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Properties and Pathologies of the Composition and Inversion Operators in Schauder Spaces (**)

ABSTRACT. — The composition operator T defined by $T[f,g](x) \equiv f(g(x))$, from $C^{m,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ to $C^{m,\gamma}(\mathrm{cl}\,\Omega)$ and the inversion operator J, where $J[f]$ is the inverse function of $f \in C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ are considered. Conditions on m, α, β, Ω, Ω_1 are given to ensure continuity and boundedness on bounded sets of T, J. A few counterexamples show the sharpness of such conditions.

Proprietà e patologie degli operatori di composizione e di inversione negli spazi di Schauder

RIASSUNTO. — Si considerano l'operatore di composizione T definito da $T[f,g](x) \equiv f(g(x))$, da $C^{m,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ in $C^{m,\gamma}(\mathrm{cl}\,\Omega)$ e l'operatore di passaggio all'inverso J, ove $J[f]$ è la funzione inversa della funzione $f \in C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$. Si danno condizioni su m, α, β, Ω, Ω_1 che assicurano la continuità e la limitatezza di T, J sui limitati. Alcuni controesempi mostrano l'ottimalità di tali condizioni.

1. - INTRODUCTION

In this paper we study the composition operator defined by

$$T[f,g](x) \equiv f(g(x)), \qquad f \in C^{m,\alpha}(\mathrm{cl}\,\Omega_1), \quad g \in (C^{m,\beta}(\mathrm{cl}\,\Omega))^n, \quad x \in \mathrm{cl}\,\Omega, \tag{1.1}$$

and the inversion operator J defined by

$$J[f] \equiv f^{(-1)}, \qquad f \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n, \tag{1.2}$$

where Ω, Ω_1 are open subsets of $\mathbf{R}^n$, $g(\mathrm{cl}\,\Omega) \subseteq \mathrm{cl}\,\Omega_1$, f in (1.2) is one to one with inverse $f^{(-1)}$, and where $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ is a Schauder space. Operators such as T occur in the study of nonlinear differential equations and are well-known in the literature. We mention the work of Berkolajko (1969), Berkolajko & Rutitskji (1971), Bondarenko & Zabrejko (1975), Drábec (1975), Valent (1988). Extensive references can be found in Appel

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(1988), Appel & Zabrejko (1990). Particular attention has been devoted to find sharp statements on the continuity and boundedness of T, J. We show the sharpness of these theorems by producing several counterexamples. We encounter several pathologies of T and J. Similar facts for T in the case $m=0$ were pointed out by Berkolajko (1969) (as reported by Appel (1988, p. 240)), and our counterexamples have that of Berkolajko as a starting point. Most of this paper is devoted to the case $m \geqslant 1$, and is to the best of the author's knowledge, new. Our methods are different from those used for the «Nemytskii» operator $g \mapsto T[f,g]$ when $f \in C^{m+1}(\mathrm{cl}\,\Omega_1)$ (cf. Drábek (1975)). Finally, we mention that the study of J has required a characterization of injective transformations which extends that of Lanza (1987, Lemmas 13.22, 13.27) (cf. Lanza & Antman (1991a, Lemma 4.11; 1991b, Lemma 3.4)). The author was motivated to prove a part of the statements contained in this paper in order to support the analysis of Lanza (1991).

2. - Preliminaries and notation

We denote the norm on a Banach space $\mathcal{X}$, by $\|\cdot : \mathcal{X}\|$. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. We equip the product space $\mathcal{X} \times \mathcal{Y}$ with the norm $\|\cdot : \mathcal{X} \times \mathcal{Y}\| \equiv \|\cdot : \mathcal{X}\| + \|\cdot : \mathcal{Y}\|$, while we use the Euclidean norm for $\mathbf{R}^n$. We say that $\mathcal{X}$ is imbedded in $\mathcal{Y}$ provided that there exists a continuous injective map of $\mathcal{X}$ into $\mathcal{Y}$. Let $\mathcal{B} \subseteq \mathcal{X}$. We say that a nonlinear operator $N: \mathcal{B} \to \mathcal{Y}$ is bounded provided that N maps bounded subsets of $\mathcal{B}$ into bounded subsets of $\mathcal{Y}$. The inverse function of a function f is denoted $f^{(-1)}$ as opposed to the reciprocal of a real valued function g or the inverse of a matrix A, which are denoted g^{-1} and A^{-1} respectively. Let Ω be an open subset of $\mathbf{R}^n$. The space of m-times continuously differentiable functions on Ω, is denoted with $C^m(\Omega)$. The space of those functions of $C^\infty(\Omega)$ which have compact support contained in Ω is denoted $\mathcal{D}(\Omega)$. Let $f \in (C^m(\Omega))^n$. Df denotes the gradient matrix $(\partial f_i / \partial x_j)_{i,j=1,\ldots,n}$. Let $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbf{N}^n$, $|\eta| \equiv \eta_1 + \ldots + \eta_n$. Then D^η denotes $\partial^{|\eta|} / (\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n})$. The subspace of $C^m(\Omega)$ of those functions which are uniformly continuous in Ω together with their derivatives $D^\eta f$ of order $|\eta| \leqslant m$ is denoted $C^m(\mathrm{cl}\,\Omega)$. Let $f \in C^m(\mathrm{cl}\,\Omega)$. The unique continuous extension of $D^\eta f$, $|\eta| \leqslant m$ to $\mathrm{cl}\,\Omega$ is still denoted by the same symbol. Let Ω be a bounded open subset of $\mathbf{R}^n$. $C^m(\mathrm{cl}\,\Omega)$ equipped with the norm $\|f: C^m(\mathrm{cl}\,\Omega)\| \equiv \sum_{|\eta| \leqslant m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$ is a Banach space. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in (0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, (cf. Kufner, John & Fučik (1977)). Let $B \subseteq \mathbf{R}^n$. Then $C^{m,\alpha}(\mathrm{cl}\,\Omega, B)$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : f(\mathrm{cl}\,\Omega) \subseteq B\}$. If $f \in C^{0,\alpha}(\mathrm{cl}\,\Omega)$, then its Hölder quotient is

$$|f: \mathrm{cl}\,\Omega|_\alpha \equiv \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} : x, y \in \mathrm{cl}\,\Omega,\ x \neq y \right\}.$$

The space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ is equipped with its usual norm

$$\|f: C^{m,\alpha}(\mathrm{cl}\,\Omega)\| = \|f: C^m(\mathrm{cl}\,\Omega)\| + \sum_{|\eta| = m} |D^\eta f: \mathrm{cl}\,\Omega|_\alpha .$$

Let G be a closed subset of $\mathbf{R}^n$. $C^m_W(G)$ denotes the subspace of $C^0(G)$ of those functions f which are restrictions of elements F of $C^m(\mathbf{R}^n)$. Namely $C^m_W(G) \equiv \{f \in C^0(G): \exists F \in C^m(\mathbf{R}^n): F_{|G} = f\}$. Let Ω be a bounded, open connected subset of $\mathbf{R}^n$. For every $x, y \in \Omega$, there exists an arc $\gamma_{x,y}$ of class C^1 such that

$$\gamma_{x,y}: [0,1] \to \Omega, \qquad \gamma_{x,y}(0) = x, \qquad \gamma_{x,y}(1) = y. \tag{2.1}$$

The geodesic distance $\lambda(x,y)$ is defined as

$$\lambda(x,y) \equiv \inf\{\text{length of } \gamma_{x,y}: \gamma_{x,y} \text{ is of class } C^1 \text{ and (2.1) holds}\}. \tag{2.2}$$

Let

$$c[\Omega] \equiv \sup\left\{\frac{\lambda(x,y)}{|x-y|}: x, y \in \Omega,\ x \neq y\right\}. \tag{2.3}$$

If $c[\Omega] < \infty$, then Ω is said to be regular in the sense of H. Whitney. Note that $c[\Omega] = 1$ if Ω is convex. The following Lemma collects a few elementary inequalities involving Hölder norms.

2.4. Lemma: *Let $m, n \in \mathbf{N}$, $n \geq 1$, $\alpha, \beta \in (0,1]$, $\gamma \equiv \min\{\alpha, \beta\}$. Let Ω be a connected bounded open subset of $\mathbf{R}^n$ such that $c[\Omega] < \infty$. Let $\tilde{c}[\Omega] \equiv 2c[\Omega](2 + \operatorname{diam}[\Omega])$. Then*

(i) $$|f: \Omega|_\alpha \leq c[\Omega](\operatorname{diam}[\Omega])^{1-\alpha} \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i}: C^0(\operatorname{cl}\Omega) \right\| \leq$$
$$\leq \tilde{c}[\Omega] \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i}: C^0(\operatorname{cl}\Omega) \right\|, \qquad \forall f \in C^1(\operatorname{cl}\Omega).$$

(ii) $$\|f: C^{m,\alpha}(\operatorname{cl}\Omega)\| \leq \tilde{c}[\Omega] \|f: C^{m+1}(\operatorname{cl}\Omega)\|, \qquad \forall f \in C^{m+1}(\operatorname{cl}\Omega).$$

(iii) $$|f: \operatorname{cl}\Omega|_\alpha \leq (\operatorname{diam}[\Omega])^{\beta-\alpha} |f: \operatorname{cl}\Omega|_\beta \leq \tilde{c}[\Omega] |f: \operatorname{cl}\Omega|_\beta, \qquad \forall \alpha < \beta,\ f \in C^{0,\beta}(\operatorname{cl}\Omega).$$

(iv) $$\|f: C^{m,\alpha}(\operatorname{cl}\Omega)\| \leq (1 + (\operatorname{diam}[\Omega])^{\beta-\alpha}) \|f: C^{m,\beta}(\operatorname{cl}\Omega)\| \leq \tilde{c}[\Omega] \|f: C^{m,\beta}(\operatorname{cl}\Omega)\|,$$
$$\forall \alpha < \beta,\ f \in C^{m,\beta}(\operatorname{cl}\Omega).$$

(v) $$\|uv: C^{m,\gamma}(\operatorname{cl}\Omega)\| \leq (\tilde{c}[\Omega])^{m+1} \|u: C^{m,\alpha}(\operatorname{cl}\Omega)\| \cdot \|v: C^{m,\beta}(\operatorname{cl}\Omega)\|,$$
$$\forall u \in C^{m,\alpha}(\operatorname{cl}\Omega),\ v \in C^{m,\beta}(\operatorname{cl}\Omega).$$

Proof: Proof of (i)-(iv) is elementary and is accordingly omitted. It is well-known that (v) holds for some constant. We include a few lines of proof to show that the constant is $(\tilde{c}[\Omega])^{m+1}$. We proceed by induction on m. Case $m = 0$ can be verified by a simple computation and is well known (cf. *e.g.* Gilbarg & Trudinger (1977, p. 52)). Now assume the statement holds for $m - 1 \geq 0$. Then by inductive hypothesis and the

definition of Hölder norms, and (i)-(iv), we have

$$(2.5)\qquad \|uv\colon C^{m,\gamma}(\mathrm{cl}\,\Omega)\| \leqslant \|u\colon C^0(\mathrm{cl}\,\Omega)\|\,\|v\colon C^0(\mathrm{cl}\,\Omega)\| + \tilde{c}[\Omega]^m \cdot$$

$$\cdot \sum_{i=1}^{n}\left\{\left\|\frac{\partial u}{\partial x_i}\colon C^{m-1,\alpha}(\mathrm{cl}\,\Omega)\right\|\|v\colon C^{m-1}(\mathrm{cl}\,\Omega)\| + \|u\colon C^{m-1}(\mathrm{cl}\,\Omega)\|\left\|\frac{\partial v}{\partial x_i}\colon C^{m-1,\beta}(\mathrm{cl}\,\Omega)\right\|\right\} +$$

$$+\tilde{c}[\Omega]^m\left\{\|u\colon C^{m,\alpha}(\mathrm{cl}\,\Omega)\|\,c[\Omega](\mathrm{diam}\,[\Omega])^{1-\alpha}\left(\sum_{|\eta|=m}\|\partial^\eta v\colon C^0(\mathrm{cl}\,\Omega)\|\right) +\right.$$

$$\left.+\left(\sum_{|\eta|=m}\|\partial^\eta u\colon C^0(\mathrm{cl}\,\Omega)\|\right)c[\Omega](\mathrm{diam}\,[\Omega])^{1-\beta}\|v\colon C^{m,\beta}(\mathrm{cl}\,\Omega)\|\right\} \leqslant$$

$$\leqslant 2\tilde{c}[\Omega]^m\|u\colon C^{m,\alpha}(\mathrm{cl}\,\Omega)\|\,\|v\colon C^{m,\beta}(\mathrm{cl}\,\Omega)\| +$$

$$+2\tilde{c}[\Omega]^m(1+\mathrm{diam}\,[\Omega])\,c[\Omega]\|u\colon C^{m,\alpha}(\mathrm{cl}\,\Omega)\|\,\|v\colon C^{m,\beta}(\mathrm{cl}\,\Omega)\|.$$

Hence, (v) holds for m. ■

By using (i), a simple inductive argument, and Ascoli-Arzelà Theorem, we easily see that if Ω is a bounded open connected subset of $\mathbf{R}^n$ such that $c[\Omega]<\infty$, $m\geqslant 0$, then $C^{m+1}(\mathrm{cl}\,\Omega)$ is compactly imbedded in $C^m(\mathrm{cl}\,\Omega)$. Then by the well-known compactness of the imbedding $C^{0,\alpha}(\mathrm{cl}\,\Omega)\subseteq C^{0,\beta}(\mathrm{cl}\,\Omega)$, $\alpha>\beta$ (which holds also if $c[\Omega]=\infty$) we deduce the compactness of the imbedding $C^{m,\alpha}(\mathrm{cl}\,\Omega)\subseteq C^{m,\beta}(\mathrm{cl}\,\Omega)$ for $\alpha>\beta$. By a simple contradiction argument, it is then easy to realize that the following holds.

2.6. Lemma: *Let $0<\beta<\alpha\leqslant 1$ and let m, n be nonnegative integers. Let Ω be an open bounded subset of $\mathbf{R}^n$. If $m>1$, we further assume that Ω is connected and that $C[\Omega]<\infty$. Let $\{f_k\}$ be a sequence of functions on $\mathrm{cl}\,\Omega$ that is bounded in $C^{m,\alpha}(\mathrm{cl}\,\Omega)$. If f_k converges to $f\in C^0(\mathrm{cl}\,\Omega)$ either pointwise a.e. or in the sense of distributions in Ω, then $f\in C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and f_k converges to f in the norm of $C^{m,\beta}(\mathrm{cl}\,\Omega)$.*

3. - The composition operator

We now consider the composition operator T defined in (1.1)

3.1. Lemma: *Let $\mathbf{N}\ni n\geqslant 1$, $\alpha,\beta\in(0,1]$, $\gamma\equiv\min\{\alpha,\beta\}$. Let Ω be a connected bounded open subset of $\mathbf{R}^n$ such that $c[\Omega]<\infty$. Let $u\in C^{m,\alpha}(\mathrm{cl}\,\Omega)$, $v\equiv(v_1,\dots,v_n)\in(C^{m,\beta}(\mathrm{cl}\,\Omega))^n$, $v(\mathrm{cl}\,\Omega)\subseteq\mathrm{cl}\,\Omega_1$,*

$$\|v\colon (C^{m,\beta}(\mathrm{cl}\,\Omega))^n\| \equiv \sum_{l=1}^{n}\|v_l\colon C^{m,\beta}(\mathrm{cl}\,\Omega)\|.$$

If $m=0$, *then*

$$(3.2a)\qquad \|u(v(\cdot)): C^{0,\alpha\beta}(\mathrm{cl}\,\Omega)\| \leqslant \|u: C^{0,\alpha}(\mathrm{cl}\,\Omega_1)\|(1+\|v: (C^{0,\beta}(\mathrm{cl}\,\Omega))^n\|^{\alpha}) \leqslant$$
$$\leqslant \|u: C^{0,\alpha}(\mathrm{cl}\,\Omega_1)\|(2+\|v: (C^{0,\beta}(\mathrm{cl}\,\Omega))^n\|).$$

If $m \geqslant 1$, *then*

$$(3.2b)\qquad \|u(v(\cdot)): C^{m,\gamma}(\mathrm{cl}\,\Omega)\| \leqslant \tilde{c}[\Omega]^{m(m+1)}\|u: C^{m,\alpha}(\mathrm{cl}\,\Omega_1)\|(1+\|v: (C^{m,\beta}(\mathrm{cl}\,\Omega))^n\|)^{m+1}.$$

Proof: The statement is trivial when $m=0$. We consider $m=1$. By using Lemma 2.4(v) and the inductive hypothesis together with Lemma 2.4(i) for $\alpha=1$, we obtain

$$(3.3)\qquad \|u(v(\cdot)): C^{1,\gamma}(\mathrm{cl}\,\Omega)\| \leqslant$$
$$\leqslant \|u: C^0(\mathrm{cl}\,\Omega_1)\| + \tilde{c}[\Omega]\sum_{i,l=1}^{n}\left\|\frac{\partial u}{\partial y_l}(v(\cdot)): C^{0,\alpha}(\mathrm{cl}\,\Omega)\right\|\left\|\frac{\partial v_l}{\partial x_i}: C^{0,\beta}(\mathrm{cl}\,\Omega)\right\| \leqslant$$
$$\leqslant \|u: C^0(\mathrm{cl}\,\Omega_1)\| + \tilde{c}[\Omega]\|u: C^{1,\alpha}(\mathrm{cl}\,\Omega_1)\|\,\|v: (C^{1,\beta}(\mathrm{cl}\,\Omega))^n\| +$$
$$+\tilde{c}[\Omega]^2\|u: C^{1,\alpha}(\mathrm{cl}\,\Omega_1)\|\,\|v: (C^1(\mathrm{cl}\,\Omega))^n\|\,\|v: (C^{1,\beta}(\mathrm{cl}\,\Omega))^n\| \leqslant$$
$$\leqslant \|u: C^{1,\alpha}(\mathrm{cl}\,\Omega_1)\|\tilde{c}[\Omega]^2(1+\|v: (C^{1,\beta}(\mathrm{cl}\,\Omega))^n\|)^2.$$

Now, let the statement be true for $m-1$, with $m \geqslant 2$. Then

$$(3.4)\qquad \|u(v(\cdot)): C^{m,\gamma}(\mathrm{cl}\,\Omega)\| \leqslant$$
$$\leqslant \|u: C^0(\mathrm{cl}\,\Omega_1)\| + \tilde{c}[\Omega]^m\sum_{i,l=1}^{n}\left\|\frac{\partial u}{\partial y_l}(v(\cdot)): C^{m-1,\alpha}(\mathrm{cl}\,\Omega)\right\|\left\|\frac{\partial v_l}{\partial x_i}: C^{m-1,\beta}(\mathrm{cl}\,\Omega)\right\| \leqslant$$
$$\leqslant \|u: C^0(\mathrm{cl}\,\Omega_1)\| + \tilde{c}[\Omega]^{m^2}\sum_{i,l=1}^{n}\left\|\frac{\partial u}{\partial y_l}(\cdot): C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1)\right\|\cdot$$
$$\cdot\left\|\frac{\partial v_l}{\partial x_i}: C^{m-1,\beta}(\mathrm{cl}\,\Omega)\right\|(1+\|v: (C^{m-1,1}(\mathrm{cl}\,\Omega))^n\|)^m \leqslant$$
$$\leqslant \|u: C^{m,\alpha}(\mathrm{cl}\,\Omega_1)\|\tilde{c}[\Omega]^{m(m+1)}(1+\|v: (C^{m,\beta}(\mathrm{cl}\,\Omega))^n\|)^{m+1}.$$

and the proof is complete. ■

From Lemma 2.6 we easily deduce the following

3.5. Theorem: *Let* $m, n \in \mathbf{N}$, $n \geqslant 1$, $\alpha, \beta \in (0,1]$, $\gamma \equiv \min\{\alpha,\beta\}$. *Let* Ω, Ω_1 *be connected bounded open subsets of* $\mathbf{R}^n$. *Let* $c[\Omega] < \infty$. *Then the following hold.*

(i) *If* $m=0$, T *is bounded from* $C^{0,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{0,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ *to* $C^{0,\alpha\beta}(\mathrm{cl}\,\Omega)$ *and is continuous from* $C^{0,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{0,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ *to* $C^{0,\theta}(\mathrm{cl}\,\Omega)$, *for all* $\theta \in (0, \alpha\beta)$.

(ii) *If* $m \geq 1$, T *is bounded from* $C^{m,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ *to* $C^{m,\gamma}(\mathrm{cl}\,\Omega)$ *and is continuous from* $C^{m,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ *to* $C^{m,\theta}(\mathrm{cl}\,\Omega)$ *for all* $\theta \in (0, \gamma)$.

(iii) *If* $m \geq 1$, $\alpha > \beta$, *then* T *is continuous from* $C^{m,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ *to* $C^{m,\gamma}(\mathrm{cl}\,\Omega)$.

Proof: We first prove (i), (ii). By virtue of Lemmas 2.6 and 3.1, it suffices to show that if $\lim_n (f_n, g_n) = (f, g)$ in $C^{0,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{0,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$, then $\lim_n T[f_n, g_n] = T[f, g]$ pointwise in $\mathrm{cl}\,\Omega$. This can be easily inferred by using the elementary inequality

$$|f_n(g_n(x)) - f(g(x))| \leq |f_n(g_n(x)) - f(g_n(x))| + |f(g_n(x)) - f(g(x))|, \tag{3.6}$$

We now prove (iii). By Lemma 2.4(ii), $C^{m,\beta}(\mathrm{cl}\,\Omega)$ is imbedded in $C^{m-1,1}(\mathrm{cl}\,\Omega)$. Then by (ii) of this Theorem, T is continuous from $C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{m-1,1}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ to $C^{m-1,\theta}(\mathrm{cl}\,\Omega)$, for all $\theta < \alpha = \min\{\alpha, 1\}$, and accordingly for $\theta = \beta$. Then, by Lemma 2.4(v), the nonlinear operator

$$(f, g) \mapsto \left(\sum_{l=1}^{n} T\left[\frac{\partial f}{\partial y_l}, g \right] \frac{\partial g_l}{\partial x_i} \right)_{i=1,\ldots,n}$$

is continuous from $C^{m,\alpha}(\mathrm{cl}\,\Omega) \times C^{m,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ to $(C^{m-1,\beta}(\mathrm{cl}\,\Omega))^n$. Hence, by (ii) the proof is complete. ■

It is now natural to ask whether the continuity statements contained in (i), (ii) hold for $\vartheta = \alpha\beta$, $\vartheta = \gamma$ respectively. We now produce two counterexamples to show that the answer is no. The following example shows that in general T is not continuous from $C^{0,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{0,\beta}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ to $C^{0,\alpha\beta}(\mathrm{cl}\,\Omega)$ even though T is bounded.

Example: We consider the operator T from $C^{0,\alpha}([-2,2], \mathbf{R}) \times C^{0,\beta}([0,1], [-2,2])$ to $C^{0,\alpha\beta}([0,1])$, $\alpha, \beta \in (0,1]$, $\beta < 1$. Let

$$f(u) \equiv \sup{}^{\alpha}\left\{\left(u - \frac{1}{2}\right), 0\right\}, \qquad g_n(x) = (1 + n^{-\beta})\left[x - \sup{}^{\beta}\left\{\left(x - \frac{1}{2}\right), 0\right\}\right],$$

$$g(x) = x - \sup{}^{\beta}\left\{\left(x - \frac{1}{2}\right), 0\right\}.$$

Clearly, $\lim_n (f, g_n) = (f, g)$ in $C^{0,\alpha}([-2,2]) \times C^{0,\beta}([0,1], [-2,2])$. However, for n sufficiently large, we have

$$f\left(g_n\left(\frac{1}{2} + \frac{1}{n}\right)\right) = f\left(g\left(\frac{1}{2} + \frac{1}{n}\right)\right) = 0.$$

Hence, for those n we have

$$|f(g_n)-f(g)\colon[-2,2]|_{\alpha\beta} \geqslant$$

$$\geqslant \frac{|[f(g_n(1/2))-f(g(1/2))]-[f(g_n(1/2+1/n))-f(g(1/2+1/n))]|}{|1/2-(1/2+1/n)|^{\alpha\beta}} = 2^{-\alpha}$$

and consequently $\{T[f,g_n]\}$ does not converge to $T[f,g]$. In this example $\beta<1$. However, in the next example we see that similar facts hold with $\beta=1$.

The following example shows that Theorem 3.5(iii) does not hold if $\beta \geqslant \alpha$.

Example: Let $0<\alpha\leqslant\beta\leqslant 1$,

$$f(x) \equiv \int_0^x \sup{}^{\alpha}\left\{\left(u-\frac{1}{2}\right),0\right\}du, \qquad g_n(x)\equiv(1+n^{-1})\,x, \qquad g(x)\equiv x.$$

Clearly g_n, $g\in C^{\infty}([0,1],[0,2])\subseteq C^{1,\beta}([0,1])$, $f\in C^{1,\alpha}([0,2])$. Since $[f(g_n(x))]' = f'(g_n(x))\,g_n'(x)$, then

$$\lim_n [f(g_n(x))]' = [f(g(x))]'$$

in $C^{0,\alpha}([0,1])$ if and only if $\lim_n T[f',g_n]=T[f',g]$. By considering the Hölder quotient at the points $1/2$, $(1/2)-(1/n)$ it can be shown, as in the example above, that this is false. (This also produces the counterexample for $\beta=1$ announced above.) Hence $\{T[f,g_n]\}$ does not converge to $T[f,g]$ in $C^{1,\alpha}([0,1])$.

Similar counterexamples can be deduced for $m>1$. It is interesting to note that in the example above, all the functions g_n are linear, g is the identity and $\lim_n g_n = g$ even in the Fréchet space $C^{\infty}([0,1])$. Hence, T is not continuous even from $C^{1,\alpha}([0,2])\times C^{\infty}([0,1],[0,2])$ to $C^{1,\alpha}([0,1])$. This fact is somewhat in contrast with what happens in Sobolev Spaces, where higher regularity and injectivity of g_n together with the condition $\sup\{|g_n'|^{-1}\}<\infty$ imply the continuity of T (cf. Lanza (1991)).

4. - The inversion operator in the case $m\geqslant 1$ and a characterization of injective nonsingular transformations

The goal of this section is to study the operator J defined in (1.2). Let Ω, Ω_1 be bounded open subsets of $\mathbf{R}^n$. Let $\delta\geqslant 0$. We set

$$(4.1)\qquad X_{m,\alpha,\delta}(\Omega,\Omega_1)\equiv$$

$$\equiv\{g\in(C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : g \text{ is injective},\ g(\mathrm{cl}\,\Omega)=\mathrm{cl}\,\Omega_1,\ |\det Dg(x)|>\delta,\ \forall x\in\mathrm{cl}\,\Omega\}.$$

4.2. Lemma: *Let $m,n\in\mathbf{N}$; $m,n\geqslant 1$, $\alpha\in(0,1]$, $0<\delta<1$. Let Ω, Ω_1 be connected bounded open subsets of $\mathbf{R}^n$ such that $c[\Omega]<\infty$, $c[\Omega_1]<\infty$. Let $\tilde{c}[\Omega]\equiv 2c[\Omega](2+\operatorname{diam}[\Omega])$, and $c^*[\Omega,\Omega_1]\equiv\sup\{n\max_{x\in\mathrm{cl}\,\Omega}|x|,\tilde{c}[\Omega],\tilde{c}[\Omega_1]\}$. Let $v\in X_{m,\alpha,\delta}(\Omega,\Omega_1)$. Let*

$\tau \geqslant c^*[\Omega, \Omega_1]$, $\sigma \geqslant \|v\colon (C^{m,\alpha}(\mathrm{cl}\,\Omega_1))^n\|$. *Then there exists a constant* $c(\delta, m, n, \tau, \sigma)$ *depending only on* δ, m, n, τ, σ, *and otherwise independent of* Ω, Ω_1, v, *such that* $\|v^{(-1)}\colon (C^{m,\alpha}(\mathrm{cl}\,\Omega_1))^n\| \leqslant c(\delta, m, n, \tau, \sigma)$.

Proof: We proceed by induction on m. By virtue of Lemma 3.1 and the well-known formula for the jacobian matrix of an inverse function, we have

$$(4.3)\qquad \|v^{(-1)}\colon (C^{m,\alpha}(\mathrm{cl}\,\Omega_1))^n\| \leqslant$$

$$\leqslant c^*[\Omega, \Omega_1] + \tilde{c}[\Omega_1]^{m(m-1)} \|(Dv)^{-1}(\cdot)\colon (C^{m-1,\alpha}(\mathrm{cl}\,\Omega))^{n^2}\| (1 + \|v^{(-1)}\colon (C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1))^n\|)^m$$

when $m \geqslant 2$ and

$$(4.4)\qquad \|v^{(-1)}\colon (C^{1,\alpha}(\mathrm{cl}\,\Omega_1))^n\| \leqslant$$

$$\leqslant c^*[\Omega, \Omega_1] + \|(Dv)^{-1}(\cdot)\colon (C^{0,\alpha}(\mathrm{cl}\,\Omega))^{n^2}\| (1 + \|v^{(-1)}\colon (C^{0,1}(\mathrm{cl}\,\Omega_1))^n\|)$$

when $m = 1$. Now, let $(Dv)^{-1}(x) \equiv ((-1)^{i+j} V_{j,i}(x)/\det(Dv)(x))$. Then we have the following inequalities for all $m \geqslant 1$

$$(4.5a)\qquad \|V_{i,j}\colon C^{m-1,\alpha}(\mathrm{cl}\,\Omega)\| \leqslant (n-1)!\,\tilde{c}[\Omega]^{m \sup\{n-2,0\}} \|v\colon (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n\|^{n-1},$$

$$(4.5b)\qquad \|\det Dv(\cdot)\colon C^{m-1,\alpha}(\mathrm{cl}\,\Omega)\| \leqslant n!\,\tilde{c}[\Omega]^{m(n-1)} \|v\colon (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n\|^n.$$

By Lemma 2.4(v), Lemma 3.1, (4.5), and inequality

$$\left\| \frac{1}{x}\colon C^{m-1,1}\Big[\delta, \max_{x \in \mathrm{cl}\,\Omega} |\det Dv(x)|\Big] \right\| \leqslant (m+1)!\,\delta^{-(m+1)},$$

we conclude that for all $m \geqslant 1$

$$(4.6)\qquad \|(Dv)^{-1}(\cdot)\colon (C^{m-1,\alpha}(\mathrm{cl}\,\Omega))^{n^2}\| \text{ can be estimated in terms of } m,\ n,\ \delta,\ \sigma,\ \tau.$$

We now prove the statement for $m = 1$. Note that in this case $\sigma \geqslant \|v\colon (C^{1,\alpha}(\mathrm{cl}\,\Omega))^n\|$. By virtue of (4.4), (4.6), we can estimate $\|v^{(-1)}\colon (C^{1,\alpha}(\mathrm{cl}\,\Omega_1))^n\|$ in terms of n, δ, τ, σ and $\|v^{(-1)}\colon (C^{0,1}(\mathrm{cl}\,\Omega_1))^n\|$. By Lemma 2.4(i) we have

$$(4.7)\qquad \|v^{(-1)}\colon (C^{0,1}(\mathrm{cl}\,\Omega_1))^n\| \leqslant c^*[\Omega, \Omega_1] + \tilde{c}[\Omega_1] \|(Dv)^{-1}(\cdot)\colon (C^0(\mathrm{cl}\,\Omega))^{n^2}\|.$$

Hence, by (4.6), $\|v^{(-1)}\colon C^{0,1}(\mathrm{cl}\,\Omega_1)\|$ can be estimated in terms of n, δ, τ, σ. Now, let the statement hold for $m - 1$. Then (4.3) and (4.6) together with the inductive hypothesis imply the validity of the statement for m. ■

For a proof of the following cf. Lanza (1991, Lemma 3.16)

4.8. Lemma: *Let* Ω, Ω_1 *be open subsets of* $\mathbb{R}^n$, $n \geqslant 1$, Ω *bounded. Let* ϕ, ϕ_j *continuous and one to one mappings of* Ω *onto* Ω_1. *Let* ϕ, $\phi_j \in C^0(\Omega)$. *If* $\{\phi_j\}$ *converges uniformly to* ϕ *in* Ω, *then* $\{\phi_j^{(-1)}\}$ *converges to* $\phi^{(-1)}$ *pointwise in* Ω_1. ■.

By combining Lemmas 2.6, 4.2, 4.8, we easily obtain the following

4.9. Proposition: *Let $m, n \in \mathbf{N} \setminus \{0\}$, $\alpha \in (0,1]$, $0 < \delta < 1$. Let Ω, Ω_1 be connected open bounded subsets of $\mathbf{R}^n$ such that $c[\Omega], c[\Omega_1] < \infty$. Then the operator J defined in* (1.2) *is bounded from $X_{m,\alpha,\delta}(\Omega,\Omega_1)$ into $X_{m,\alpha,0}(\Omega_1,\Omega)$. Furthermore, J is continuous from $X_{m,\alpha,0}(\Omega,\Omega_1)$ to $X_{m,\theta,0}(\Omega_1,\Omega)$ for all $0 < \theta < \alpha$.*

It is at this point natural to ask whether Proposition 4.9 holds for $\theta = \alpha$. As the following Example shows, the answer is no.

Example: Let

$$h_n(u) = \frac{1}{2} \quad \text{if } u \in \left[0, \frac{1}{2}\right],$$

$$h_n(u) = \frac{1}{n}[2^\alpha(\alpha+1) + (n+1)]\left(u - \frac{1}{2}\right)^\alpha + \frac{1}{2} \quad \text{if } u \in \left[\frac{1}{2}, 1\right];$$

$$h(u) = \frac{1}{2} \quad \text{if } u \in \left[0, \frac{1}{2}\right], \qquad h(u) = \left(u - \frac{1}{2}\right)^\alpha + \frac{1}{2} \quad \text{if } u \in \left[\frac{1}{2}, 1\right];$$

$$g_n(x) = \left\{\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right\}^{-1} \frac{n}{n+1} \int_0^x h_n(u)\,du;$$

$$g(x) = \left\{\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right\}^{-1} \int_0^x h(u)\,du.$$

Clearly $\lim_n g_n = g$ in $C^{1,\alpha}([0,1],[0,1])$, $g(0) = g_n(0) = 0$, $g(1) = g_n(1) = 1$, $g_n' > 0$, $g' > 0$. We now show that the sequence $\{g_n^{(-1)}\}$ does not converge to $g^{(-1)}$. Since $\lim_n g_n' = g'$ in $C^{0,\alpha}([0,1])$, a simple computation shows that

$$\lim_n \Big\{\sup_{x \neq y} \{|[g_n'(g_n^{(-1)}(x))^{-1} - g'(g_n^{(-1)}(x))^{-1}] -$$

$$-[g_n'(g_n^{(-1)}(y))^{-1} - g'(g_n^{(-1)}(y))^{-1}]|\,|x-y|^{-\alpha}\}\Big\} \leq$$

$$\leq \lim_n \Big\{[\min|g_n'|]^{-\alpha} \sup_{u \neq v} \{|[g_n'(u)^{-1} - g'(u)^{-1}] - [g_n'(v)^{-1} - g'(v)^{-1}]|\,|u-v|^{-\alpha}\}\Big\} = 0.$$

Then it suffices to show that $\liminf_n \mathfrak{J}_n > 0$, with

$$\mathfrak{J}_n \equiv \sup_{x \neq y} \{|[g'(g_n^{(-1)}(x))^{-1} - g'(g^{(-1)}(x))^{-1}] -$$

$$-[g'(g_n^{(-1)}(y))^{-1} - g'(g^{(-1)}(y))^{-1}]|\,|x-y|^{-\alpha}\}.$$

Let $x = g(1/2)$, $y_n = g_n(1/2 - 1/n)$, $n > 1$. Clearly

$$g'(g_n^{(-1)}(y_n)) = \frac{1}{2}\left\{\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right\}^{-1} = g'(g^{(-1)}(x)).$$

It easy to verify that $g_n(1/2 - 1/n) < g_n(1/2) < g(1/2)$. Then

$$g'(g^{(-1)}(y_n)) = \frac{1}{2}\left\{\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right\}^{-1}$$

and

$$g'(g^{(-1)}(x))^{-1} - \left[g'\left(g_n^{(-1)}\left(g\left(\frac{1}{2}\right)\right)\right)\right]^{-1} > 0.$$

It follows that

$$\mathfrak{I}_n \geqslant |[g'(g_n^{(-1)}(x))^{-1} - g'(g^{(-1)}(x))^{-1}] - [g'(g_n^{(-1)}(y_n))^{-1} - g'(g^{(-1)}(y_n))^{-1}]|\,|x - y_n|^{-\alpha} \geqslant$$

$$\geqslant \left\{2\left\{\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right\} - \left[g'\left(g_n^{(-1)}\left(g\left(\frac{1}{2}\right)\right)\right)\right]^{-1}\right\}|x - y_n|^{-\alpha}.$$

It is easy to verify that for every $c \in (0, 1/2)$ there exists $n_0 \in \mathbf{N}$ such that

$$2\left[\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right] - \left[g'\left(g_n^{(-1)}\left(g\left(\frac{1}{2}\right)\right)\right)\right]^{-1} >$$

$$> 2\left[\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right] - \left[g'\left(\frac{1}{2} + \frac{c}{n}\right)\right]^{-1}, \qquad \forall n \geqslant n_0.$$

Then a simple computation shows that

$$\liminf_n \mathfrak{I}_n \geqslant \liminf_n \left\{2\left[\frac{1}{2} + (\alpha+1)^{-1}2^{-(\alpha+1)}\right] - \left[g'\left(\frac{1}{2} + \frac{c}{n}\right)\right]^{-1}\right\}|x - y_n|^{-\alpha} > 0.$$

However, the following partial result holds.

4.10. Lemma: *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in (0, 1)$. Let Ω, Ω_1 be bounded open connected subsets of $\mathbf{R}^n$, $c[\Omega] < \infty$, $c[\Omega_1] < \infty$. The operator J defined in (1.2) from $X_{m,\alpha,0}(\Omega, \Omega_1)$ into $X_{m,\alpha,0}(\Omega_1, \Omega)$ is continuous at g if the nonlinear operator $T_{Dg}[\cdot]$ from $X_{m,\alpha,0}(\Omega_1, \Omega)$ to $(C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1))^{n^2}$ defined by $T_{Dg}[h] = Dg(h)$, $\forall h \in X_{m,\alpha,0}(\Omega_1, \Omega)$, is continuous at $g^{(-1)}$ when $X_{m,\alpha,0}(\Omega_1, \Omega)$ is equipped with the norm $\|\cdot : C^{m,\gamma}(\mathrm{cl}\,\Omega_1)\|$ for some $\gamma \in [0, \alpha)$.*

Proof: Let $m \geqslant 1$. Let $\lim_n g_n = g$ in $X_{m,\alpha,0}(\Omega, \Omega_1)$. By Proposition 4.9, we have

$$\lim_n g_n^{(-1)} = g^{(-1)} \quad \text{in } (C^{m,\gamma}(\mathrm{cl}\,\Omega_1))^n, \ \forall \gamma \in (0, \alpha). \tag{4.11}$$

Hence, it suffices to show that

$$\lim_n D(g_n^{(-1)}) = D(g^{(-1)}) \quad \text{in } (C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1))^{n^2}. \tag{4.12}$$

Note that $D(g^{(-1)})(y) = (Dg)^{-1}(g^{(-1)}(y))$. Then

$$\|D(g_n^{(-1)}) - D(g^{(-1)}): C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1)^{n^2}\| \leqslant \tag{4.13}$$

$$\leqslant \|(Dg_n)^{-1}(g_n^{(-1)}(\cdot)) - (Dg)^{-1}(g_n^{(-1)}(\cdot)): C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1)^{n^2}\| +$$

$$+\|(Dg)^{-1}(g_n^{(-1)}(\cdot)) - (Dg)^{-1}(g^{(-1)}(\cdot)): C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1)^{n^2}\|.$$

Let $0 < \delta < \min_{x \in \mathrm{cl}\,\Omega} |\det Dg(x)|$, $\delta < 1$. For n sufficiently large, $\min_{x \in \mathrm{cl}\,\Omega} |\det Dg_n(x)| > \delta$. Clearly the map $\{A \in \mathbf{R}^{n^2} : |\det A| > \delta\} \ni A \mapsto A^{-1} \in \mathbf{R}^{n^2}$ is of class C^∞. Hence, Theorem 3.5(iii), (4.11) and the assumption on T_{Dg} imply that $\lim_n (Dg)^{-1}(g_n^{(-1)}(\cdot)) = (Dg)^{-1}(g^{(-1)}(\cdot))$ in $C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1)^{n^2}$. Moreover, Lemma 3.1 and the boundedness of J stated in Proposition 4.9 and (4.13) imply that $\lim_n (Dg_n)^{-1}(g_n^{(-1)}(\cdot)) = (Dg)^{-1}(g^{(-1)}(\cdot))$ in $C^{m-1,\alpha}(\mathrm{cl}\,\Omega_1)^{n^2}$. Hence, (4.12) holds. ■

Let $m \geqslant 1$, $\delta \geqslant 0$. We set

$$Y_{m,\alpha,\delta}(\Omega) \equiv \{g \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : g \text{ is injective}, \ |\det Dg(x)| > \delta, \ \forall x \in \mathrm{cl}\,\Omega\}. \tag{4.14}$$

Let $A \subseteq Y_{m,\alpha,\delta}(\Omega)$. We say that JA is bounded, provided that

$$\sup_{g \in A} \{\|g^{(-1)} : (C^{m,\alpha}(g(\mathrm{cl}\,\Omega)))^n\|\} < \infty.$$

We say that J is bounded on $Y_{m,\alpha,\delta}(\Omega)$ provided that JA is bounded for all bounded subsets A of $Y_{m,\alpha,\delta}(\Omega)$. To study the boundedness of J, we need to characterize the elements of $Y_{m,\alpha,0}(\Omega)$. We first introduce a few definitions and preliminary lemmas.

4.15. Definition: *Let $B \subseteq \mathbf{R}^n$, $x \in \mathbf{R}^n$. We define $\mathfrak{A}(B,x)$ to be the set $v \in \mathbf{R}^n \setminus \{0\}$ such that there exist sequences $\{x_n\}$, $\{y_n\}$ in B with $x_n \neq y_n$,*

$$\lim_n x_n = x = \lim_n y_n, \qquad \lim_n \frac{x_n - y_n}{|x_n - y_n|} = v.$$

It is easy to verify that the following holds

4.16. Lemma: *Let Ω be an open subset of $\mathbf{R}^n$. Then $\mathfrak{A}(B,x) = \{v \in \mathbf{R}^n : |v| = 1\}$ for all $x \in \mathrm{cl}\,\Omega$.*

Let $B \subseteq \mathbf{R}^n$, $f: B \to \mathbf{R}^n$. We set

$$l_B[f] \equiv \inf\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in B, \ x \neq y\right\}. \tag{4.17}$$

Note that $l_{\mathrm{cl}B}[f] = l_B[f]$. Clearly f is injective on B if $l_B[f] > 0$. We now show that under suitable circumstances, the converse is true.

4.18. THEOREM: *Let K be a compact subset of $\mathbf{R}^n$. Let $f \in (C^1_W(K))^n$. The following statements are equivalent.*

(i) $l_K[f] > 0$.

(ii) *f is injective and for at least an element $F \in (C^1(\mathbf{R}^n))^n$ such that $F_{|K} = f$, the following condition holds*

$$DF(x) \cdot v \neq 0, \quad \forall x \in K, \ \forall v \in \mathcal{A}(K,x). \tag{4.19}$$

(iii) *f is injective and for all elements $F \in (C^1(\mathbf{R}^n))^n$ such that $F_{|K} = f$, condition* (4.19) *holds.*

PROOF: (i) $\Rightarrow$ (iii). If $l_K[f] > 0$ then f is clearly injective. Assume that for some $F \in (C^1(\mathbf{R}^n))^n$, $F_{|K} = f$ and that there exist $x \in K$, $v \in \mathcal{A}(K,x)$ such that $DF(x) \cdot v = 0$. By definition of v, there exist sequences $\{x_n\}$, $\{y_n\}$ in K such that

$$\lim_n x_n = x = \lim_n y_n, \qquad \lim_n \frac{x_n - y_n}{|x_n - y_n|} = v.$$

Then

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \left| \int_0^1 DF(y_n + t(x_n - y_n)) \cdot \frac{x_n - y_n}{|x_n - y_n|} \, dt \right|.$$

By taking the limit as $n \to \infty$, we conclude that

$$\lim_n \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = 0,$$

in contradiction with the assumption $l_K[f] > 0$. The implication (iii) $\Rightarrow$ (ii) is obvious. We now prove (ii) $\Rightarrow$ (i). Assume by contradiction that there exists a sequence $\{(x_n, y_n)\}_{n \in \mathbf{N}}$ in $K \times K$ such that

$$x_n \neq y_n, \qquad \lim_n \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = 0.$$

By selecting a suitable subsequence, we can assume that there exist $\bar{x}, \bar{y} \in K$, $v \in \mathbf{R}^n \setminus \{0\}$ such that

$$\lim_k x_{n_k} = \bar{x}, \qquad \lim_k y_{n_k} = \bar{y}, \qquad \lim_n \frac{x_{n_k} - y_{n_k}}{|x_{n_k} - y_{n_k}|} = v.$$

By injectivity and continuity of f, we must have $\bar{x} = \bar{y}$. Moreover $v \in \mathcal{A}(K, \bar{x})$ by defini-

tion. Now let $F \in (C^1(\mathbf{R}^n))^n$, $F_{|K} = f$, then

$$\frac{|f(x_{n_k}) - f(y_{n_k})|}{|x_{n_k} - y_{n_k}|} = \left| \int_0^1 DF(y_{n_k} + t(x_{n_k} - y_{n_k})) \cdot \frac{x_{n_k} - y_{n_k}}{|x_{n_k} - y_{n_k}|}\, dt \right|.$$

By taking the limit as $k \to \infty$, we deduce that $0 = DF(\bar{x}) \cdot v$, which contradicts (ii). ■

4.20. Lemma: *Let Ω be a bounded connected open subset of $\mathbf{R}^n$. If $c[\Omega] < \infty$, then $C^1_W(\mathrm{cl}\,\Omega) = C^1(\mathrm{cl}\,\Omega)$.*

Proof: By definition $C^1_W(\mathrm{cl}\,\Omega) \subseteq C^1(\mathrm{cl}\,\Omega)$. Now, let $f \in C^1(\mathrm{cl}\,\Omega)$. By virtue of the Whitney characterization, it suffices to show that $\lim_{\delta \to 0} \rho_{\mathrm{cl}\,\Omega}(\delta) = 0$, where

$$\rho_{\mathrm{cl}\,\Omega}(\delta) \equiv \sup\{|R(y,x)| : 0 < |x - y| \leq \delta,\ x, y \in \mathrm{cl}\,\Omega\},$$

$$R(y,x) = \{[f(y) - f(x) - Df(x) \cdot (y - x)]|y - x|^{-1}\}.$$

By continuity of f, Df on $\mathrm{cl}\,\Omega$, it suffices to show that $\lim_{\delta \to 0} \rho_\Omega(\delta) = 0$. Let $\varepsilon > 0$. By uniform continuity of Df in $\mathrm{cl}\,\Omega$, there exists $\delta > 0$ such that $|Df(v) - Df(x)| \leq \varepsilon(2c[\Omega])^{-1}$ whenever $|v - x| < 2c[\Omega]\,\delta$, $v, x \in \mathrm{cl}\,\Omega$. Let $x, y \in \Omega$, $|y - x| < \delta$. By the connectedness of Ω and the definition of geodesic distance, there exists $\gamma \in C^1([0,1], \Omega)$ as in (2.1) such that $\int_0^1 |\gamma'|\, dt < 2\lambda(x,y)$. We now consider the function defined by $g_x(y) = f(y) - Df(x) \cdot (y - x)$. Clearly $g_x(\cdot) \in C^1(\mathrm{cl}\,\Omega)$, $Dg_x(y) = Df(y) - Df(x)$, $g_x(x) = f(x)$. Hence

$$(4.21) \qquad |R(y,x)| = \frac{|g_x(y) - g_x(x)|}{|y - x|} \leq |y - x|^{-1} \sup_{t \in [0,1]} |Dg_x(\gamma(t))| \int_0^1 |\gamma'|\, dt \leq$$

$$\leq 2\,\frac{\lambda(x,y)}{|y - x|} \sup_{t \in [0,1]} |Dg_x(\gamma(t))| \leq 2c[\Omega] \sup_{t \in [0,1]} |Dg_x(\gamma(t))|.$$

Clearly $|\gamma(t) - x| \leq \int_0^1 |\gamma'|\, dt \leq 2c[\Omega]\,\delta$. Hence,

$$|R(y,x)| \leq 2c[\Omega] \sup\{|Dg_x(v)| : v \in \mathrm{cl}\,\Omega, |v - x| \leq 2c[\Omega]\,\delta\} \leq \varepsilon,$$

and the proof is complete. ■

4.22. Lemma: *Let Ω be an open subset of $\mathbf{R}^n$, $p \in \Omega$, $f \in (C^1(\Omega))^n$. Then $|\det Df(p)|^{1/n} \geq l_\Omega[f]$.*

Proof: Let $v \in \mathbf{R}^n \setminus \{0\}$, $|v| = 1$. For all t such that $p + tv \in \Omega$, we have $l_\Omega[f] \leq |f(p + tv) - f(p)|\,|tv|^{-1}$. Hence, $|Df(p) \cdot v| \geq l_\Omega[f]$ and

$$(4.23) \qquad l_\Omega[f]^2 \leq v \cdot [(Df(p))^T (Df(p))\, v],$$

for all unit vectors $v \in \mathbf{R}^n$. Since the matrix $(Df(p))^T(Df(p))$ is symmetric, it has n real eigenvalues $\lambda_1, \ldots, \lambda_n$. By (4.23), $\lambda_i \geq 0$. Let $\lambda_1 = \min\{\lambda_i\}$. Clearly $l_\Omega[f]^2 \leq \lambda_1$. Then $l_\Omega[f]^{2n} \leq \lambda_1^n \leq (\det Df(p))^2$. ■

4.24. Corollary: *Let Ω be a bounded connected open subset of $\mathbf{R}^n$ such that $c[\Omega] < \infty$. Let $f \in (C^1(\mathrm{cl}\,\Omega))^n$. Then $l_{\mathrm{cl}\,\Omega}[f] > 0$ if and only if f is injective and $\det Df(x) \neq 0$ for all $x \in \mathrm{cl}\,\Omega$.*

Proof: By Lemmas 4.16, 4.20 and Theorem 4.18, $l_{\mathrm{cl}\,\Omega}[f] > 0$ if and only if f is injective and $Df(x) \cdot v \neq 0$ for all $x \in \mathrm{cl}\,\Omega$ and for all unit vectors $v \in \mathbf{R}^n$. ■

By Corollary 4.24, we clearly have

$$Y_{m,\alpha,0}(\Omega) = \{g \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : l_{\mathrm{cl}\,\Omega}[g] > 0\}. \tag{4.25}$$

4.26. Lemma: *Let Ω be a bounded connected open subset of $\mathbf{R}^n$ such that $c[\Omega] < \infty$. Let $g \in (C^1(\mathrm{cl}\,\Omega))^n$ satisfy $l_\Omega[g] > 0$. Then $g(\Omega)$ is open and*

$$c[g(\Omega)] \leq l_\Omega[g]^{-1} \|Dg: (C^0(\mathrm{cl}\,\Omega))^{n^2}\| \, c[\Omega]. \tag{4.27}$$

Proof: That $g(\Omega)$ is open is a well-known consequence of the injectivity and continuity of g. We denote by λ_Ω, $\lambda_{g(\Omega)}$ the geodesic distance in Ω, $g(\Omega)$ respectively. Clearly g establishes a one to one corrispondence between the C^1-curves in Ω and the C^1-curves in $g(\Omega)$. Then

$$\lambda_{g(\Omega)}(g(x), g(y)) = \inf\left\{\int_0^1 |(g \circ \gamma)'| \, dt : \gamma \in C^1([0,1], \Omega),\ \gamma(0) = x,\ \gamma(1) = y\right\} \leq$$

$$\leq \|Dg: (C^0(\mathrm{cl}\,\Omega))^{n^2}\| \, \lambda_\Omega(x, y),$$

and

$$c[g(\Omega)] \leq \|Dg: (C^0(\mathrm{cl}\,\Omega))^{n^2}\| \sup\left\{\frac{\lambda_\Omega(x,y)}{|x-y|} : x, y \in \Omega,\ x \neq y\right\} \cdot$$

$$\cdot \sup\left\{\frac{|x-y|}{|g(x) - g(y)|} : x, y \in \Omega,\ x \neq y\right\},$$

which implies (4.27). ■

By combining Lemmas 4.2, 4.22, 4.26, it is easy to deduce the following

4.28. Theorem: *Let Ω be a bounded connected open subset of $\mathbf{R}^n$ such that $c[\Omega] < \infty$. Let $\delta > 0$, $\tau > 0$, $\sigma > 0$, $g \in ((C^{m,\alpha}(\mathrm{cl}\,\Omega))^n$, $l_{\mathrm{cl}\,\Omega}[g] > \delta$, $\|g: (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n\| \leq \tau$, $\sup_{x \in \mathrm{cl}\,\Omega} |x| \leq \sigma$, $c[\Omega] < \sigma$. Then there exists a constant $c(\delta, m, n, \tau, \sigma)$*

depending only on δ, m, n, τ, σ, *such that* $\|g^{(-1)}: (C^{m,\alpha}(\mathrm{cl}\, g(\Omega)))^n\| \leq c(\delta, m, n, \tau, \sigma)$ *and the operator* J *defined in* (1.2) *is bounded on* $\{g \in ((C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : l_{\mathrm{cl}\,\Omega}[g] > \delta\}$.

Finally, we note that the following holds

4.29. Proposition: *Let* Ω *be a bounded connected open subset of* $\mathbf{R}^n$ *such that* $c[\Omega] < \infty$. *Then* $g \mapsto l_\Omega[g]$ *is continuous from* $(C^1(\mathrm{cl}\,\Omega))^n$ *equipped with the (gradient) seminorm* p *defined by* $p(g) \equiv \sum_{i=1}^{n} \|\partial g/\partial x_i : (C^0(\mathrm{cl}\,\Omega))^n\|$ *to* $\mathbf{R}$.

Proof: Let $\{g_n\}$ be a sequence in $(C^1(\mathrm{cl}\,\Omega))^n$, $g \in (C^1(\mathrm{cl}\,\Omega))^n$, $\lim_n p(g_n - g) = 0$. Let $\varepsilon > 0$. By definition of infimum, there exist $\bar{x}, \bar{y} \in \mathrm{cl}\,\Omega$ such that $|g(\bar{x}) - g(\bar{y})| \cdot |\bar{x} - \bar{y}|^{-1} < l_\Omega[g] + \varepsilon/2$. By assumption, there exists n_0 such that $p(g_n - g) < \varepsilon/(2c[\Omega])$ for all $n \geq n_0$. Then clearly $l_\Omega[g_n] \leq c[\Omega]\, p(g_n - g) + |g(\bar{x}) - g(\bar{y})|\, |\bar{x} - \bar{y}|^{-1} \leq l_\Omega[g] + \varepsilon$. If $l_\Omega[g] = 0$, then clearly $|l_\Omega[g_n] - l_\Omega[g]| \leq \varepsilon$. Now let $l_\Omega[g] > 0$. We can assume that $2\varepsilon < l_\Omega[g]$. Then for all $n \geq n_0$, $x \neq y$, $x, y \in \mathrm{cl}\,\Omega$, we have

$$\frac{|g_n(x) - g_n(y)|}{|x - y|} \geq \left| \frac{|g(x) - g(y)|}{|x - y|} - \frac{|[g_n - g](x) - [g_n - g](y)|}{|x - y|} \right| \geq$$

$$\geq l_\Omega[g] - c[\Omega]\, p(g_n - g) > l_\Omega[g] - \varepsilon/2\,,$$

and the proof is complete. ■

From Proposition 4.29 we can also deduce the following variant of Hirsch (1988, Lemma 1.3, p. 36).

4.30. Corollary: *Under the same assumptions of Proposition* 4.29, *the set* $Y_{1,0,0}(\Omega)$ *is open in* $(C^1(\mathrm{cl}\,\Omega))^n$ *equipped by the seminorm* p.

Remark: Theorem 4.18 is to the best of the author's knowledge new, although some of the arguments used to prove it resemble those used in Hirsch (1988) to prove Corollary 4.30 in a slightly different setting. Corollary 4.30 finds application in Elastostatics, where $Y_{1,0,0}(\Omega)$ represents the set of admissible deformations of a body Ω. That the affine maps of $\mathbf{R}^n$ into itself were interior points of $Y_{1,0,0}(\Omega)$, was observed by Valent (1988), who used a different method.

5. - The inversion operator J in the case $m = 0$

Considerably different from case $m \geq 1$ is the case $m = 0$. We first introduce the following Lemma.

5.1. Lemma: *Let* $\alpha \in (0, 1]$. *Let* Ω, Ω_1 *be bounded open subsets of* $\mathbf{R}^n$. *Let*

$f \in C^0(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$ *be a bijection of* $\mathrm{cl}\,\Omega$ *onto* $\mathrm{cl}\,\Omega_1$. *Let*

$$(5.2) \qquad l_{\mathrm{cl}\,\Omega}[\alpha^{-1}, f] \equiv \inf\left\{ \frac{|f(x)-f(y)|}{|x-y|^{1/\alpha}} : x, y \in \mathrm{cl}\,\Omega,\ x \neq y \right\}.$$

Then $l_{\mathrm{cl}\,\Omega}[\alpha^{-1}, f] = l_{\Omega}[\alpha^{-1}, f]$, *and* $f^{(-1)} \in C^{0,\alpha}(\mathrm{cl}\,\Omega_1, \mathrm{cl}\,\Omega)$ *if and only* $l_{\mathrm{cl}\,\Omega}[\alpha^{-1}, f] > 0$.

Proof: The continuity of $f^{(-1)}$ is well-known. Then the statement follows immediately from the following inequality

$$(5.3) \qquad \sup\left\{ \frac{|f^{(-1)}(u) - f^{(-1)}(v)|}{|u-v|^{\alpha}} : u, v \in \mathrm{cl}\,\Omega_1,\ u \neq v \right\} =$$

$$= \sup{}^{\alpha}\left\{ \frac{|x-y|^{1/\alpha}}{|f(x)-f(y)|} : x, y \in \mathrm{cl}\,\Omega,\ x \neq y \right\}. \qquad \blacksquare$$

Accordingly, we introduce the sets

$$(5.4) \qquad X_{0,\alpha,\delta}(\Omega, \Omega_1) \equiv \{ f \in C^{0,\alpha}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1) : f(\mathrm{cl}\,\Omega) = \mathrm{cl}\,\Omega_1, \text{ is injective, } l_{\mathrm{cl}\,\Omega}[\alpha^{-1}, f] > \delta \}.$$

From Lemmas 2.6, 4.8, and from (5.3), we deduce the following

5.5. Proposition: *Let* $\delta > 0$, $\alpha \in (0, 1]$. *Let* Ω, Ω_1 *be bounded open subsets of* $\mathbf{R}^n$. *The operator* J *defined in* (1.2) *is bounded from* $X_{0,\alpha,\delta}(\Omega, \Omega_1)$ *to* $C^{0,\alpha}(\mathrm{cl}\,\Omega_1, \mathrm{cl}\,\Omega)$ *and is continuous from* $X_{0,\alpha,\delta}(\Omega, \Omega_1)$ *to* $C^{0,\theta}(\mathrm{cl}\,\Omega_1, \mathrm{cl}\,\Omega)$ *for all* $\theta \in (0, \alpha)$.

It is now natural to ask whether the continuity holds for $\theta = \alpha$. The following counterexample shows that the answer is no and that Proposition 5.5 is sharp.

Example: Let $\alpha \in (0, 1)$, $n > 2$. Let $f(x) = 1 - (1-x)^{\alpha}$, if $x \in [0, 1]$, $f(x) = (x - 1)^{1/\alpha} + 1$ if $x \in [1, 2]$, $f(x) = x$ if $x \in [2, 3]$. Let $f_n(x) = (1 + n^{-\alpha}) f(x)$ if $x \in [0, 2]$, $f_n(x) = x(1 - 2n^{-\alpha}) + 6n^{-\alpha}$ if $x \in [2, 3]$. It is easy to recognize that $\lim_n f_n = f$ in $X_{0,\alpha,0}((0,3),(0,3))$ and that $f_n^{(-1)}, f^{(-1)} \in C^{0,\alpha}([0,3],[0,3])$. By using Lemmas 2.4, 3.1 it is easy to verify that $\sup_n \{ |f_n^{(-1)} : [0, 2(1 + n^{-\alpha})]|_{\alpha} \} < \infty$ and that $\sup_n \{ |f_n^{(-1)} : [2(1 + n^{-\alpha}), 3]|_{\alpha} \} < \infty$. Hence $\sup_n \{ |f_n^{(-1)} : [0,3]|_{\alpha} \} < \infty$ and by (5.3) there exists $\delta > 0$ such that $\inf_n l_{(0,3)}[\alpha^{-1}, f] > \delta > 0$. Hence $\lim_n f_n = f$ in $X_{0,\alpha,\delta}((0,3),(0,3))$. We now show that $\{ f_n^{(-1)} \}$ does not converge to $f^{(-1)}$ in $C^{0,\alpha}([0,3])$:

$$\liminf_n |f_n^{(-1)} - f^{(-1)} : [0,3]|_{\alpha} \geq \liminf_n \{ n^{\alpha^2} |f_n : [0,3]|_{\alpha}^{-\alpha} \cdot$$

$$\cdot |[f^{(-1)}(f(1)) - f^{(-1)}(f_n(1))] - [f^{(-1)}(f(1 - n^{-1})) - f^{(-1)}(f_n(1 - n^{-1}))]| \} =$$

$$= \liminf_n \{ |f_n : [0,3]|_{\alpha}^{-\alpha} | - 1 + n^{\alpha^2 - 1} - n^{\alpha^2 - 2} | \} \geq 2^{-1} \liminf_n |f_n : [0,3]|_{\alpha}^{-\alpha} > 0.$$

For $\alpha = 1$, a similar example can be constructed by considering $f_n(x) = (1 + (1/n))[x2^{-1} + \sup(x-1, 0)]$ if $x \in [0,2]$; $f_n(x) = x(1 - 2n^{-1}) + 6n^{-1}$, if $x \in [2,3]$.

REFERENCES

J. Appel (1988), *The superposition operator in function spaces. A survey*, Expo Math., 6, 209-270.

J. Appel - P. P. Zabrejko (1990), *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics 95, Cambridge University Press.

M. Z. Berkolajko (1969), *On a nonlinear operator acting in generalized Hölder spaces*, (Russian) Voronezh. Gos. Univ. Trudy Sem. Funk. Anal., 12, 96-104.

M. Z. Berkolajko - Ja. B. Rutitskij (1971), *Operators in generalized Hölder spaces*, (Russian) Sibir. Mat. Zhurn. 12, 5, 1015-1025; (= Siber. Math. J., 12, 5 (1971), 731-738).

V. A. Bondarenko - P. P. Zabrejko (1975), *The superposition operator in Hölder function spaces*, (Russian) Doklady Akad. Nauk SSSR, 222, 6, 1265-1268; (= Soviet. Math. Doklady, 16, 3 (1975), 739-743).

P. Drábek (1975), *Continuity of Nemyckii's operator in Hölder spaces*, Comm. Math. Univ. Carolinae, 16, 1, 37-57.

D. Gilbarg - N. S. Trudinger (1977), *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag.

M. W. Hirsch (1988), *Differential Topology*. Third Printing. Graduate texts in mathematics, Springer-Verlag.

A. Kufner - O. John - S. Fučik (1977), *Function Spaces*, Noordhoff International Publishing.

M. Lanza de Cristoforis (1987), *Nonlinear Deformation of Structures in Perfect Flows*, (Doctoral Dissertation), University Microfilm International.

M. Lanza de Cristoforis (1990), *A note on composition operators in Sobolev spaces and an extension of the chain rule*, Rend. Acc. Naz. delle Sci. detta dei XL, XIV, 18, pp. 341-360.

M. Lanza de Cristoforis (1991), *A note on conformal representation in Sobolev spaces*, to appear.

M. Lanza de Cristoforis - S. S. Antman (1991a), *The large deformation of nonlinearly elastic tubes in two dimensional flows*, to appear on SIAM J. Math. Anal. issue 5, vol. 22.

M. Lanza de Cristoforis - S. S. Antman (1991b), *The large deformation of nonlinearly elastic shells in axisymmetric flows*, to appear on Annales de l'Institut Henri Poincaré, Analyse non Linèaire.

T. Valent (1988), *Boundary value problems of finite elasticity—local theorems on existence, uniqueness and analytic dependence on data*, Springer Tract Nat. Philos, 31, Springer.

H. Whitney (1934), *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., 36, 63-89.