\leq N_5.$$

Hence, by (6.5) and (6.8) and well known embedding theorems,

$$\left|\int_0^T\langle u_\varepsilon',h\rangle_\varphi\,dt\right|\leq N_6\|h\|_{H^{1-s}(0,T;V_\varphi^{2-s})}$$

and, consequently, $\|u_\varepsilon'\|_{H^{s-1}(0,T;V_\varphi^{s-2})}\leq N_6$, *i.e.* the second of (6.8).

LEMMA 4: *Assume that $\varphi\in K_2$, $\bar{u}\in V^0_{\varphi,0}\cap K_1$, $f(t)\in L^2(0,T;V'_\varphi)$. There exists then a unique function $u(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^s(0,T;V_\varphi^{s-2})$ $(s<1/2)$ which satisfies,*

$\forall h(t) \in L^2(0,T;V_\varphi \cap K_1) \cap H^1(0,T;V'_\varphi)$, *a.e. on* $(0,T)$ *the inequality*

$$\frac{1}{2}\|u(t)-h(t)\|_{V^0_\varphi} - \frac{1}{2}\|\bar{u}-h(0)\|_{V^0_\varphi} +$$

$$+\int_0^t \Big\{ \langle h', u-h\rangle_\varphi + \mu a_\varphi(u,u-h) - \frac{1}{2}(d_\varphi(u-h),u-h)_{L^2_\varphi(\Omega)} +$$

$$+ (g_\varphi u, u-h)_{L^2(\Gamma_1)} - \mu(\rho_\varphi u, u-h)_{L^2(\Gamma_3)} - \langle f, u-h\rangle_\varphi \Big\} d\eta \leq 0 .$$

The existence of the solution can be proved by a standard procedure (see, for instance [5], ch 3, Th. 6.2) utilizing the results contained in Lemmas 2 and 3.

The uniqueness is obtained by a standard procedure in the theory of parabolic inequalities (see, for instance [6]).

7. - Proof of Theorems *a*)

Theorem a_1): We shall divide the proof in three parts.

a) Let $v \in L^4(Q)$ and G_δ be a smoothing operator such that $G_\delta v \in H^{1,4}(Q)$, $G_\delta v \xrightarrow[\delta\to 0]{} v$ strongly in $L^4(Q)$. We begin to prove that there exists, $\forall \delta > 0$, a function

$$u_\delta \in L^2(0,T;V_\varphi \cap K_1) \cap H^s(0,T;V^{s-2}_\varphi),$$

which satisfies a.e. in $(0,T)$, the inequality

$$(7.1) \qquad \frac{1}{2}\|u_\delta(t)-h(t)\|^2_{V^0_\varphi} - \frac{1}{2}\|\bar{u}-h(0)\|^2_{V^0_\varphi} + \int_0^t \Big\{ \langle h', u_\delta - h\rangle_\varphi +$$

$$+ \mu a_\varphi(u_\delta, u_\delta - h) - \frac{1}{2}(d_\varphi(u_\delta - h), u_\delta - h)_{L^2_\varphi(\Omega)} + b_\varphi(G_\delta u_\delta, u_\delta - h, G_\delta u_\delta) +$$

$$+ (g_\varphi u_\delta, u_\delta - h)_{L^2(\Gamma_1)} - \mu(\rho_\varphi u_\delta, u_\delta - h)_{L^2(\Gamma_3)} - \langle f, u_\delta - h\rangle_\varphi \Big\} d\eta \leq 0,$$

$\forall h(t) \in L^2(0,T;V_\varphi \cap K_1) \cap H^1(0,T;V'_\varphi)$.

Consider, in fact, $\forall$ fixed δ, the transformation $v \to S_\delta v = u$ from $L^4(Q)$ to itself,

where u is the solution of the inequality

$$(7.2)\qquad \frac{1}{2}\|u(t)-h(t)\|^2_{V^0_\varphi}-\frac{1}{2}\|\bar{u}-h(0)\|^2_{V^0_\varphi}+\int_0^t\Big\{\langle h',u-h\rangle_\varphi+$$

$$-\frac{1}{2}(d_\varphi(u-h),u-h)_{L^2_\varphi(\Omega)}+b_\varphi(G_\delta v,u-h,G_\delta v)+(g_\varphi u,u-h)_{L^2(\Gamma_1)}+$$

$$-\mu(\rho_\varphi u,u-h)_{L^2(\Gamma_3)}-\langle f,u-h\rangle_\varphi\Big\}\,d\eta\leqslant 0\,.$$

Since

$$\left|\int_0^T b_\varphi(G_\delta v,u-h,G_\delta v)\,dt\right|\leqslant C_1\|G_\delta v\|_{L^4(\Omega)}\|u-h\|_{L^2(0,T;V_\varphi)}\,,$$

it follows from Lemma 4 that such a solution is uniquely determined; moreover

$$u(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^s(0,T;V^{s-2}_\varphi)$$

and satisfies (7.2) $\forall h(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^1(0,T;V'_\varphi)$, while, since $|u|\leqslant M_2$ a.e., S_δ transforms in itself every sphere of $L^4(Q)$ with sufficiently large radius.

Let $\{v_j\}$ be an $L^4(Q)$-weakly convergent sequence; by definition of G_δ,

$$(7.3)\qquad \lim_{j\to\infty} G_\delta v_j=G_\delta v$$

weakly in $H^{1,4}(Q)$ and strongly in $L^4(Q)$. Setting $u_j=S_\delta v_j$, the sequence $\{u_j\}$ is by (7.3) and Lemma 4, uniformly bounded in $L^2(0,T;V_\varphi\cap K_1)\cap H^s(0,T;V^{s-2}_\varphi)$; by well known interpolation, embedding and trace theorems, it is then possible to select from $\{u_j\}$ a subsequence (again denoted by $\{u_j\}$) such that

$$(7.4)\qquad \lim_{j\to\infty} u_j(t)=u(t)$$

in the weak topology of $L^2(0,T;V_\varphi)\cap H^s(0,T;V^{s-2}_\varphi)$, the weak* topology of $L^\infty(Q)$ and the strong topology of $H^\sigma(0,T;V^{\sigma+1/2}_\varphi)$ $(\sigma<1/12)$; indicating by $\gamma_i u(t)$ the trace of u on Γ_i, we have also

$$(7.5)\qquad \lim_{j\to\infty}\gamma_1 u_j(t)=\gamma_1 u(t)\,,$$

in the strong topology of $L^2(0,T;L^2(\Gamma_1))$ and

$$(7.6)\qquad \lim_{j\to\infty}\gamma_3 u_j(t)=\gamma_3 u(t)$$

in the strong topology of $L^2(0,T;L^2(\Gamma_3))$.

Observe moreover that, since $|u_j|$, $|u|\leqslant M_2$ a.e. in Q, it follows from (7.4)

that

$$\lim_{j\to\infty} u_j = u \tag{7.7}$$

in the strong topology of $L^4(Q)$.

Writing (7.2) for u_j, v_j letting $j\to\infty$ and bearing in mind (7.3)-(7.7), it follows then by the semicontinuity of the weak limit, that $u(t)$, $v(t)$ satisfy (7.2). Since however, by Lemma 4, the solution $u(t)$ is unique, we conclude that the whole sequence $\{u_j\}$ converges to u; hence S_δ is, $\forall\delta>0$, completely continuous from $L^4(Q)$ to itself.

By the Tychonof fixed point theorem, there exists then $u_\delta = S_\delta u_\delta$; this function is obviously the solution of (7.1).

b) We now prove that, when $\delta\to 0$, the sequence $\{u_\delta\}$ defined in *a*) converges to a solution of (4.1).

By the same procedure followed in *a*) (bearing in mind that $u_\delta \leq M_2$ a.e.) it can be shown that the sequence $\{u_\delta(t)\}$ is uniformly bounded in $L^2(0,T;V_\varphi\cap K_1)\cap$ $\cap H^s(0,T;V_\varphi^{s-2})$; hence (cfr. (7.5), (7.7))

$$\lim_{\delta\to 0} u_\delta(t) = u(t) \tag{7.8}$$

in the weak topology of $L^2(0,T;V_\varphi)\cap H^s(0,T;V_\varphi^{s-2})$, in the weak* topology of $L^\infty(Q)$ and the strong topology of $H^\sigma(0,T;V_\varphi^{\sigma+1/2})$ $(\sigma<1/12)$ and of $L^4(Q)$; moreover

$$\lim_{\delta\to 0} \gamma_1 u_\delta(t) = \gamma_1 u(t) \tag{7.9}$$

in the strong topology of $L^2(0,T;L^2(\Gamma_1))$, and

$$\lim_{\delta\to 0} \gamma_3 u_\delta(t) = \gamma_3 u(t) \tag{7.10}$$

in the strong topology of $L^2(0,T;L^2(\Gamma_3))$.

Letting $\delta\to 0$ in (7.1) we obtain then, by (7.8), (7.9), (7.10) and the semicontinuity of the weak limit, relation (4.1). The existence of a solution is therefore proved.

c) The uniqueness of the solution can be proved by exactly the same procedure mentioned in Lemma 4.

THEOREM a_2): Observe, to begin with, that, by Theorem a_1) and the assumption that $\varphi_n \in K_2$,

$$\|u_n\|_{L^\infty(0,T;L^\infty(\Omega))\cap L^2(0,T;V_{\varphi_n})\cap H^s(0,T;V_{\varphi_n}^{s-2})} \leq N_6$$

with N_6 independent of n. Hence, since $V_{\varphi_n}\subset H^1(\Omega)$, and

$$L^2(0,T;V_{\varphi_n})\cap H^s(0,T;V_{\varphi_n}^{s-2})\subset H^\sigma(0,T;H^{\sigma+1/2}(\Omega)) \qquad (\sigma<1/12),$$

with embedding constants independent of n, (5.1) is, following the proof given in Theorem a_1) satisfied for an appropriate subsequence of $\{u_n\}$. Moreover, by the as-

sumptions made on $\{\varphi_n\}$,

$$\lim_{n\to\infty} \operatorname{div}_{\varphi_n} u_n = \operatorname{div}_\varphi u \quad \text{weakly in } L^2(Q), \tag{7.11}$$

so that $u(t) \in L^2(0,T; V_\varphi \cap K_1)$.

We must now show that $u(t)$ is a solution of (4.1).

Setting $\overset{\circ}{K}_1 = \{v \in H^1(\Omega): |v| < M_2 \text{ a.e.}\}$, let $h(t)$ be an arbitrary function $\in L^2(0,T; V_{\varphi_n} \cap \overset{\circ}{K}_1)$ and $\{h_n(t)\}$ a sequence such that

$$h_n(t) \in L^2(0,T; V_{\varphi_n} \cap K_1) \cap H^1(0,T; V'_{\varphi_n}),$$

$$\lim_{n\to\infty} h_n(t) = h(t),$$

in the strong topology of $L^2(0,T; H^1(\Omega))$ (cfr. [3]).

The functions u_n satisfy, by definition, the inequality

$$\begin{aligned} \frac{1}{2}\|u_n(t) - h_n(t)\|^2_{V^0_{\varphi_n}} - \frac{1}{2}\|\bar{u} - h_n(0)\|^2_{V^0_{\varphi_n}} + \int_0^t \Big\{ \langle h'_n, u_n - h_n \rangle_{\varphi_n} + \\ + \mu a_{\varphi_n}(u_n, u_n - h_n) - \frac{1}{2}(d_{\varphi_n}(u_n - h_n), u_n - h_n)_{L^2_\varphi(\Omega)} - b_\varphi(u_n, u_n - h_n, u_n) + \\ + (g_{\varphi_n} u_n, u_n - h_n)_{L^2(\Gamma_1)} - \mu(\rho_{\varphi_n} u_n, u_n - h_n)_{L^2(\Gamma_3)} - \langle f, u_n - h_n \rangle_{\varphi_n} \Big\} d\eta \leq 0. \end{aligned} \tag{7.11}$$

Observe now that, by the assumption made, $g_{\varphi_n} \to g_\varphi$, $d_{\varphi_n} \to d_\varphi$ strongly in $C^0(\bar{\Lambda})$ and $\rho_{\varphi_n} \to \rho_\varphi$ strongly in $L^\infty(\bar{\Gamma}_3 \times [0,T])$, moreover

$$\begin{aligned} a_{\varphi_n}(u_n, u_n - h_n) - a_\varphi(u, u - h) = a_{\varphi_n}(u_n, u_n - h_n) + \\ - a_\varphi(u_n, u_n - h_n) + a_\varphi(u_n, u_n - h_n) - a_\varphi(u, u - h), \end{aligned} \tag{7.12}$$

and consequently, by (5.1), and semicontinuity of the weak limit (since $a_\varphi(v,v)$ is equivalent, $\forall v \in V^1$, to $\|v\|^2_{H^1(Q)}$)

$$\lim_{n\to\infty} \int_0^t a_{\varphi_n}(u_n, u_n - h_n)\, d\eta \leq \int_0^t a_\varphi(u, u - h)\, d\eta. \tag{7.13}$$

On the other hand, by (2.21) and (5.1)

$$\begin{aligned} \lim_{n\to\infty} \int_0^t b_{\varphi_n}(u_n, u_n - h_n, u_n)\, d\eta = \lim_{n\to\infty} \int_0^t \Big\{ -b_{\varphi_n}(u_n, h_n, u_n) + \frac{1}{2}(g_{\varphi_n} u_n, u_n)_{L^2(\Gamma_1)} \Big\} d\eta = \\ = \int_0^t \Big\{ -b_\varphi(u, h, u) + \frac{1}{2}(g_\varphi u, u)_{L^2(\Gamma_1)} \Big\} d\eta = \int_0^t b_\varphi(u, u - h, u)\, d\eta. \end{aligned} \tag{7.14}$$

Letting now $n \to \infty$ in (7.11) we have by (5.1), (7.12), (7.13), (7.14),

$$\frac{1}{2}\|u(t)-h(t)\|^2_{V^0_\varphi} - \frac{1}{2}\|\bar{u}-h(0)\|^2_{V^0_\varphi} + \int_0^t \Big\{ \langle h', u-h\rangle_\varphi +$$

$$+\mu a_\varphi(u, u-h) - \frac{1}{2}(d_\varphi(u-h), u-h)_{L^2_\varphi(\Omega)} - b_\varphi(u, u-h, u) +$$

$$+ (g_\varphi u, u-h)_{L^2(\Gamma_1)} - \mu(\rho_\varphi u, u-h)_{L^2(\Gamma_3)} - \langle f, u-h\rangle_\varphi \Big\} d\eta \leq 0,$$

a.e. in $(0,T)$, $\forall h(t) \in L^2(0,T; V_\varphi \cap \overset{\circ}{K}_1) \cap H^1(0,T; V'_\varphi)$.

Since the space of such functions is dense in $L^2(0,T; V_\varphi \cap K_1) \cap H^1(0,T; V'_\varphi)$ we conclude that u is the unique solution of (4.1) corresponding to φ. Hence (5.1) holds for the whole sequence $\{u_n\}$ and the theorem is proved.

THEOREM b_1): Consider, at first, $\forall \varepsilon > 0$, the inequality

$$\int_{\Lambda_t} \left\{ \frac{\partial \varphi}{\partial \eta} - \varepsilon \Delta \varphi + u_1(x,y,\eta)\varphi_x + u_2(x,y,\eta)\varphi_y - u_3(x,y,\eta) \right\} (\varphi - l)\, d\Lambda \leq 0. \tag{7.15}$$

It is well known (see, for instance [6], ch. 6, Th. 6.2) that (7.15) admits, $\forall \varepsilon > 0$, a solution $\varphi_\varepsilon(t) \in L^2(0,T; \mathcal{N}_0(A)) \cap K_2$, $\forall l \in K_2$ and $t \in [0,T]$, such that $\varphi_\varepsilon(0) = \bar{\varphi}$.

Letting $\varepsilon \to 0$ it is then obviously possible, since $\varphi_\varepsilon \in K_2$, to select from $\{\varphi_\varepsilon\}$ a sequence which converges to a solution of (4.2).

The uniqueness of this solution can easily be proved by setting in (4.2), $\varphi = \varphi_1$, $l = \varphi_2$ and $\varphi = \varphi_2$, $l = \varphi_1$ and adding.

THEOREM b_2): Observe that since, by Theorem b_1), $\varphi_n \in K_2$, (5.2) certainly holds for an appropriate subsequence of $\{\varphi_n\}$.

On the other hand

$$\int_{\Lambda_t} \left\{ \frac{\partial \varphi_n}{\partial t} + u_{n1}\varphi_{nx} + u_{n2}\varphi_{ny} - u_{n3} \right\} (\varphi_n - l)\, d\Lambda \leq 0,$$

and, letting $n \to \infty$, it is obvious that φ satisfies (5.2).

Since, however, the solution is unique, the whole sequence must tend to φ. This proves our theorem.

8. - PROOF OF THEOREM 1

Let us consider a function

$$\Phi = \{u, \varphi\} \in H^\sigma(0,T; H^{\sigma+1/2}(\Omega)) \times C^1(\bar{\Lambda}) \cap L^2(0,T; \mathcal{N}_0(A)) \cap K_2,$$

and two transformations $S_{i,\lambda}$ ($i = 1,2$, λ real parameter $\in [0,1]$) defined in the follow-

ing way

$$\Phi^{(1)} = S_{1,\lambda}(\Phi) = \{u^{(1)}, \varphi^{(1)}\} = \{u^{(1)}, \varphi\}, \tag{8.1}$$

where $u^{(1)}$ satisfies (4.1).

We observe that Theorems *a*) guarantee existence, uniqueness and continuous dependence on the data of $u^{(1)}$, and, moreover, that

$$u^{(1)} \in K_1 \cap H^{\sigma}((0,T); H^{\sigma+1/2}(\Omega)).$$

We define now $S_2(\Phi)$ as it follows

$$\Phi^{(2)} = S_2 \Phi^{(1)} = \{u^{(2)}, \varphi^{(2)}\} = \{u^{(1)}, \varphi^{(2)}\}, \tag{8.2}$$

where $\varphi^{(2)}$ is the solution of (4.2) having substituted $u^{(1)}$ to u.

Theorems *b*) guarantee existence, uniqueness and continuous dependence on the data of $\varphi^{(2)}$. Moreover

$$\varphi^{(2)} \in C^1(\bar{\Lambda}) \cap L^2(0,T; \mathcal{N}_0(A)) \cap K_2.$$

We can consider now the transformation

$$\Psi = S(\Phi) = \prod_{i=1,2} S_i(\Phi)$$

and its restriction

$$\psi = \varphi^{(2)} = s(\varphi).$$

By (8.1), (8.2), we have that $\psi \in K_2$. We observe now that K_2 is a compact set in $C^1(\bar{\Lambda})$ then by Shauder theorem there exists a fixed point φ^* such that

$$\varphi^* = s(\varphi^*).$$

Now if we consider a function $\tilde{\Phi} = \{u, \varphi^*\}$ the function

$$\Phi^* = \{u^*, \varphi^*\} = S(\tilde{\Phi})$$

is a fixed point of the transformation S.

By Theorems *a*), and *b*), Φ^* is a solution of the problem in the sense indicated in § 4.

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