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On the Lower Semicontinuity of Certain Integral Functionals (**) (***)

ABSTRACT. — We study the lower semicontinuity of some functionals of the type

$$\int_\Omega f(x,u,Du)\,dx + \int_{\bar\Omega} g(x,u)\,d\mu\,,$$

where $f(x,s,\cdot)$ is a convex non-homogeneous function, and μ is a Radon measure. In a particular case, an integral representation formula for the relaxed functional is also proved.

Sulla semicontinuità inferiore di alcuni funzionali integrali

RIASSUNTO. — Si studia la semicontinuità inferiore di alcuni funzionali del tipo

$$\int_\Omega f(x,u,Du)\,dx + \int_{\bar\Omega} g(x,u)\,d\mu\,,$$

dove $f(x,s,\cdot)$ è una funzione convessa non omogenea, e μ è una misura di Radon. In un caso particolare, si prova anche una formula di rappresentazione per il funzionale rilassato.

0. - INTRODUCTION

In [2], [3] M. Carriero, A. Leaci and E. Pascali have studied the lower semicontinuity of functionals of the following type:

$$\mathcal{F}(u) = \int_\Omega f(x,u,Du)\,dx + \int_{\bar\Omega} g(x,u)\,d\mu\,, \tag{0.1}$$

where $\Omega \subset \boldsymbol{R}^n$ is open and bounded, μ is a non negative Radon measure in $\boldsymbol{R}^n$, and f is a

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Borel function verifying the following coercitivity condition:

$$f(x,s,z) \geqslant \psi(z) \qquad \forall (x,s,z) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n, \tag{0.2}$$

where ψ is a non negative positively p-homogeneous convex function ($p \geqslant 1$).

Under the hypothesis that the first integral in (0.1) is L^p-lower semicontinuous on $H^{1,p}(\Omega)$, in [2], [3] conditions are found on g and μ (depending on f) which ensure the L^p-lower semicontinuity of $\mathcal{F}$. Under some further hypotheses, in the same papers an integral representation formula for the relaxed functional is proved when the lower semicontinuity condition fails.

On the other hand, there is an increasing interest on wider classes of functionals, *i.e.* functionals exhibiting different degrees of homogeneity with respect to (the components of) Du, both in the scalar (see *e.g.* [11], [12]) and in the vector valued case (see *e.g.* [10]). One is then led to consider functionals analogous to (0.1), with f satisfying a condition like (0.2) as well, but allowing ψ to have different degrees of homogeneity in different directions.

The main goal of this paper is to prove a L^1 lower semicontinuity theorem for such non homogeneous functionals. If the degrees of homogeneity are all strictly larger than 1 (*i.e.* f dominates a convex function with superlinear growth) we are also able to prove an integral representation formula for the relaxed of $\mathcal{F}$.

Let us present an example of application of the results of this paper.

(0.3). Example: Let be $\Omega = [0,1]^3 \subset \boldsymbol{R}^3$, and $f(z_1, z_2, z_3) = z_1^+ + |z_2|^2 + (z_3^-)^{3/2}$, ($z_i^+ = \max\{z_i, 0\}$, $z_i^- = \max\{-z_i, 0\}$) and consider the functional

$$\mathcal{F}(u) = \int_\Omega f(Du)\, dx + \int_{\partial\Omega} g(x,u)\, d\mathcal{H}^2,$$

where $\mathcal{H}^2$ is the Hausdorff surface measure and $g(x,s)$ is a Borel function lower semicontinuous with respect to s. Set

$$P_0^i = \partial\Omega \cap \{z_i = 0\}, \qquad P_1^i = \partial\Omega \cap \{z_i = 1\}.$$

Theorem (3.14) below then yields the following conditions on g in order to get L^1 lower semicontinuity of $\mathcal{F}$ on $H^{1,2}(\Omega)$:

on P_0^1	$0 \leqslant g(x,u) - g(x,v) \leqslant (u-v)$	$\forall v \leqslant u$;
on P_1^1	$-(u-v) \leqslant g(x,u) - g(x,v) \leqslant 0$	$\forall v \leqslant u$;
on P_0^3	$g(x,u) - g(x,v) \leqslant 0$	$\forall v \leqslant u$;
on P_1^3	$0 \leqslant g(x,u) - g(x,v)$	$\forall v \leqslant u$;
on $P_0^2 \cup P_1^2$	no condition on g.	

Notice that in the homogeneous case we slightly improve the quoted results: in fact our lower semicontinuity condition agrees with that one of [3], but we obtain L^1 lower semicontinuity, rather than L^p, independently of the homogeneity degree.

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1. - Notation

Let $\Omega \subset \boldsymbol{R}^n$ be open and bounded, and let $\psi_i: \boldsymbol{R}^n \to [0, +\infty[$ $(i = 0, \ldots, k)$ be convex functions such that

$$\psi_i(tz) = t^{p_i}\psi_i(z) \qquad \forall z \in \boldsymbol{R}^n, \quad t \geqslant 0, \quad i = 0, \ldots, k,$$

with $p_0 = 1$ and $p_i > 1$ for $i \geqslant 1$. Set

$$p = \max_{i=1,\ldots k} p_i, \qquad \psi = \sum_{i=1}^{k} \psi_i.$$

(C) Convention: *In the following we shall use the letter φ to indicate one of the functions ψ_0 or ψ. We set also*

$$H(t) = \begin{cases} t^p & \text{if } 0 \leqslant t \leqslant 1 \\ t & \text{if } t > 1. \end{cases} \tag{1.1}$$

2. - Some families of capacities

In this section we shall construct some families of set functions and discuss some of their properties. Our technique is similar to the methods of [2], [3], which in turn are refinements and generalizations of ideas already exploited, *e.g.* in dealing with Plateau's problem with thin obstacles, see [6, Cap. IV]. Therefore, for the sake of brevity, we shall omit most of the proofs, whenever they can be obtained by the proofs of the analogous results quoted everytime without any essential new idea, and we shall confine ourselves to point out the novelty of the statements.

(2.1). Definition: *Let be $c \in \boldsymbol{R}$, $\varepsilon > 0$ and $K \subset \boldsymbol{R}^n$ compact. Set, for $\varphi = \psi_0$ or $\varphi = \psi$ according to convention* (C) *above*

$$\text{(i)} \qquad \sigma_\varepsilon^\varphi(K, c, \Omega) = \inf\left\{ \int_\Omega \left[\varphi(Du) + \frac{|u|}{\varepsilon} \right] dx; \; u \in \operatorname{Lip}(\overline{\Omega}), \; u = c \text{ on } K \cap \overline{\Omega} \right\};$$

if $A \subset \boldsymbol{R}^n$ is open, set

$$\text{(ii)} \qquad \sigma_\varepsilon^\varphi(A, c, \Omega) = \sup\{\sigma_\varepsilon^\varphi\{K, c, \Omega), \; K \text{ compact}, \; K \subset A\};$$

and for general $E \subset \boldsymbol{R}^n$

$$\text{(iii)} \qquad \sigma_\varepsilon^\varphi(E, c, \Omega) = \inf\{\sigma_\varepsilon^\varphi\{A, c, \Omega), \; A \text{ open}, \; E \subset A\}.$$

(2.2). Remark: It can be easily checked, *e.g.* by adapting the arguments

of [4, pg. 671-672], that Definition (2.2)(iii) is coherent with (i) when E is compact.

(2.3). Remark: The set functions $\sigma_\varepsilon^\varphi(\cdot, c, \Omega)$ are clearly non negative and non decreasing; one can verify by standard techniques (see *e.g.* [8]) that they are Choquet capacities.

(2.4). Definition: *We set, for c, Ω, φ as in Definition* (2.1), *and $E \subset \boldsymbol{R}^n$:*

$$\sigma^\varphi(E, c, \Omega) = \lim_{\varepsilon \to 0^+} \sigma_\varepsilon^\varphi(E, c, \Omega) = \sup_{\varepsilon > 0} \sigma_\varepsilon^\varphi(E, c, \Omega).$$

The following lemma extends Lemma 3.2 in [3] and can be proved in essentially the same way.

(2.5). Lemma: *Let Ω_1 and Ω_2 be open subsets of $\boldsymbol{R}^n$, and let $E \subset \Omega_2$ be such that* $\operatorname{dist}(E, \Omega_1 \setminus \Omega_2) > 0$; *then for every $\varepsilon > 0$ and for every $c \in \boldsymbol{R}$ there exists $\gamma_c(\varepsilon) > 0$ such that*

$$\sigma_\varepsilon^\varphi(E, c, \Omega_1) \leq (1 + \gamma_c(\varepsilon))\, \sigma_\varepsilon^\varphi(E, c, \Omega_1 \cap \Omega_2),$$

and moreover $\gamma_c(\varepsilon) \to 0$ as $\varepsilon \to 0$, hence

$$\sigma^\varphi(E, c, \Omega_1) = \sigma^\varphi(E, c, \Omega_1 \cap \Omega_2).$$

Lemma (2.5) allows one to prove the following result (see [6, Cap. IV Teor. 2.7], as well as [2, Prop. 3.5] and [3, Th. 3.4]).

(2.6). Theorem: *Let φ, c, Ω as in Definition* (2.1); *then the set functions $\sigma^\varphi(\cdot, c, \Omega)$ are Borel regular measures.*

(2.7). Remark: The measures $\sigma^\varphi(\cdot, c, \Omega)$ can be viewed as Γ-limits (see [7], in particular Prop. 3.1); in fact, it is possible to prove the following statement (cf. also [2, Lemmas 3.7 and 3.8] and [3, Remark 3.3])

for every compact $K \subset \boldsymbol{R}^n$ such that $\sigma^\varphi(K, c, \Omega) < +\infty$ there exists a sequence $(u_h) \subset \operatorname{Lip}(\overline{\Omega})$ such that $u_h \to 0$ in $L^1(\Omega)$ and

$$\sigma^\varphi(K, c, \Omega) = \lim_{h \to +\infty} \int_\Omega \varphi(Du_h)\, dx.$$

(2.8). Remark: By a careful inspection of the proof of Theorem 3.5 in [3], it can be checked that for any $E \subset \boldsymbol{R}^n$ one has either $\sigma_\varepsilon^\psi(E, c, \Omega) = 0$ or $\sigma_\varepsilon^\psi(E, c, \Omega) = +\infty$.

(2.9). Remark: For each function ψ_i one could define a Borel measure $\sigma^{\psi_i}(\cdot, c, \Omega)$, but in general $\sigma^\psi(\cdot, c, \Omega) > \sum \sigma^{\psi_i}(\cdot, c, \Omega)$. For instance, if $(0,0) \in \Omega \subset \boldsymbol{R}^2$, $E = \{(0,0)\}$,

$\psi_1(z_1, z_2) = |z_1|^2$, $\psi_2(z_1, z_2) = |z_2|^3$, it is easy to compute

$$\sigma^{\psi_1}(E, 1, \Omega) = \sigma^{\psi_2}(E, 1, \Omega) = 0, \qquad \text{but } \sigma^{\psi}(E, 1, \Omega) = +\infty .$$

In this connexion, let us notice that in the semicontinuity theorem we are going to prove, the condition on g (see (3.11) below, and Lemma (3.5), where the functions $\lambda^{\pm}$ are introduced) will be less restrictive in so far as the measures $\sigma^{\psi}(\cdot, \pm 1, \Omega)$ (hence the functions $\lambda^{\pm}$) are large.

(2.10). Remark: Set, for $K \subset \boldsymbol{R}^n$ compact

$$\operatorname{cap}_{\Omega}(K) = \inf\left\{ \int_{\Omega} [|Du|^p + |u|^p]\, dx;\ u \in C^1(\overline{\Omega}),\ u \geqslant 1 \text{ on } K \cap \overline{\Omega} \right\},$$

and extend the set function $\operatorname{cap}_{\Omega}(\cdot)$ first to the open sets and next to every $E \subset \boldsymbol{R}^n$ as in Definition (2.1); then, it is well-known (see *e.g.* [4]) that for every $u \in H^{1,p}(\Omega)$ there exist $\tilde{u} \in H^{1,p}(\Omega)$ such $u = \tilde{u}$ a.e. with respect to the Lebesgue measure, and each $\tilde{u}$ is $\operatorname{cap}_{\Omega}$-quasi continuous (*i.e.* $\forall \delta > 0\ \exists A$ relatively open in Ω such that $\operatorname{cap}_{\Omega}(A) < \delta$ and $\tilde{u}$ restricted to $\overline{\Omega} \setminus A$ is continuous); a function $\tilde{u}$ as above is called a $\operatorname{cap}_{\Omega}$-quasi continuous representative of u and it is determined up to a null capacity set. Henceforth, for every $u \in H^{1,p}(\Omega)$ we shall use the notation $\tilde{u}$ to denote a $\operatorname{cap}_{\Omega}$-quasi continuous representative of u. Since $\operatorname{cap}_{\Omega}(\mathrm{E}) = 0$ implies $\sigma^{\varphi}(E, c, \Omega) = 0$ for every $E \subset \boldsymbol{R}^n$, functions $u \in H^{1,p}(\Omega)$ are determined up to null $\sigma^{\varphi}(\cdot, c, \Omega)$ sets.

In view of Remark (2.10) we can put the following definition (cf. [3, § 4]).

(2.11). Definition: *Let be* $\varepsilon > 0$, φ *as in convention* (C), *and* $w: \boldsymbol{R}^n \to [0, +\infty]$; *for every* $E \subset \boldsymbol{R}^n$ *we set*

$$\tau_{\varepsilon}^{\varphi}(E, w, \Omega) = \inf\left\{ \int_{\Omega} \left[\varphi(Du) + \frac{|u|}{\varepsilon} \right] dx;\ u \in H^{1,p}(\Omega),\ \tilde{u} \geqslant w \text{ in } E \cap \overline{\Omega}\ \sigma^{\varphi}(\cdot, 1, \Omega) - \text{a.e.} \right\};$$

$$\tau_{\varepsilon}^{\varphi}(E, -w, \Omega) =$$

$$= \inf\left\{ \int_{\Omega} \left[\varphi(Du) + \frac{|u|}{\varepsilon} \right] dx;\ u \in H^{1,p}(\Omega),\ \tilde{u} \leqslant -w \text{ in } E \cap \overline{\Omega}\ \sigma^{\varphi}(\cdot, -1, \Omega) - \text{a.e.} \right\}.$$

(2.12). Remark: If $w \equiv c$, arguing as in [3, Prop. 4.2], we can prove that $\forall \varepsilon > 0$ and $\forall E \subset \boldsymbol{R}^n$ the following inequality holds:

$$\sigma_{\varepsilon}^{\varphi}(E, c, \Omega) \leqslant \tau_{\varepsilon}^{\varphi}(E, c, \Omega) . \tag{2.13}$$

(2.14). Lemma: *Let be* $\varepsilon > 0$, $w: \boldsymbol{R}^n \to [0, +\infty]$, $E \subset \boldsymbol{R}^n$, *and* H *given by* (1.1); *set*

$E_s = \{x \in E: w(x) > s\}$, *the following inequalities hold*:

$$\int_0^{+\infty} \tau_\varepsilon^{\psi_0}(E_s, 1, \Omega)\, ds \leqslant \tau_\varepsilon^{\psi_0}(E, w, \Omega), \tag{2.15}$$

$$\int_0^{+\infty} \tau_\varepsilon^{\psi}(E_s, 1, \Omega)\, dH(s) \leqslant c\tau_\varepsilon^{\psi}(E, w, \Omega), \tag{2.16}$$

where the constant c depends only on p.

Proof: Inequality (2.15) is proved in [2, Lemma 3.9]; notice that in [2] a coerciveness condition on ψ is assumed (cf. (3.1)(iv)), but is not used in the proof of Lemma 3.9.

Let us then go directly to the proof of (2.16), which will be obtained by using methods of [1, Prop. 1.2]; our statement is similar to [3, (4.4)], the novelty relying in the use of the function H, which allows to deal with functions ψ which are sum of functions with different degrees of homogeneity. Fix $\eta > 0$, and choose $u \in H^{1,p}(\Omega)$ such that $\tilde{u} \geqslant w$ $\sigma_\varepsilon^{\psi}(\cdot, 1, \Omega)$ a.e. and

$$\int_\Omega \left[\psi(Du) + \frac{|u|}{\varepsilon}\right] dx \leqslant \tau_\varepsilon^{\psi}(E, w, \Omega) + \eta; \tag{2.17}$$

set also

$$T(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \leqslant t \leqslant 1 \\ 1 & \text{for } t > 1 \end{cases}$$

and, for any $h \in \mathbf{Z}$:

$$S_h = \{x \in \Omega: 2^{h-1} \leqslant \tilde{u}(x) < 2^h\}; \qquad T_h(t) = T(2^{1-h}t - 1),$$

and notice that $T_h(u) \geqslant 1$ on E_{2^h}. Then compute

$$\int_0^{+\infty} \tau_\varepsilon^{\psi}(E_s, 1, \Omega)\, dH(s) \leqslant \sum_{h<0} \int_{2^h}^{2^{h+1}} \tau_\varepsilon^{\psi}(E_s, 1, \Omega)\, ds^p + \sum_{h \geqslant 0} \int_{2^h}^{2^{h+1}} \tau_\varepsilon^{\psi}(E_s, 1, \Omega)\, ds \leqslant$$

$$\leqslant \sum_{h<0} (2^{(h+1)p} - 2^{hp})\, \tau_\varepsilon^{\psi}(E_{2^h}, 1, \Omega) + \sum_{h \geqslant 0} (2^{h+1} - 2^h)\, \tau_\varepsilon^{\psi}(E_{2^h}, 1, \Omega) \leqslant$$

$$\leqslant \sum_{h<0} (2^{(h+1)p} - 2^{hp}) \int_\Omega \psi(DT_h(u)) + \frac{|T_h(u)|}{\varepsilon} dx + \sum_{h \geqslant 0} 2^h \int_\Omega \psi(DT_h(u)) + \frac{|T_h(u)|}{\varepsilon} dx \leqslant$$

$$\leqslant (2^{2p} - 2^p) \int_\Omega \psi(Du)\, dx + \frac{1}{\varepsilon}\left[\sum_{h<0} (2^{(h+1)p} - 2^{hp}) \int_\Omega |T_h(u)|\, dx + \sum_{h \geqslant 0} 2^h \int_\Omega |T_h(u)|\, dx\right] =$$

$$= (2^{2p} - 2^p) \int_\Omega \psi(Du)\, dx + \frac{1}{\varepsilon} \left\{ \sum_{h<0} (2^{(h+1)p} - 2^{hp}) \left[\int_{\{\tilde{u} \geq 2^h\}} dx + \int_{S_h} |2^{1-h} u - 1|\, dx \right] + \right.$$

$$\left. + \sum_{h \geq 0} 2^h \left[\int_{\{\tilde{u} \geq 2^h\}} dx + \int_{S_h} |2^{1-h} u - 1|\, dx \right] \right\} \leq c \int_\Omega \left[\psi(Du) + \frac{|u|}{\varepsilon} \right] dx\,;$$

from (2.17) and the arbitrariness of η the thesis follows.

By Lemmas (2.5) and (2.14) we can deduce the following result.

(2.18). THEOREM: *Let $\Omega \subset \boldsymbol{R}^n$ be open bounded, and $A \subset \boldsymbol{R}^n$ open; let $E \subset A$ be such that* $\operatorname{dist}(E, \Omega \setminus A) > 0$; *then for every $\varepsilon > 0$ there exists $\gamma(\varepsilon) > 0$, $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every $v \in H^{1,p}(\Omega)$ verifying $\tilde{v}(x) \geq 0$ for* $\operatorname{cap}_\Omega$*-a.e. x it holds*

$$\int_E \tilde{v}(x)\, d\sigma_\varepsilon^{\psi_0}(x, 1, \Omega) \leq (1 + \gamma(\varepsilon)) \left\{ \int_{A \cap \Omega} \left[\psi_0(Dv) + \frac{|v|}{\varepsilon} \right] dx \right\}; \tag{2.19}$$

$$\int_E \tilde{v}(x)\, d\sigma_\varepsilon^{\psi_0}(x, -1, \Omega) \leq (1 + \gamma(\varepsilon)) \left\{ \int_{A \cap \Omega} \left[\psi_0(-Dv) + \frac{|v|}{\varepsilon} \right] dx \right\}; \tag{2.20}$$

moreover, there exists $c > 0$ (depending only on p) such that

$$\int_E H(\tilde{v}(x))\, d\sigma_\varepsilon^{\psi}(x, 1, \Omega) \leq c(1 + \gamma(\varepsilon)) \left\{ \int_{A \cap \Omega} \left[\psi(Dv) + \frac{|v|}{\varepsilon} \right] dx \right\}; \tag{2.21}$$

$$\int_E H(\tilde{v}(x))\, d\sigma_\varepsilon^{\psi}(x, -1, \Omega) \leq c(1 + \gamma(\varepsilon)) \left\{ \int_{A \cap \Omega} \left[\psi(-Dv) + \frac{|v|}{\varepsilon} \right] dx \right\}. \tag{2.22}$$

PROOF: Inequalities (2.19), (2.20) are proved in [3, Theorem 4.4]. Let us prove (2.21). From Lemma (2.5), and from (2.13), (2.16) it follows

$$\int_E H(\tilde{v}(x))\, d\sigma_\varepsilon^{\psi}(x, 1, \Omega) = \int_0^{+\infty} \sigma_\varepsilon^{\psi}(\{H(\tilde{v}) > s\} \cap E, 1, \Omega)\, ds \leq$$

$$\leq (1 + \gamma(\varepsilon)) \int_0^{+\infty} \sigma_\varepsilon^{\psi}(\{H(\tilde{v}) > s\} \cap E, 1, A \cap \Omega)\, ds \leq$$

$$\leqslant p(1+\gamma(\varepsilon)) \int_0^{+\infty} \sigma_\varepsilon^\psi(\{\tilde{v}>t\} \cap E, 1, A \cap \Omega)\, dH(t) \leqslant$$

$$\leqslant p(1+\gamma(\varepsilon)) \int_0^{+\infty} \tau_\varepsilon^\psi(\{\tilde{v}>t\} \cap E, 1, A \cap \Omega)\, dH(t) \leqslant$$

$$\leqslant c(1+\gamma(\varepsilon))\, \tau_\varepsilon^\psi(E, \tilde{v}, A \cap \Omega) \leqslant c(1+\gamma(\varepsilon)) \left\{ \int_{A \cap \Omega} \left[\psi(Dv) + \frac{|v|}{\varepsilon} \right] dx \right\}.$$

The proof of (2.22) is analogous.

3. - The semicontinuity theorem

In this section we shall prove a lower semicontinuity theorem for functionals which are sum of an integral with respect to the Lebesgue measure and an integral with respect to a Radon measure.

Let $f: \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}$ be a Borel function such that

$$f(x,s,z) \geqslant \psi_0(z) + \psi(z) \qquad \forall (x,s,z) \in \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n, \tag{3.1}$$

where the functions ψ_0, ψ share the properties discussed in section 1.

Let further $g: \Omega \times \boldsymbol{R} \to [0, +\infty]$ be a Borel function lower semicontinuous with respect to $s \in [0, +\infty]$ (a normal proper integrand in the terminology of [13]) and not identically $+\infty$, and μ a non negative Radon measure on $\boldsymbol{R}^n$. Set, for $A \subset \boldsymbol{R}^n$ open, and $u \in H^{1,p}(\Omega)$:

$$F_A(u) = \int_A f(x, u, Du)\, dx, \tag{3.2}$$

$$G(u) = \int_{\bar{\Omega}} g(x, u)\, d\mu, \tag{3.3}$$

$$\mathcal{F}(u) = F_\Omega(u) + G(u). \tag{3.4}$$

Following [2], [3], we shall construct some Borel functions ($\lambda_{\psi_0}^+$, $\lambda_{\psi_0}^-$, λ_ψ^+, λ_ψ^-, and then λ^+, λ^-, see (3.7), (3.8) below) through Radon-Nykodym densities of truncated measures connected to $\sigma^\varphi(\cdot, 1, \Omega)$ (resp. $\sigma^\varphi(\cdot, -1, \Omega)$) and μ. By means of such functions we shall express a condition on g (see (3.11) below) which will ensure the L^1 semicontinuity of $\mathcal{F}$. Condition (3.11) is formally analogous to [3, (5.3)]: the novelty is in the construction of the functions $\lambda^\pm$ which take into account the possibility of different degrees of homogeneity of the ψ_i's used in their construction. Notice also that the use of the function H (see (1.1)) in the construction of the approximating g_m's (see (3.13)) allows to get a L^1 semicontinuity result independently of the homogeneity degrees of the ψ_i's.

The following result is an easy consequence of the properties of the operation $\wedge$ between measures, as discussed *e.g.* in [9] and, in the present context, [2]. We state it as a lemma to simplify future references.

(3.5). Lemma: *For every $\rho > 0$ the set functions defined by the following formulas (φ as in convention* (C))

$$(\sigma^{\varphi}(\cdot, 1, \Omega) \wedge \rho\mu)(E) = \inf\{\sigma^{\varphi}(E_1, 1, \Omega) + \rho\mu(E_2),\ E_1 \cap E_2 = \emptyset,\ E_1 \cup E_2 = E\},$$

$$(\sigma^{\varphi}(\cdot, -1, \Omega) \wedge \rho\mu)(E) = \inf\{\sigma^{\varphi}(E_1, -1, \Omega) + \rho\mu(E_2),\ E_1 \cap E_2 = \emptyset,\ E_1 \cup E_2 = E\}$$

are Radon measures absolutely continuous with respect to μ, hence the following functions are well defined and they belong to $L^1(\mu)$:

$$\lambda_{\varphi}^{+}(\cdot, \rho) = \frac{d}{d\mu}(\sigma^{\varphi}(\cdot, 1, \Omega) \wedge \rho\mu),$$

$$\lambda_{\varphi}^{-}(\cdot, \rho) = \frac{d}{d\mu}(\sigma^{\varphi}(\cdot, -1, \Omega) \wedge \rho\mu),$$

and the following hold:

$$\rho_1 < \rho_2 \Rightarrow \begin{cases} \lambda_{\varphi}^{+}(\cdot, \rho_1) \leq \lambda_{\varphi}^{+}(\cdot, \rho_2), \\ \lambda_{\varphi}^{-}(\cdot, \rho_1) \leq \lambda_{\varphi}^{-}(\cdot, \rho_2), \end{cases}$$

From Lemma (3.5) it follows that the definition below does make sense.

(3.6). Definition: *For φ as in convention* (C), *we put*

$$\lambda_{\varphi}^{+}(x) = \sup_{\rho > 0} \lambda_{\varphi}^{+}(x, \rho), \qquad \lambda_{\varphi}^{-}(x) = \sup_{\rho > 0} \lambda_{\varphi}^{-}(x, \rho);$$

we define also the functions:

$$\lambda^{+}(x) = \max\{\lambda_{\psi_0}^{+}(x), \lambda_{\psi}^{+}(x)\}, \tag{3.7}$$

$$\lambda^{-}(x) = \max\{\lambda_{\psi_0}^{-}(x), \lambda_{\psi}^{-}(x)\}. \tag{3.8}$$

(3.9). Remark: Taking into account Remark (2.8) it readily follows that both λ_{ψ}^{+} and λ_{ψ}^{-} assume only the values 0 or $+\infty$.

(3.10). Lemma: *Let λ^{+}, λ^{-} be given by* (3.7), (3.8); *if g verifies the condition*

$$-\lambda^{+}(x)(u - v) \leq g(x, u) - g(x, v) \leq \lambda^{-}(x)(u - v), \qquad \forall v \leq u \ \ \mu \text{ a.e.} \tag{3.11}$$

then the functional G defined in (3.3) *is well defined for every $u \in H^{1,p}(\Omega)$*

Proof: We must show that if $w = \tilde{u}$ a.e. with respect to cap_{Ω} then $G(w) = G(u)$. To this end, notice that, according to Remark (2.10), if $\mathrm{cap}_{\Omega}(\{\tilde{u} \neq w\}) = 0$, then $\sigma^{\varphi}(\{\tilde{u} \neq w\}, 1, \Omega) = \sigma^{\varphi}(\{\tilde{u} \neq w\}, -1, \Omega) = 0$ for both $\varphi = \psi_0$ and $\varphi = \psi$, hence $\lambda^{+} = \lambda^{-} =$

$= 0$ μ-a.e. in the set $\{\tilde{u} \neq w\}$; by virtue of (3.11), $g(x, \cdot) = 0$ in $\{\tilde{u} \neq w\}$, therefore

$$\int_{\bar{\Omega}} g(x, \tilde{u}(x))\, d\mu = \int_{\bar{\Omega}} g(x, w(x))\, d\mu\,,$$

and the thesis follows.

The following lemma is the key step in the proof of our semicontinuity theorems.

(3.12). Lemma: *Let $(u_h) \subset H^{1,p}(\Omega)$ $u_h \to u \in H^{1,p}(\Omega)$ in the topology of $L^1(\Omega)$. If g fulfils condition* (3.11), *then for every open set $A \subset \boldsymbol{R}^n$ the following inequality holds*:

$$\int_{\bar{\Omega} \cap A} g(x, \tilde{u})\, d\mu \leqslant \int_{\Omega \cap A} [\psi_0(Du) + \psi(Du)]\, dx + 2 \int_{\Omega \cap A} [\psi_0(-Du) + \psi(-Du)]\, dx +$$

$$+ \liminf_{h \to +\infty} \left\{ \int_{\Omega \cap A} [\psi_0(Du_h) + \psi(Du_h)]\, dx + \int_{\bar{\Omega} \cap A} g(x, \tilde{u}_h)\, d\mu \right\}.$$

Proof: Set

$$\Omega_0^+ = \{x \in \Omega : \lambda^+(x) = \lambda_{\psi_0}^+(x)\}\,, \qquad \Omega_0^- = \{x \in \Omega : \lambda^-(x) = \lambda_{\psi_0}^-(x)\}\,,$$

$$\Omega_1^+ = \Omega \setminus \Omega_0^+\,, \qquad \Omega_1^- = \Omega \setminus \Omega_0^-\,;$$

set also, for $m \in \boldsymbol{N}$:

$$H_m^+(x, t) = \begin{cases} (\lambda^+(x) \wedge m)\, t & \text{if } x \in \Omega_0^+\,,\ t \geqslant 0 \\ \dfrac{2^{1-p}}{c} (\lambda^+(x) \wedge m)\, H(t) & \text{if } x \in \Omega_1^+\,,\ t \geqslant 0 \end{cases}$$

$$H_m^-(x, t) = \begin{cases} (\lambda^-(x) \wedge m)\, t & \text{if } x \in \Omega_0^-\,,\ t \geqslant 0 \\ \dfrac{2^{1-p}}{c} (\lambda^-(x) \wedge m)\, H(t) & \text{if } x \in \Omega_1^-\,,\ t \geqslant 0 \end{cases}$$

(where H is the function defined in (1.1) and c is the constant in (2.21), (2.22)), and finally:

$$g_m(x, s) = \min\left\{ \inf_{t \geqslant s} [g(x, t) + H_m^+(t - s)],\ \inf_{t \leqslant s} [g(x, t) + H_m^-(t - s)] \right\}. \tag{3.13}$$

Taking into account (3.11) and using Prop. 2R, 2L in [13] it is not difficult to prove that the g_m's are Carathéodory functions and converge increasingly to g as m goes to $+\infty$. Remark also that, since $u_h = u_h \vee u + u_h \wedge u - u$, it suffices to prove the thesis separately for sequences u_h such that either $\tilde{u}_h \geqslant \tilde{u}$ or $\tilde{u}_h \leqslant \tilde{u}$. Let be then $\tilde{u}_h \geqslant \tilde{u}$ (in the opposite case the proof is analogous). Since $\sigma^{\varphi}(\{\tilde{u}_h < \tilde{u}\}, 1, \Omega) = 0$, and then $\lambda^+ = 0$ in

$\{\tilde{u} < \tilde{u}\}$, we have

$$\int_{\{\tilde{u}_h < \tilde{u}\}} g(x, \tilde{u}_h)\, d\mu = \int_{\{\tilde{u}_h < \tilde{u}\}} g(x, \tilde{u})\, d\mu\,,$$

hence we can suppose $\lambda^+ > 0$ μ-a.e. in $\overline{\Omega}$.

Fix $m \in N$ and define the set functions:

$$\mu_\varepsilon^{(0)}(B, \Omega) = (\sigma_\varepsilon^{\psi_0}(\cdot, 1, \Omega) \wedge m\mu)(B \cap \Omega_0^+)\,,$$

$$\mu_\varepsilon^{(1)}(B, \Omega) = (\sigma_\varepsilon^{\psi}(\cdot, 1, \Omega) \wedge m\mu)(B \cap \Omega_1^+)\,;$$

from the results in the appendix of [2] it follows that such functions verify for every compact set K

$$\lim_{\varepsilon \to 0} \int_K g_m(x, \tilde{u}(x)) \frac{1}{\lambda^+(x) \wedge m}\, d\mu_\varepsilon^{(i)} = \int_{K \cap \Omega_i^+} g_m(x, \tilde{u}(x))\, d\mu \qquad (i = 0, 1)\,.$$

Since moreover it holds

$$\int_K g_m(x, \tilde{u}(x))\, d\mu = \int_{K \cap \Omega_0^+} g_m(x, \tilde{u}(x))\, d\mu + \int_{K \cap \Omega_1^+} g_m(x, \tilde{u}(x))\, d\mu\,,$$

we must evaluate the quantities:

$$J_i = \int_K g_m(x, \tilde{u}(x)) \frac{1}{\lambda^+(x) \wedge m}\, d\mu_\varepsilon^{(i)} \qquad (i = 0, 1)\,.$$

Take $K \subset A \cap \overline{\Omega}$ in such a way that $\operatorname{dist}(K, \overline{\Omega} \setminus A) > 0$; from the definition of g_m and $\mu_\varepsilon^{(i)}$, and Theorem (2.18) we infer

$$J_0 + J_1 \leqslant \int_{K \cap \Omega_0^+} g(x, \tilde{u}_h(x)) \frac{1}{\lambda_{\psi_0}^+(x) \wedge m}\, d\mu_\varepsilon^{(0)} + \int_{K \cap \Omega_0^+} (\tilde{u}_h - \tilde{u})\, d\mu_\varepsilon^{(0)} +$$

$$+ \int_{K \cap \Omega_1^+} g(x, \tilde{u}_h(x)) \frac{1}{\lambda_\psi^+(x) \wedge m}\, d\mu_\varepsilon^{(1)} + \frac{2^{1-p}}{c} \int_{K \cap \Omega_1^+} H(\tilde{u}_h - \tilde{u})\, d\mu_\varepsilon^{(1)} \leqslant$$

$$\leqslant \int_{K \cap \Omega_0^+} g(x, \tilde{u}_h(x)) \frac{1}{\lambda_{\psi_0}^+(x) \wedge m}\, d[\sigma_\varepsilon^{\psi_0} \wedge m\mu] + \int_{K \cap \Omega_0^+} (\tilde{u}_h - \tilde{u})\, d\sigma_\varepsilon^{\psi_0}(\cdot, 1, \Omega) +$$

$$+ \int_{K \cap \Omega_1^+} g(x, \tilde{u}_h(x)) \frac{1}{\lambda_\psi^+(x) \wedge m}\, d[\sigma_\varepsilon^{\psi} \wedge m\mu] + \frac{2^{1-p}}{c} \int_{K \cap \Omega_1^+} H(\tilde{u}_h - \tilde{u})\, d\sigma_\varepsilon^{\psi}(\cdot, 1, \Omega) \leqslant$$

$$\leqslant \int_K g(x, \tilde{u}_h(x))\, d\mu + (1+\gamma(\varepsilon)) \int_{A\cap\Omega} \left[\psi_0(Du_h - Du) + \frac{|u_h - u|}{\varepsilon}\right] dx +$$

$$+2^{1-p}(1+\gamma(\varepsilon)) \int_{A\cap\Omega} \left[\psi(Du_h - Du) + \frac{|u_h - u|}{\varepsilon}\right] dx \leqslant$$

$$\leqslant \int_K g(x, \tilde{u}_h(x))\, d\mu + (1+\gamma(\varepsilon)) \left\{ \int_{A\cap\Omega} [\psi_0(Du_h) + \psi_0(-Du)]\, dx + \right.$$

$$\left. + \int_{A\cap\Omega} [\psi(Du_h) + \psi(-Du)]\, dx + \frac{2^{1-p}+1}{\varepsilon} \int_{A\cap\Omega} |u_h - u| \right\} dx,$$

whence, taking the $\liminf_{h\to+\infty}$

$$J_0 + J_1 \leqslant (1+\gamma(\varepsilon)) \left\{ \int_{A\cap\Omega} (\psi_0(-Du) + \psi(-Du)\, dx + \right.$$

$$\left. + \liminf_{h\to+\infty} \left[\int_{A\cap\Omega} (\psi_0(Du_h) + \psi(Du_h))\, dx + \int_{A\cap\bar{\Omega}} g(x, \tilde{u}_h(x))\, d\mu \right] \right\}.$$

Letting now $\varepsilon \to 0$ and $m \to +\infty$, and taking into account the arbitrariness of K we deduce

$$\int_{A\cap\bar{\Omega}} g(x, \tilde{u}(x))\, d\mu \leqslant \int_{A\cap\Omega} (\psi_0(-Du) + \psi(-Du))\, dx +$$

$$+ \liminf_{h\to+\infty} \left[\int_{A\cap\Omega} (\psi_0(Du_h) + \psi(Du_h))\, dx + \int_{A\cap\bar{\Omega}} g(x, \tilde{u}_h(x))\, d\mu \right].$$

An analogous result holds for $\tilde{u} \geqslant \tilde{u}_h$, so that, by combining them, the thesis follows.

We are now ready to prove the main result of this paper.

(3.14). Theorem: *Suppose that f, g, μ satisfy the hypotheses stated at the beginning of this section, and moreover that g satisfies condition* (3.11); *then the functional $\mathcal{F}$ defined by* (3.4) *is lower semicontinuous in $H^{1,p}(\Omega)$ with respect to the topology induced by $L^1(\Omega)$.*

Proof: The proof is by now standard (cf. [2, Teor. 4.2], or [3, Th. 5.2]); we sketch it for completeness.

Let $\mu = \mu_a + \mu_s$ the Lebesgue decomposition of μ, μ_a being the absolutely conti-

nuous part of μ with respect to the Lebesgue measure, and μ_s the singular one. Since

$$u \mapsto \int_{\bar{\Omega}} g(x, \tilde{u}(x))\, d\mu_a$$

is trivially lower semicontinuous by the Fatou's lemma, in order to simplify the exposition we can suppose without loss of generality that $\mu = \mu_s$, *i.e.* that $\operatorname{spt}\mu$ is Lebesgue negligible. Fix then $u \in H^{1,p}(\Omega)$ such that $\mathcal{F}(u) < +\infty$, and let be $(u_h) \subset H^{1,p}(\Omega)$, $u_h \to u$ in L^1. Let further $\varepsilon > 0$ be given, and choose two open sets A_1, A_2, with $A_1 \subset \Omega$, such that

$$F_\Omega(u) \leq F_{A_1}(u) + \varepsilon, \qquad F_\Omega(-u) \leq F_{A_1}(-u) + \varepsilon, \qquad G(u) \leq \int_{A_2} g(x, \tilde{u})\, d\mu + \varepsilon.$$

From (3.1) and Lemma (3.12) it then readily follows

$$\mathcal{F}(u) \leq F_{A_1}(u) + \int_{\bar{\Omega} \cap A_2} g(x, \tilde{u})\, d\mu + 2\varepsilon \leq \liminf_{h \to +\infty} \int_{\Omega \cap A_1} f(x, u_h, Du_h)\, dx +$$

$$+ \liminf_{h \to +\infty} \left[\int_{\Omega \cap A_2} f(x, u_h, Du_h)\, dx + \int_{\bar{\Omega} \cap A_2} g(x, \tilde{u})\, d\mu \right] + 5\varepsilon \leq \mathcal{F}(u_h) + 5\varepsilon,$$

and, in view of the arbitrariness of ε, the theorem is proved.

In the particular case $\psi_0 \equiv 0$ (whence of course $\lambda^+_{\psi_0} \equiv \lambda^-_{\psi_0} \equiv 0$), *i.e.* f does not have linear growth in any direction, one has obviously $\lambda^\pm = \lambda^\pm_\psi$; taking into account Remark (3.9), condition (3.11) means that

$$\text{if } \lambda^+(x) = 0 \text{ then } g(c, \cdot) \text{ is non decreasing},$$

$$\text{if } \lambda^-(x) = 0 \text{ then } g(c, \cdot) \text{ is non increasing}.$$

In this case we are able to prove an integral representation formula for the relaxed functional, *i.e.* the greatest lower semicontinuous functional which is less than or equal to $\mathcal{F}$. Precisely, we set:

$$\mathrm{sc}^- \mathcal{F}(u) = \inf \left\{ \liminf_{h \to +\infty} \mathcal{F}(u_h),\ (u_h) \subset H^{1,p}(\Omega),\ u_h \to u \text{ in } L^1(\Omega) \right\};$$

then, under some further hypothesis, it is possible to give an integral formula for $\mathrm{sc}^- \mathcal{F}$ as in the following theorem; it can be proved in exactly the same way as [3, Th. 6.1, case $p > 1$], which relies essentially on [3, Remark3.3], analogous to our Remark (3.9).

(3.15). THEOREM: *Let $\Omega \subset \boldsymbol{R}^n$ be open and bounded, let ψ be as in section* 1, *and let f satisfy* (3.1) *with $\psi_0 \equiv 0$; suppose that μ is a non negative Radon measure verifying $\mu(E) = 0$ for every Borel set E such that* $\mathrm{cap}_\Omega(E) = 0$; *let g, T be Borel functions such that*

$T(\cdot, s) \in L^1(\mu)$ for any $s \in \boldsymbol{R}$, $T(x, \cdot)$ is non decreasing for μ-a.e. x, and $g(x,s) \leq T(x, |s|)$. Set

$$\gamma^+(x,s) = \begin{cases} g(x,s) & \text{if } \lambda^+(x) = +\infty \\ \inf\limits_{t \in \boldsymbol{R}} g(x,t) & \text{if } \lambda^+(x) = 0 \end{cases}$$

$$\gamma^-(x,s) = \begin{cases} g(x,s) & \text{if } \lambda^-(x) = +\infty \\ \inf\limits_{t \in \boldsymbol{R}} g(x,t) & \text{if } \lambda^-(x) = 0 \end{cases}$$

$$\gamma(x,s) = \min\{\gamma^+(x,s), \gamma^-(x,s)\},$$

we have

$$\mathrm{sc}^- \mathcal{F}(x) = \int_\Omega f(x,u,Du)\,dx + \int_{\overline{\Omega}} \gamma(x,\tilde{u})\,d\mu$$

for every $u \in H^{1,p}(\Omega) \cap L^\infty(\Omega)$.

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