Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
109° (1991), Vol. XV, fasc. 4, pagg. 71-91

ANGELA IANNELLI (*)

Bounded and Almost-Periodic Solutions of a Navier-Stokes Type Equation (**) (***)

Soluzioni limitate e quasi-periodiche di un'equazione di tipo Navier-Stokes

RIASSUNTO. — Si illustrano alcune proprietà delle soluzioni limitate di un'equazione del tipo Navier-Stokes in tre dimensioni e si dànno alcune condizioni sufficienti a garantire che la soluzione limitata esista, sia unica e sia quasi-periodica.

1. - INTRODUCTION

In a recent work [5], Prouse has studied the Cauchy-Dirichlet problem in a bounded 3-dimensional domain for the Navier-Stokes type equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{\varphi}(\boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} + \nabla p - \nabla \operatorname{div} \boldsymbol{\varphi}(\boldsymbol{u}) = \boldsymbol{f}, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.2}$$

having denoted by $\boldsymbol{u}, p, \mu, \boldsymbol{f}$ respectively the fluid velocity, its pressure, its viscosity, the mass forces, and by $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ a function from R^3 in itself, of the form $\boldsymbol{\varphi}(\boldsymbol{u}) = \sigma(|\boldsymbol{u}|)\, \boldsymbol{u}$, with $\sigma \in C^1([0, +\infty))$.

The equations (1.1), (1.2) are deduced from the general equations of conservation of momentum by assuming that the relationship between the stress tensor $\mathrm{T} = \{\tau_{ij}\}$ and the deformation velocity tensor

$$\mathrm{S} = \{\eta_{ij}\} = \left\{ \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\}$$

(*) Indirizzo dell'Autore: Dipartimento di Matematica, Politecnico di Milano, Piazzale L. da Vinci 32, 20133 Milano.
(**) Memoria presentata il 19 dicembre 1990 da Luigi Amerio, uno dei XL.
(***) Lavoro eseguito nell'ambito dei gruppi M.U.R.S.T. 60%.

ISSN-0392-4106

is given by

$$(1.3) \qquad \tau_{ij} = \frac{1}{2}\left(\frac{\partial \varphi_i(\boldsymbol{u})}{\partial x_j} + \frac{\partial \varphi_j(\boldsymbol{u})}{\partial x_i}\right) - p\delta_{ij}, \qquad i,j = 1,2,3 .$$

It is obvious that, if $\sigma(\xi) = \mu$, (1.3) reduces to the usual linear relationship and equations (1.1), (1.2) to the classical Navier-Stokes equations.

While we refer to [5] for a more detailed discussion of the physical significance of (1.3), it should be pointed out that, by an appropriate choice of the function σ, it is possible to modify the classical linear relationship between **T** and **S** only for «large» values of the velocity, for which there is no experimental evidence that a linear law holds.

The main result proved in [5] consists in a global existence and uniqueness theorem for a weak solution of a Cauchy-Dirichlet problem posed for (1.1), (1.2).

The aim of the present paper is to illustrate some properties of the bounded solutions of (1.1), (1.2) and to state some conditions which allow to guarantee that these solutions exist and are almost periodic. More precisely, we shall study at first (sections 4 and 5) some asymptotic properties of the solutions of (1.1), (1.2) and prove, in particular, that if f is bounded and «small», there exists a unique solution bounded on **R** and «small» (sections 6 and 7). Subsequently (section 8), we shall study the almost-periodicity of the solutions, proving that, if f is «small» and almost-periodic, there exists a unique almost-periodic solution.

2. - Notations and basic definitions

Let Ω be an open, bounded domain in $\mathbf{R}^3$ with boundary $\Gamma = \partial\Omega$ of class C^2 (*) and denote by $\mathcal{V}$ the space of vectors $\boldsymbol{v} \in (C^\infty(\bar{\Omega}))^3$ with compact support and null divergence in Ω and by H, V respectively the closures of V in $(L^2(\Omega))^3$ and in $(H^1(\Omega))^3$, with $(\boldsymbol{u},\boldsymbol{v})_H = (\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)}$, $(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{u},\boldsymbol{v})_{H^1_0(\Omega)}$.

Assume moreover that the function $\sigma = \sigma(\xi)$ satisfies the following conditions:

$$(2.1) \qquad \sigma \in C^1([0,\bar{\xi})), \qquad 0 < \bar{\xi} \leq +\infty ;$$

$$(2.2) \qquad \sigma \geq \mu > 0, \qquad \sigma' \geq 0 ;$$

$$(2.3) \qquad \text{if } \bar{\xi} = +\infty, \text{ then } \sigma(\xi) \geq \alpha\xi^3 \ (\alpha > 0), \text{ when } \xi \geq \chi_1 ;$$

$$(2.4) \qquad \text{if } \bar{\xi} < +\infty, \text{ then } \lim_{\xi \to \bar{\xi}} \sigma(\xi) = +\infty .$$

We turn now the the definition of solution: assuming that $f \in L^2_{\mathrm{loc}}(-\infty, +\infty; V')$, we

(*) This assumption on $\Delta\Omega$ is made in order to ensure that $\boldsymbol{v} \in (H^2_0(\Omega))^3$, $\Delta\boldsymbol{v} \in (L^2(\Omega))^3$ imply $\boldsymbol{v} \in (H^2(\Omega))^3$; less stringent conditions are given by Grisvard [3].

shall say that $\boldsymbol{u}$ *is a weak solution of* (1.1), (1.2) *in* $\Omega \times \mathbf{R}$, *satisfying the boundary condition*

$$u|_{\Gamma \times \mathbf{R}} = 0, \tag{2.5}$$

if:

$$\begin{gathered} u \in L^2_{\mathrm{loc}}(-\infty, +\infty; V) \cap L^5_{\mathrm{loc}}(-\infty, +\infty; (L^5(\Omega))^3) \cap H^1_{\mathrm{loc}}(-\infty, +\infty; H), \\ \Delta\varphi(u) = \Delta\sigma(|u|)\, u \in L^2_{\mathrm{loc}}(-\infty, +\infty; (H^2(\Omega))'); \end{gathered} \tag{2.6}$$

$$\int_{t_1}^{t_2} \left\{ \left(\frac{\partial u}{\partial \eta}(\eta), b(\eta) \right)_{L^2(\Omega)} - (\varphi(u(\eta)), \Delta b(\eta))_{L^2(\Omega)} + \right. \\ \left. + b(u(\eta), u(\eta), b(\eta)) - (f(\eta), b(\eta))_{L^2(\Omega)} \right\} d\eta = 0, \tag{2.7}$$

$\forall t_1, t_2 \in \mathbf{R}$, $t_1 < t_2$, $\forall b \in L^2_{\mathrm{loc}}(-\infty, +\infty; (H^2(\Omega))^3 \cap V)$, having denoted by b the nonlinear form

$$b(u, v, w) = \int_\Omega \left(\sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j \right) d\Omega .$$

Remark 2.1: If u satisfies (2.6), then by Sobolev imbedding theorems (see [1], ch. V), $(u \cdot \nabla) u \in L^2_{\mathrm{loc}}(-\infty, +\infty; (L^1(\Omega))^3)$ and consequently $\int_{t_1}^{t_2} b(u, u, b)\, d\eta$ exists.

3. - Some already known results

We collect here some already known results that we will use in the sequel. First, we recall a relation that is a special case of a more general one, due to Prodi [4], and, for this, denote by G the Green's operator, from V' to V, relative to $-\Delta$. Then

Lemma 3.1 (see [5], sect. 3, lemma 1): *Let* $v \in (L^5(\Omega))^3$, $u \in H$: *then*, $\forall \mu > 0$,

$$|\langle (u \cdot \nabla) v, Gu \rangle| \leq \frac{\mu}{4} \|u\|^2_{L^2(\Omega)} + \delta \|v\|^5_{L^5(\Omega)} \|u\|^2_{V'}, \tag{3.1}$$

$$|\langle (u \cdot \nabla) v, Gv \rangle| \leq \frac{\mu}{4} \|v\|^2_{L^2(\Omega)} + \delta \|u\|^5_{L^5(\Omega)} \|v\|^2_{V'}, \tag{3.2}$$

where δ *is an appropriate positive constant depending on* μ *and* Ω. ■

Now, we turn to the main result of [5]:

Theorem 3.1 (see [5], sect. 4, theorems 2, 3, 4): *Assume that* σ *satisfies assumptions* (2.1)-(2.3) *or* (2.1), (2.2), (2.4) *and that* $f \in H^1(0, T; V')$, $u_0 \in V \cap (L^\infty(\Omega))^3$. *The-*

re exists then in $Q = [0,T] \times \Omega$ one and only one function $\boldsymbol{u}$ satisfying:

$$
\begin{aligned}
&\boldsymbol{u} \in L^2(0,T;V) \cap L^5(0,T;(L^5(\Omega))^3) \cap H^1(0,T;H), \\
&\Delta\varphi(\boldsymbol{u}) \in L^2(0,T;(H^2(\Omega))');
\end{aligned}
\tag{3.3}
$$

$$
\int_0^T \left\{ \left(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{f}, \boldsymbol{h} \right)_{L^2(\Omega)} + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{h}) - (\varphi(\boldsymbol{u}), \Delta \boldsymbol{h})_{L^2(\Omega)} \right\} d\eta = 0, \tag{3.4}
$$

$$
\forall \boldsymbol{h} \in L^2(0,T;(H^2(\Omega))^3 \cap V),
$$

$$
\boldsymbol{u}(0) = \boldsymbol{u}_0. \quad \blacksquare \tag{3.5}
$$

Remark 3.1: Since the function $\boldsymbol{u}$, whose existence and uniqueness are proved in Theorem 3.1, is a global solution of the Cauchy-Dirichlet problem and $[0,T]$ is an arbitrary interval, then, by an appropriate choice of the test function, one can deduce that $\boldsymbol{u}$ satisfies (2.6), (2.7), that is $\boldsymbol{u}$ is a weak solution of (1.1)-(1.2). $\blacksquare$

Remark 3.2: Again in ([5], sect. 4, remark 2), the solution $\boldsymbol{u}$ is stressed to verify the following «energy» estimates:

$$
\frac{1}{2}\|\boldsymbol{u}(t_2)\|_H^2 - \frac{1}{2}\|\boldsymbol{u}(t_1)\|_H^2 + \mu \int_{t_1}^{t_2} \|\boldsymbol{u}(\eta)\|_V^2 \, d\eta \le \int_{t_1}^{t_2} |\langle \boldsymbol{f}(\eta), \boldsymbol{u}(\eta) \rangle| \, d\eta, \tag{3.6}
$$

$$
\forall t_1, t_2 \in \mathbf{R}, \; t_1 < t_2;
$$

$$
\frac{1}{2}\|\boldsymbol{u}(t_2)\|_{V'}^2 - \frac{1}{2}\|\boldsymbol{u}(t_1)\|_{V'}^2 + \alpha \int_{t_1}^{t_2} \|\boldsymbol{u}(\eta)\|_{L^5(\Omega)}^5 \, d\eta \le \tag{3.7}
$$

$$
\le \int_{t_1}^{t_2} |b(\boldsymbol{u}(\eta), \boldsymbol{u}(\eta), G\boldsymbol{u}(\eta)) - \langle \boldsymbol{f}(\eta), G\boldsymbol{u}(\eta) \rangle| \, d\eta, \quad \forall t_1, t_2 \in \mathbf{R}, \; t_1 < t_2;
$$

$$
\frac{1}{2}\|\boldsymbol{u}'(t_2)\|_{V'}^2 - \frac{1}{2}\|\boldsymbol{u}'(t_1)\|_{V'}^2 + \mu \int_{t_1}^{t_2} \|\boldsymbol{u}'(\eta)\|_H^2 \, d\eta \le \tag{3.8}
$$

$$
\le \int_{t_1}^{t_2} |b(\boldsymbol{u}'(\eta), \boldsymbol{u}(\eta), G\boldsymbol{u}'(\eta)) + b(\boldsymbol{u}(\eta), \boldsymbol{u}'(\eta), G\boldsymbol{u}'(\eta))| \, d\eta + \int_{t_1}^{t_2} |\langle \boldsymbol{f}'(\eta), G\boldsymbol{u}'(\eta) \rangle| \, d\eta,
$$

for almost all $t_1, t_2 \in \mathbf{R}$, $t_1 < t_2$. $\blacksquare$

4. - Some auxiliary results

We prove here some preliminary lemmas and theorems. In the sequel, we shall suppose that the function σ satisfies (2.1)-(2.3) or (2.1), (2.2), (2.4).

Lemma 4.1: *Let $f \in L^2_{\text{loc}}(-\infty, +\infty; (L^2(\Omega))^3) \cap H^1_{\text{loc}}(-\infty, +\infty; V')$ and let $\boldsymbol{u}$ be a weak solution of* (1.1), (1.2). *Then $\boldsymbol{u}$ satisfies the following inequalities:*

$$\text{(4.1)} \qquad \frac{1}{2}\|\boldsymbol{u}(t_2)\|_H^2 - \frac{1}{2}\|\boldsymbol{u}(t_1)\|_H^2 + \mu \int_{t_1}^{t_2} \|\boldsymbol{u}(\eta)\|_V^2 \, d\eta \le$$

$$\le \int_{t_1}^{t_2} \|\boldsymbol{u}(\eta)\|_H \|f(\eta)\|_{L^2(\Omega)}, \qquad \forall t_1, t_2 \in \mathbf{R}, \ t_1 < t_2;$$

$$\text{(4.2)} \qquad \frac{1}{2}\|\boldsymbol{u}(t_2)\|_{V'}^2 - \frac{1}{2}\|\boldsymbol{u}(t_1)\|_{V'}^2 + \alpha \int_{t_1}^{t_2} \|\boldsymbol{u}(\eta)\|_{L^5(\Omega)}^5 \, d\eta \le$$

$$\le c_1 \int_{t_1}^{t_2} \|\boldsymbol{u}(\eta)\|_H^2 \|\boldsymbol{u}(\eta)\|_V \, d\eta + \int_{t_1}^{t_2} \|\boldsymbol{u}(\eta)\|_{V'} \|f(\eta)\|_{V'} \, d\eta, \qquad \forall t_1, t_2 \in \mathbf{R}, \ t_1 < t_2,$$

with c_1 positive constant;

$$\text{(4.3)} \qquad \frac{1}{2}\|\boldsymbol{u}'(t_2)\|_{V'}^2 - \frac{1}{2}\|\boldsymbol{u}'(t_1)\|_{V'}^2 + \frac{\mu}{2} \int_{t_1}^{t_2} \|\boldsymbol{u}'(\eta)\|_H^2 \, d\eta \le$$

$$\le c_2 \int_{t_1}^{t_2} \|\boldsymbol{u}'(\eta)\|_{V'}^2 \|\boldsymbol{u}(\eta)\|_{L^5(\Omega)}^5 \, d\eta + \int_{t_1}^{t_2} \|\boldsymbol{u}'(\eta)\|_{V'} \|f'(\eta)\|_{V'} \, d\eta,$$

for almost all $t_1, t_2 \in \mathbf{R}$, $t_1 < t_2$, and with c_2 positive constant.

Proof: The function $\boldsymbol{u}$, being a weak solution of (1.1), (1.2), satisfies (3.6), (3.7), (3.8) by Remark 3.2. The deduction of (4.1) from (3.6) is trivial. About (4.2) observe that, by Holder's inequality and Sobolev imbedding theorems, it follows:

$$\text{(4.4)} \qquad |b(\boldsymbol{u}, \boldsymbol{u}, G\boldsymbol{u})| = |b(\boldsymbol{u}, G\boldsymbol{u}, \boldsymbol{u})| = \left| \int_\Omega (\boldsymbol{u} \cdot \nabla)\, G \cdot \boldsymbol{u} \, d\Omega \right| \le$$

$$\le \|\boldsymbol{u}\|_H \|\boldsymbol{u}\|_H \|G\boldsymbol{u}\|_{W^{1,\infty}} = \|\boldsymbol{u}\|_H^2 \|\boldsymbol{u}\|_{W^{-1,\infty}} \le c_1 \|\boldsymbol{u}\|_H^2 \|\boldsymbol{u}\|_V,$$

with c_1 imbedding constant of $H^1(\Omega)$ into $W^{-1,\infty}(\Omega)$.

Moreover,

$$\text{(4.5)} \qquad |\langle f, G\boldsymbol{u} \rangle| \le \|f\|_{V'} \|G\boldsymbol{u}\|_V = \|f\|_{V'} \|\boldsymbol{u}\|_{V'}.$$

Relations (4.4), (4.5) allow us to deduce (4.2) from (3.7). The inequality (4.3) follows by (3.8), observing that, by Lemma 3.1,

$$(4.6)\qquad |b(u',u,Gu')+b(u,u',Gu')|\le \frac{\mu}{2}\|u'\|_H^2+2\delta\|u\|_{L^5(\Omega)}^5\|u'\|_{V'}^2. \qquad ■$$

LEMMA 4.2: *Assume that $f\in L^2_{\rm loc}(0,+\infty;(L^2(\Omega))^3)$, $f'\in L^2_{\rm loc}(0,+\infty;V')$and*

$$(4.7)\qquad \sup_{t\ge 1}\int_{t-1}^{t}\|f(\eta)\|_{L^2(\Omega)}\,d\eta=K<+\infty.$$

Assume moreover that u is a weak solution of (1.1), (1.2) *in* $[0,+\infty)\times\Omega$. *Then, denoting by $[\bar t-1,\bar t]$ an arbitrary interval in $[0,+\infty)$, the only two following cases can be verified:*

i) $\displaystyle\max_{\bar t-1\le t\le \bar t}\|u(t)\|_H\le K/c,$

for an appropriate constant c, sufficiently small, depending only on Ω and μ,

ii) $\displaystyle\max_{\bar t-1\le t\le \bar t}\|u(t)\|_H^2\ge\beta\|u(\bar t)\|_H^2,$

for an appropriate value $\beta>1$.

PROOF: Let $[\bar t-1,\bar t]$ be an arbitrary interval with $\bar t\ge 1$; moreover, let us fix arbitrarily $\beta>1$ and a positive constant c smaller than $1/2\beta$. Setting

$$U=\max_{\bar t-1\le t\le \bar t}\|u(t)\|_H,$$

assume that

$$(4.8)\qquad U>\frac{K}{c},$$

$$(4.9)\qquad U^2<\beta\|u(\bar t)\|_H^2.$$

By (4.1), (4.8), (4.9), we have for any $\tilde t\in[\bar t-1,\bar t]$

$$\|u(\tilde t)\|_H^2\ge\|u(\bar t)\|_H^2-2UK>\|u(\bar t)\|_H^2-2K\beta\|u(\bar t)\|_H^2/U>(1-2\beta c)\|u(\bar t)\|_H^2.$$

Consequently,

$$\int_{\bar t-1}^{\bar t}\|u(\eta)\|_V^2d\eta\ge\frac{1}{\rho^2}(1-2\beta c)\|u(\bar t)\|_H^2,$$

being ρ the imbedding constant of $H^1(\Omega)$ into $L^2(\Omega)$. If we take

$$(4.10)\qquad \lambda\ge\frac{\rho^2}{\mu(1-2\beta c)},$$

then

$$\int_{\bar{t}-1}^{\bar{t}} \|u(\eta)\|_H^2 d\eta \geq \frac{1}{\lambda\mu} \|u(\bar{t})\|_H^2 . \tag{4.11}$$

Therefore, if λ satisfies (4.10), by (4.1), (4.8), (4.9), (4.11), we have

$$\|u(\bar{t})\|_H^2 \leq \|u(\bar{t}-1)\|_H^2 - \frac{2}{\lambda} \|u(\bar{t})\|_H^2 + 2\beta c \|u(\bar{t})\|_H^2 , \tag{4.12}$$

and consequently

$$(1 - 2\lambda - 2\beta c)\|u(\bar{t})\|_H^2 \leq \|u(\bar{t}-1)\|_H^2 .$$

Hence, if $1 + 2/\lambda - 2\beta c \geq \beta$, that is

$$\lambda \leq \frac{2}{\beta + 2\beta c - 1} , \tag{4.13}$$

by (4.12), we obtain

$$\|u(\bar{t}-1)\|_H^2 \geq \beta \|u(\bar{t})\|_H^2 . \tag{4.14}$$

We remark that (4.14) holds provided that both (4.10) and (4.13) are verified, that is

$$\frac{\rho^2}{\mu(1-2\beta c)} \leq \frac{2}{\beta + 2\beta c - 1} ,$$

and this last inequality holds provided that c is taken «sufficiently small» and β is «sufficiently close» to 1.

Relation (4.14) contradicts (4.9); hence, (4.8), (4.9) cannot hold at the same time. The theorem is proved. ∎

Lemma 4.3: *If the assumptions made in Lemma 4.2 hold, then*

$$M = \max \lim_{t \to +\infty} \|u(t)\|_H \leq \frac{K}{c} . \tag{4.15}$$

Proof: Let $\max \lim_{t \to +\infty} \|u(t)\|_H > K/c$. Then, there exists a sequence $\{t_n\}$ divergent to $+\infty$, such that

$$\|u(t_n)\|_H > \frac{K}{c} , \qquad \forall n \in \mathrm{N}.$$

Denote by t^* a point of this sequence and examine the interval $[t^* - 1, t^*]$. By Lemma 4.2, case ii), there exists $t_1^* \in [t^* - 1, t^*)$, such that:

$$\|u(t_1^*)\|_H^2 \geq \beta \|u(t^*)\|_H^2 > \beta \frac{K^2}{c^2} . \tag{4.16}$$

Since by (4.16)

$$\|u(t_1^*)\|_H^2 > \beta \frac{K^2}{c^2} ,$$

we can repeat the same argument about the interval $[t_1^* - 1, t_1^*]$. Therefore, there exists $t_2^* \in [t_1^* - 1, t_1^*)$ such that

$$\|u(t_2^*)\|_H^2 \geq \beta \|u(t_1^*)\|_H^2 > \beta^2 \frac{K^2}{c^2}.$$

In this way, one can construct a sequence of points $\{t_n^*\}$, such that $t_{n+1}^* < t_n^*$, $t_n^* - t_{n+1}^* \leq 1$ and

$$\|u(t_n^*)\|_H^2 \geq \beta^n \frac{K^2}{c^2}.$$

Since t_1^* is arbitrarily large, $\|u(t)\|_H$ is unbounded in a neighbourhood of $t = 0$; but this is absurd, since an existence theorem in $[0, +\infty) \times \Omega$ (see Theorem 3.1) holds. ■

Lemma 4.4: *If the assumptions of Lemma* 4.2 *hold, then*

$$M^* = \sup_{t \geq 0} \|u(t)\|_H \leq \max\left(\max_{0 \leq t \leq 1} \|u(t)\|_H,\ K/c \right).$$

Proof: Assume that

$$M^* > \max_{0 \leq t \leq 1} \|u(t)\|_H, \tag{4.17}$$

$$M^* > \frac{K}{c}. \tag{4.18}$$

Then, by Lemma 4.3, there exists an interval $[\bar{t} - 1, \bar{t}]$ with $\bar{t} \geq 2$, such that

$$\max_{\bar{t} - 1 \leq t \leq \bar{t}} \|u(t)\|_H = M^*.$$

Therefore, there exists $t^* \in [\bar{t} - 1, \bar{t}]$ and $\beta > 1$ such that

$$\beta \|u(t^*)\|_H^2 > M^{*2} \geq \max_{t^* - 1 \leq t \leq t^*} \|u(t)\|_H^2.$$

This last inequality contradicts Lemma 4.2, case ii); on the other hand, (4.18) excludes case i). Hence, (4.17), (4.18) cannot hold at the same time. This proves the assert. ■

Lemma 4.5: *Assume that* $f \in L^2_{\text{loc}}(0, +\infty; (L^2(\Omega))^3)$, $f' \in L^2_{\text{loc}}(0, +\infty; V')$ *and*

$$\sup_{t \geq 1} \int_{t-1}^{t} \|f'(\eta)\|_{V'}\, d\eta < +\infty.$$

Assume moreover that u *is a weak solution of* (1.1), (1.2) *in* $[0, +\infty) \times \Omega$ *and*

$$\sup_{t \geq 1} \int_{t-1}^{t} \|u(\eta)\|_{L^5(\Omega)}^5\, d\eta \leq c',$$

with c' *positive constant. Then denoting by* $[\bar{t} - 1, \bar{t}]$ *an arbitrary interval in* $[0, +\infty)$, *if* c'

is sufficiently small, the two following cases only can be verified:

i) $\operatorname*{ess\,sup}_{\bar t-1\le t\le \bar t} \|\boldsymbol{u}'(t)\| \le K'/c,$

for an appropriate constant $c>0$ depending only on Ω and μ,

ii) $\operatorname*{ess\,sup}_{\bar t-1\le t\le \bar t} \|\boldsymbol{u}'(t)\|^2_{V'} \ge \beta \|\boldsymbol{u}'(\bar t)\|^2_{V'},$

for an appropriate value $\beta>1$ and for almost all $\bar t \geq 1$.

PROOF: Let $[\bar t-1,\bar t]$ be an interval with $\bar t\ge 1$ and let us fix $\beta>1$ and $c>0$ arbitrarily for the time being. Setting

$$U=\operatorname*{ess\,sup}_{\bar t-1\le t\le \bar t} \|\boldsymbol{u}'(t)\|_{V'},$$

assume that

$$U>K'/c, \tag{4.19}$$

$$U^2<\beta\|\boldsymbol{u}'(\bar t)\|^2_{V'}. \tag{4.20}$$

Since Lemma 4.1 holds, by (4.3), (4.20), for almost all $t\in[\bar t-1,\bar t]$ we have:

$$\|\boldsymbol{u}'(t)\|^2_{V'} \ge \|\boldsymbol{u}'(\bar t)\|^2_{V'} - 2(c_2c'+K'/U)\,U^2 > (1-2\beta c_2 c'-2\beta c)\|\boldsymbol{u}'(\bar t)\|^2_{V'}. \tag{4.21}$$

On the other hand, if c' is sufficiently small, we can take c such that $1-2\beta c_2c'-2\beta c>0$ in order to have

$$\int_{\bar t-1}^{\bar t} \|\boldsymbol{u}'(\eta)\|^2_H d\eta \ge \frac{1}{\rho^2}(1-2\beta c_2 c'-2\beta c)\|\boldsymbol{u}'(\bar t)\|^2_{V'},$$

having denoted by ρ the imbedding constant of $L^2(\Omega)$ in $H^{-1}(\Omega)$. Therefore, if

$$\lambda>\frac{\rho^2}{\mu(1-2\beta c_2c'-2\beta c)}, \tag{4.22}$$

then

$$\int_{\bar t-1}^{\bar t} \|\boldsymbol{u}'(\eta)\|^2_H d\eta > \frac{1}{\lambda\mu}\|\boldsymbol{u}'(\bar t)\|^2_{V'}. \tag{4.23}$$

Consequently, if λ satisfies (4.22), by (4.3), (4.19), (4.20), (4.23), we have

$$\|\boldsymbol{u}'(\bar t)\|^2_{V'} \le U^2-\frac{1}{\lambda}\|\boldsymbol{u}'(\bar t)\|^2_{V'}+2\beta(c_2c'+c)\|\boldsymbol{u}'(\bar t)\|^2_{V'}$$

that is

$$\left(1+\frac{1}{\lambda}-2\beta c_2 c'-2\beta c\right)\|\boldsymbol{u}'(\bar t)\|^2_{V'} \le U^2. \tag{4.24}$$

Hence, if $1+1/\lambda-2\beta c_2 c'-2\beta c\geq\beta$, that is

$$\lambda\leq(\beta+2\beta c_2 c'+2\beta c-1)^{-1}, \tag{4.25}$$

by (4.24), it follows that

$$U^2\geq\beta\|u'(\bar{t})\|^2_{V'}. \tag{4.26}$$

We remark that (4.26) holds provided that both (4.22) and (4.25) are verified, that is

$$\frac{\rho^2}{\mu(1-2\beta c_2 c'-\beta c)}<\frac{1}{\beta+2\beta c_2 c'+2\beta c-1}$$

and this last inequality holds provided that c' and c are «sufficiently small» and β is «sufficiently close» to 1.

Observe now that (4.26) contradicts (4.20); hence, (4.19) and (4.20) cannot hold at the same time. ■

Lemma 4.6: *If the assumptions of Lemma 4.5 hold, then*

$$M=\max\lim_{t\to+\infty}\|u'(t)\|_{V'}\leq K'/c.$$

Proof: Suppose $\max\lim_{t\to+\infty}\|u'(t)\|_{V'}>K'/c$. Then, there exists a sequence $\{t_n\}$ divergent to $+\infty$ such that

$$\|u'(t_n)\|_{V'}>\frac{K'}{c},\qquad\forall n\in\mathrm{N}.$$

Denote by t^* a point of this sequence and examine the interval $[t^*-1,t^*]$. By Lemma 4.5 ii), there exists $t_1^*\in[t^*-1,t^*)$ such that:

$$\|u'(t_1^*)\|^2_{V'}\geq\gamma\|u'(t^*)\|^2_{V'}>\gamma\frac{K'^2}{c^2}, \tag{4.27}$$

where γ is a constant such that $1<\gamma\leq\beta$.

Since by (4.27)

$$\|u'(t_1^*)\|_{V'}>\frac{K'}{c},$$

we can repeat the same argument about the interval $[t_1^*-1,t_1^*]$. Therefore, there exists $t_2^*\in[t_1^*-1,t_1^*)$ such that

$$\|u'(t_2^*)\|^2_{V'}>\gamma^2\frac{(K')^2}{c^2}.$$

We construct in this way a sequence of points $\{t_n^*\}$ such that $t_{n+1}^*<t_n^*$, $t_n^*-t_{n+1}^*\leq1$ and

$$\|u'(t_n^*)\|^2_{V'}>\gamma^n\frac{(K')^2}{c^2}.$$

Since t_1^* is arbitrarily large, $\|u'(t)\|_{V'}$ is essentially unbounded in a neighbourhood of $t=0$; but this is absurd, since an existence theorem holds in $[0,\infty)\times\Omega$ (see Theorem 3.1). ■

Lemma 4.7: *Under the assumptions of Lemma 4.5,*

$$M' = \operatorname*{ess\,sup}_{t\geq 0} \|u'(t)\|_{V'} \leq \max\left(\operatorname*{ess\,sup}_{0\leq t\leq 1} \|u'(t)\|_{V'},\ \frac{K'}{c}\right).$$

Proof: The proof can be carried out analogously to which of Lemma 4.4, by substituting K, $\|u\|_H$ by K', $\|u\|_{V'}$, and referring to Lemmas 4.5, 4.6 instead of Lemmas 4.2, 4.3. ■

5. - Basic results about bounded solutions

We present now some basic results about bounded solutions.

Theorem 5.1: *Assume that* $f \in L^2_{\text{loc}}(0, +\infty; (L^2(\Omega))^3)$, $f' \in L^2_{\text{loc}}(0, +\infty; V')$ *and*

$$\sup_{t\geq 1} \int_{t-1}^{t} \|f(\eta)\|_{L^2(\Omega)}\, d\eta = K < +\infty. \tag{5.1}$$

Assume moreover that σ *satisfies* (2.1)-(2.3) *or* (2.1), (2.2), (2.4), u *is the weak solution of* (1.1), (1.2) *in* $[0, +\infty) \times \Omega$ *corresponding to the initial condition* $u(0) = 0$. *Then, there exists a constant* γ_1, *depending only on* Ω *and* μ, *such that*

$$\|u\|_{L^\infty(0,+\infty;H)} \leq \gamma_1 K. \tag{5.2}$$

Moreover, the solution u *is bounded in* $L^2_{\text{loc}}(0, +\infty; V) \cap L^5_{\text{loc}}(0, +\infty, (L^5(\Omega))^3)$ *too, and precisely, for any* $t \geq 1$, *the following estimates hold:*

$$\int_{t-1}^{t} \|u(\eta)\|^2_V\, d\eta \leq \gamma_2 K^2, \tag{5.3}$$

with $\gamma_2 = (\gamma_1^2 + \gamma_1)/\mu$;

$$\int_{t-1}^{t} \|u(\eta)\|^5_{L^5(\Omega)}\, d\eta \leq \gamma_3 (K^2 + K^3), \tag{5.4}$$

with γ_3 *positive constant depending on* Ω, α *and* γ_1.

Proof: Let t^* be the point in $(0, 1]$ in which $\|u(t)\|_H$ takes the maximum value. Then, by (4.1)

$$\|u(t^*)\|^2_H \leq 2 \int_0^{t^*} \|u(\eta)\|_H \|f(\eta)\|_{L^2(\Omega)}\, d\eta \leq 2\|u(t^*)\|_H K,$$

that is

$$\|u(t^*)\|_H \leq 2K. \tag{5.5}$$

By Lemma 4.4 and inequality 5.5, it follows that (5.2) holds. By (4.1) with $t_1 = t-1$, $t_2 = t$, $t \geq 1$, bearing in mind (5.1) and (5.2), we obtain:

$$\mu \int_{t-1}^{t} \|u(\eta)\|_V^2 d\eta \leq \gamma_1^2 K^2 + \gamma_1 K^2 .$$

and, consequently, (5.3). Moreover, (4.2) implies

$$(5.6) \qquad \alpha \int_{t-1}^{t} \|u(\eta)\|_{L^5(\Omega)}^5 d\eta \leq \rho^2 \gamma_1^2 K^2 + c_1 \gamma_1^2 K^2 \int_{t-1}^{t} \|u(\eta)\|_V d\eta + \rho^2 \gamma_1 K \int_{t-1}^{t} \|f(\eta)\|_{L^2(\Omega)} d\eta ,$$

having denoted by ρ the imbedding constant of $L^2(\Omega)$ in $H^{-1}(\Omega)$. The Holder's inequality and (5.3) yield:

$$(5.7) \qquad \int_{t-1}^{t} \|u(\eta)\|_V d\eta \leq \left(\int_{t-1}^{t} \|u(\eta)\|_V^2 d\eta \right)^{1/2} \leq \sqrt{\gamma_2} K .$$

Therefore, by (5.1), (5.6), (5.7), we deduce:

$$\alpha \int_{t-1}^{t} \|u(\eta)\|_{L^5(\Omega)}^5 d\eta \leq \rho^2 \gamma_1^2 K^2 + c_1 \gamma_1^2 \sqrt{\gamma_2} K^3 + \rho^2 \gamma_1 K^2 . \quad \blacksquare$$

Theorem 5.2: *Suppose that the assumptions of Theorem* 5.1 *hold and, moreover, that* (5.1) *holds with* K *«sufficiently small» and that*

$$\sup_{t \geq 1} \int_{t-1}^{t} \|f'(\eta)\|_{V'} d\eta = K' < +\infty .$$

Then the solution u *of* (1.1), (1.2), *corresponding to the initial condition* $u(0) = 0$, *verifies for almost all* $t \in [0, +\infty)$ *the following bounds*:

$$(5.8) \qquad \|u\|_{L^\infty(0,+\infty,V')} \leq \gamma_4 ,$$

$$(5.9) \qquad \int_{t-1}^{t} \|u'(\eta)\|_H^2 d\eta \leq \gamma_5 ,$$

where γ_4, γ_5 *are positive constants depending on* μ, Ω, α, K, K' *and*

$$\lim_{K,K' \to 0} \gamma_4 = 0 .$$

Proof: By Theorem 5.1, the inequalities (5.2), (5.3), (5.4) hold. Moreover, being K small $\int_{t-1}^{t} \|u(\eta)\|_{L^5(\Omega)}^5 d\eta$ is small too; therefore we can suppose all the assumptions of Lemma 4.5 are satisfied. Consequently, Lemma 4.7 asserts that

$$\|u'\|_{L^\infty(0,+\infty;V')} \leq \max\left(\operatorname*{ess\,sup}_{0\leq t\leq 1} \|u'(t)\|_{V'},\ \frac{K'}{\nu}\right), \tag{5.10}$$

for an apposite constant ν. On the other hand, by (4.3), we have:

$$\|u'(t)\|_{V'}^2 \leq \|u'(0)\|_{V'}^2 + 2c_2 \operatorname*{ess\,sup}_{0\leq t\leq 1} \|u'(t)\|_{V'}^2 \int_0^1 \|u(\eta)\|_{L^5(\Omega)}^5 d\eta + 2K' \operatorname*{ess\,sup}_{0\leq t\leq 1} \|u'(t)\|_{V'},$$

for almost all $t \in (0,1]$. Furthermore, by (5.4) and observing that, as can be deduced directly from Faedo-Galerkin approximations (see [5])

$$\|u'(0)\|_{V'} = \|f(0)\|_{V'},$$

we obtain

$$\|u'(t)\|_{V'}^2 \leq \|f(0)\|_{V'}^2 + 2c_2\gamma_3(K^2+K^3)\,U^2 + 2K'\,U, \tag{5.11}$$

having denoted by U the number $\operatorname*{ess\,sup}_{0\leq t\leq 1} \|u'(t)\|_{V'}$. By (5.11), we have:

$$U^2 - 2c_2\gamma_3(K^2+K^3)\,U^2 \leq \|f(0)\|_{V'}^2 + 2K'^2 + \frac{U^2}{2},$$

that is

$$\{1 - 4c_2\gamma_3(K^2+K^3)\}\,U^2 \leq 2\|f(0)\|_{V'}^2 + 4K'^2.$$

More explicitly,

$$\operatorname*{ess\,sup}_{0\leq t\leq 1} \|u'(t)\|_{V'} \leq \left(\frac{2\|f(0)\|_{V'}^2 + 4K'^2}{1 - 4c_2\gamma_3(K^2+K^3)}\right)^{1/2},$$

provided that K is sufficiently small. Hence, setting

$$\gamma_4 = \max\left\{\left(\frac{2\|f(0)\|_{V'}^2 + 4K'^2}{1 - 4c_2\gamma_3(K^2+K^3)}\right)^{1/2},\ \frac{K'}{\nu}\right\},$$

by (5.10), we have (5.8). Moreover, again by (4.3), one can deduce (bearing in mind (5.4) and (5.8));

$$\mu \int_{t-1}^{t} \|u'(\eta)\|_H^2 d\eta \leq \|u'(t-1)\|_{V'}^2 + \|u'(t)\|_{V'}^2 + 2c_2 \int_{t-1}^{t} \|u'(\eta)\|_{V'}^2 \|u(\eta)\|_{L^5(\Omega)}^5 d\eta +$$

$$+2 \int_{t-1}^{t} \|u'(\eta)\|_{V'} \|f'(\eta)\|_{V'} d\eta \leq 2\gamma_4^2 + 2c_2\gamma_4^2\gamma_3(K^2+K^3) + 2\gamma_4 K'.$$

We have so (5.9). ■

6. - Existence of a bounded solution

Theorem 6.1: *Assume that* $f \in L^2_{\text{loc}}(-\infty, +\infty; (L^2(\Omega))^3)$, $f' \in L^2_{\text{loc}}(-\infty, +\infty; V')$ *and*

$$\sup_{t\in\mathbf{R}} \int_{t-1}^{t} \|f(\eta)\|_{L^2(\Omega)}\, d\eta = K < +\infty, \tag{6.1}$$

$$\sup_{t\in\mathbf{R}} \int_{t-1}^{t} \|f'(\eta)\|_{V'}\, d\eta = K' < +\infty, \tag{6.2}$$

with K «sufficiently small». Assume moreover that σ satisfies (2.1)-(2.3) *or* (2.1), (2.2), (2.4). *There exists then at least a weak bounded solution* $\boldsymbol{u}$ *of* (1.1), (1.2), *for which the following estimates hold:*

$$\|\boldsymbol{u}\|_{L^\infty(-\infty,+\infty;H)} \le \gamma_1 K, \tag{6.3}$$

$$\sup_{t\in\mathbf{R}} \int_{t-1}^{t} \|\boldsymbol{u}(\eta)\|^2_{V'}\, d\eta \le \gamma_2 K^2, \tag{6.4}$$

$$\sup_{t\in\mathbf{R}} \int_{t-1}^{t} \|\boldsymbol{u}(\eta)\|^5_{L^5(\Omega)}\, d\eta \le \gamma_3 (K^2 + K^2), \tag{6.5}$$

$$\|\boldsymbol{u}'\|_{L^\infty(-\infty,+\infty;V')} \le \gamma_4\,, \tag{6.6}$$

$$\sup_{t\in\mathbf{R}} \int_{t-1}^{t} \|\boldsymbol{u}'(\eta)\|^2_{H}\, d\eta \le \gamma_5\,, \tag{6.7}$$

where $\gamma_1, \dots, \gamma_5$ *are the positive constants defined in the preceding section.*

Proof: Let $\boldsymbol{u}_n$ denote the weak solution in $[-n, +\infty) \times \Omega$ of (1.1), (1.2), with the initial condition $\boldsymbol{u}_n(-n) = 0$. We recall that $\boldsymbol{u}_n$ exists and it is unique by Theorem 3.1 and that Theorems 5.1, 5.2 yield:

$$\sup_{t\ge -n} \|\boldsymbol{u}_n(t)\|_H \le \gamma_1 K, \tag{6.8}$$

$$\sup_{t\ge -n+1} \int_{t-1}^{t} \|\boldsymbol{u}_n(\eta)\|^2_{V}\, d\eta \le \gamma_2 K^2, \tag{6.9}$$

$$\sup_{t\ge -n+1} \int_{t-1}^{t} \|\boldsymbol{u}_n(\eta)\|^5_{L^5(\Omega)}\, d\eta \le \gamma_3 (K^2 + K^2), \tag{6.10}$$

$$\text{(6.11)} \qquad \operatorname*{ess\,sup}_{t \geq -n} \|u_n'(t)\|_{V'} \leq \gamma_4 ,$$

$$\text{(6.12)} \qquad \sup_{t \geq -n+1} \int_{t-1}^{t} \|u_n'(\eta)\|_H^2 d\eta \leq \gamma_5 .$$

As a consequence of (6.8)-(6.12), there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function u such that:

$$\lim_{k \to +\infty} u_{n_k} = u ,$$

weakly in

$$L^2_{\mathrm{loc}}(-\infty, +\infty; V) \cap H^1_{\mathrm{loc}}(-\infty, +\infty; H) \cap L^5_{\mathrm{loc}}(-\infty, +\infty; (L^5(\Omega))^3) ,$$

weakly* in

$$L^\infty(-\infty, +\infty; H) \cap W^{1,\infty}(-\infty, +\infty; V')$$

strongly in $L^2_{\mathrm{loc}}(-\infty, +\infty; H)$ being $H^{1/2}_{\mathrm{loc}}(-\infty, +\infty; H^{1/2}(\Omega))$ completely continuous imbedded in $L^2_{\mathrm{loc}}(-\infty, +\infty; L^2(\Omega))$. Therefore passing to the limit as $k \to +\infty$, and being u_n solution of (1.1), (1.2), it follows that u satisfies (2.6), (2.7), that is u is a weak solution in $\mathbf{R} \times \Omega$ of (1.1), (1.2). Moreover u is bounded, since (6.8)-(6.12) hold for u too. ■

Remark 6.1: It follows also from (6.3), (6.4), (6.5) that, if K is «small», then the bounded solution u is «small» in $L^\infty(-\infty, +\infty; H) \cap L^2(t-1, t; V) \cap L^5(t-1, t; (L^5(\Omega))^3)$, for all $t \in \mathbf{R}$. ■

7. - Uniqueness of the bounded solution

We start by proving a property of the bounded solutions and, then, we use this property to obtain the uniqueness.

Theorem 7.1: *Assume that* $f \in L^2_{\mathrm{loc}}(-\infty, +\infty; (L^2(\Omega))^3)$, $f' \in L^2_{\mathrm{loc}}(-\infty, +\infty; V')$ *and that* σ *satisfies* (2.1)-(2.3) *or* (2.1), (2.2), (2.4). *Assume moreover that* u *is a weak solution of* (1.1), (1.2) *in* $\mathbf{R} \times \Omega$ *and that, setting*

$$\text{(7.1)} \qquad K = \sup_{t \in \mathbf{R}} \int_{t-1}^{t} \|f(\eta)\|_{L^2(\Omega)} d\eta < +\infty ,$$

$$\text{(7.2)} \qquad K^* = K \frac{\rho^2}{\mu} \left(1 + 2 \frac{\mu}{\rho^2} \right) ,$$

where ρ *is the imbedding constant of* $H^1_0(\Omega)$ *into* $L^2(\Omega)$, *there exists* $t^* \in \mathbf{R}$ *such that*

$$\text{(7.3)} \qquad \|u(t^*)\|_H > K^* .$$

Then

$$\max\lim_{t\to-\infty}\|u(t)\|_H=+\infty\,.$$

PROOF: Let us assume this is not so; that is, having set

(7.4) $$M=\max\lim_{t\to-\infty}\|u(t)\|_H\,,$$

let us assume $0\le M<+\infty$.

By (7.3), we deduce that

$$\sup_{t\le t^*}\|u(t)\|_H\ge\|u(t^*)\|_H>K^*\,.$$

Then, arbitrarily fixing $\sigma>1$, there exists $\bar t\le t^*$ such that

(7.5) $$\|u(\bar t)\|_H\ge\|u(t^*)\|_H>K^*\,,\qquad \sup_{t\le t^*}\|u(t)\|_H\le\sigma^2\|u(\bar t)\|_H$$

and consequently

(7.6) $$\sup_{t\le\bar t}\|u(t)\|_H\le\sup_{t\le t^*}\|u(t)\|_H\le\sigma^2\|u(\bar t)\|_H\,.$$

Let us consider the interval $[\bar t-1,\bar t]$ and let us prove that, if σ is sufficiently close to 1, there exists $\lambda>1$ such that

(7.7) $$\int_{\bar t-1}^{\bar t}\|u(\eta)\|_V^2d\eta\ge\frac{1}{\lambda\mu}\|u(\bar t)\|_H^2\,.$$

Let $\tilde t$, in fact, a point of $[\bar t-1,\bar t]$ such that

$$\|u(\tilde t)\|_H=\min_{\bar t-1\le t\le\bar t}\|u(t)\|_H\,;$$

by (4.1), (7.1) and (7.6), we have

$$\|u(\tilde t)\|_H^2\ge\|u(\bar t)\|_H^2-2K\max_{\bar t-1\le t\le\bar t}\|u(t)\|_H\ge\|u(\bar t)\|_H^2-2K\sigma^2\|u(\bar t)\|_H=\left(1-\frac{2K\sigma^2}{\|u(\bar t)\|_H}\right)\|u(\bar t)\|_H^2\,.$$

Therefore

$$\int_{\bar t-1}^{\bar t}\|u(\eta)\|_{V'}^2d\eta\ge\frac{1}{\rho^2}\int_{\bar t-1}^{\bar t}\|u(\eta)\|_H^2d\eta\ge\frac{1}{\rho^2}\|u(\tilde t)\|_H^2\ge\frac{1}{\rho^2}\left(1-\frac{2K\sigma^2}{\|u(\bar t)\|_H}\right)\|u(\bar t)\|_H^2\,.$$

Relation (7.7) is then verified provided that σ is sufficiently close to 1 (in this case, by (7.5), $\|u(\bar t)\|_H>2K\sigma^2$) and

(7.8) $$\lambda\ge\frac{\rho^2}{\mu}\left(1-\frac{2K\sigma^2}{\|u(\bar t)\|_H}\right)^{-1}.$$

On the other hand, by (4.1), (7.6), (7.7), we may deduce that

(7.9) $$\|u(\bar t-1)\|_H^2 \geq \|u(\bar t)\|_H^2 + \frac{2}{\lambda}\|u(\bar t)\|_H^2 - 2K \max_{\bar t-1\leq t\leq \bar t}\|u(t)\|_H \geq$$

$$\geq \left(1+\frac{2}{\lambda} - \frac{2\sigma^2 K}{\|u(\bar t)\|_H}\right)\|u(\bar t)\|_H^2 .$$

Consequently, if $\beta>1$ and

(7.10) $$\lambda \leq \frac{2}{\beta - 1 + 2\sigma^2 K/\|u(\bar t)\|_H},$$

by (7.9), it follows that

(7.11) $$\|u(\bar t-1)\|_H^2 \geq \beta \|u(\bar t)\|_H^2 .$$

By (7.5), there exist $\sigma>1$ and $\beta>1$, sufficiently close to 1, in order to verify

$$\|u(\bar t)\|_H \geq 2K\sigma^2\left(1+2\frac{\mu}{\rho^2}\right)\left(1+2\frac{\mu}{\rho^2}-\beta\right)$$

or equivalently

$$\frac{\rho^2}{\mu}\left(1-\frac{2\sigma^2 K}{\|u(\bar t)\|_H}\right)^{-1} \leq \frac{2}{\beta-1+2\sigma^2 K/\|u(\bar t)\|_H};$$

then both (7.8) and (7.10) are verified and (7.11) holds. On the other hand, (7.11) is true for $\bar t-1$ too and so on. Hence, we can conclude that

$$\max_{t\to-\infty}\lim\|u(\bar t)\|_H = +\infty ,$$

that contradicts the assumption $0\leq M<+\infty$. ■

Corollary 7.1: *Assume that* $f\in L^2_{\mathrm{loc}}(-\infty,+\infty;(L^2(\Omega))^3)$, $f'\in L^2_{\mathrm{loc}}(-\infty,+\infty;V')$ *and that* σ *satisfies* (2.1)-(2.3) *or* (2.1), (2.2), (2.4). *Assume moreover that* u *is an H-bounded weak solution of* (1.1), (1.2) *in* $\mathbf{R}\times\Omega$. *Then, for any* $t\in\mathbf{R}$,

(7.12) $$\|u(t)\|_H \leq K^* ,$$

with K^* *defined by* 7.2. *Furthermore, if* K *is «small», then* $\|u(t)\|_H$ *is «small» too.* ■

Theorem 7.2: *Under the assumptions of Theorem* 6.1, *the H-bounded solution* u *is unique.*

Proof: Let u, v be two H-bounded solutions on $\mathbf{R}$ (remark that u, v are also $L^2(t-1,t;V')$ and $L^5(t-1,t;(L^5(\Omega))^3)$ bounded, for any $t\in\mathbf{R}$). Setting $w=u-v$

and reasoning as in [5], (sect. 4, th. 4), we have, $\forall t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$:

$$\|w(t_2)\|_{V'}^2 - \|w(t_1)\|_{V'}^2 + 2\mu \int_{t_1}^{t_2} \|w(\eta)\|_H^2 \, d\eta \leq$$

$$\leq 2 \int_{t_1}^{t_2} |b(w(\eta), u(\eta), Gw(\eta)) - b(v(\eta), w(\eta), Gw(\eta))| \, d\eta .$$

By Lemma 3.1, we obtain

$$\text{(7.13)} \qquad \|w(t_2)\|_{V'}^2 - \|w(t_1)\|_{V'}^2 + 2\mu \int_{t_1}^{t_2} \|w(\eta)\|_H^2 \, d\eta \leq$$

$$\leq 2\delta \max_{t_1 \leq t \leq t_2} \|w(t)\|_{V'}^2 \int_{t_1}^{t_2} (\|u(\eta)\|_{L^5(\Omega)}^5 + \|v(\eta)\|_{L^5(\Omega)}^5) \, d\eta .$$

Moreover, by Corollary 7.1, it follows that, for any $t \in \mathbb{R}$ and for an appropriate constant $\tilde{\gamma} > 0$,

$$\|u(t)\|_H \leq \tilde{\gamma} K, \qquad \|v(t)\|_H \leq \tilde{\gamma} K$$

and consequently, by the same arguments used in Theorem 5.1,

$$\Gamma = \sup_{t \in \mathbb{R}} \int_{t-1}^{t} (\|u(\eta)\|_{L^5(\Omega)}^5 + \|v(\eta)\|_{L^5(\Omega)}^5) \, d\eta \leq \gamma' (K^2 + K^3) .$$

We now apply (7.13) to the interval $[\bar{t} - 1, \bar{t}]$, $\bar{t} \in \mathbb{R}$, and obtain

$$\text{(7.14)} \qquad \|w(\bar{t} - 1)\|_{V'}^2 \geq \|w(\bar{t})\|_{V'}^2 + 2\mu \int_{\bar{t}-1}^{\bar{t}} \|w(\eta)\|_H^2 \, d\eta - 2\delta\Gamma \max_{\bar{t}-1 \leq t \leq \bar{t}} \|w(t)\|_{V'}^2 .$$

Let us prove, to begin with, that, if $\beta > 1$ is «sufficiently close» to 1 and K is «sufficiently small», depending on β, then

$$\text{(7.15)} \qquad \max_{\bar{t}-1 \leq t \leq \bar{t}} \|w(t)\|_{V'}^2 \geq \beta \|w(\bar{t})\|_{V'}^2 .$$

Assume, in fact, that (7.15) does not hold, i.e.

$$\text{(7.16)} \qquad \max_{\bar{t}-1 \leq t \leq \bar{t}} \|w(t)\|_{V'}^2 < \beta \|w(\bar{t})\|_{V'}^2 .$$

Denoting by $\tilde{t}$ a point $\in [\bar{t} - 1, \bar{t}]$, we have then, by (7.13) and (7.16)

$$\|w(\tilde{t})\|_{V'}^2 \geq \|w(\bar{t})\|_{V'}^2 - 2\delta\Gamma\beta \|w(\bar{t})\|_{V'}^2 =$$

$$= (1 - 2\delta\Gamma\beta) \|w(\bar{t})\|_{V'}^2 \geq (1 - 2\delta\beta\gamma'(K^2 + K^3)) \|w(\bar{t})\|_{V'}^2 .$$

On the other hand, assuming that K is so small that $1-2\delta\beta\gamma'(K^2+K^3)>0$,

$$\int_{\bar{t}-1}^{\bar{t}} \|w(\eta)\|_H^2 \, d\eta \geq \frac{1}{\rho^2} \int_{\bar{t}-1}^{\bar{t}} \|w(\eta)\|_{V'}^2 \, d\eta \geq \frac{1}{\rho^2} \min_{\bar{t}-1 \leq t \leq \bar{t}} \|w(t)\|_{V'}^2 \geq \frac{1}{\rho^2}(1-2\delta\Gamma\beta)\|w(\bar{t})\|_{V'}^2 .$$

Hence, if we choose a number λ such that

$$\lambda \geq \frac{\rho^2}{\mu(1-2\delta\Gamma\beta)}, \tag{7.17}$$

we obtain

$$\int_{\bar{t}-1}^{\bar{t}} \|w(\eta)\|_H^2 \, d\eta \geq \frac{1}{\lambda\mu} \|w(\bar{t})\|_{V'}^2 .$$

For such a value of λ, we have then, by (7.14), (7.15)

$$\|w(\bar{t})\|_{V'}^2 \leq \|w(\bar{t}-1)\|_{V'}^2 - \frac{1}{\lambda}\|w(\bar{t})\|_{V'}^2 + 2\delta\Gamma\beta\|w(\bar{t})\|_{V'}^2 ,$$

that is

$$\left(1+\frac{1}{\lambda}-2\delta\Gamma\beta\right)\|w(\bar{t})\|_{V'}^2 \leq \|w(\bar{t}-1)\|_{V'}^2$$

It therefore λ is such that

$$\lambda \leq (2\delta\Gamma\beta+\beta-1)^{-1}, \tag{7.18}$$

then

$$\|w(\bar{t}-1)\|_{V'}^2 \geq \beta\|w(\bar{t})\|_{V'}^2 ,$$

which contradicts (7.16).

Hence, if β is «sufficiently close» to 1 and K is so small that both (7.17) and (7.18) are satisfied, then

$$\frac{\rho^2}{\mu(1-2\delta\Gamma\beta)} \leq (2\delta\Gamma\beta+\beta-1)^{-1},$$

then, necessarily, relation (7.15) holds.

Let us fix arbitrarily a point $t_0 \in \mathbb{R}$ in which $\|w(t_0)\|_{V'} \neq 0$. By what has been already proved, there exists in $[t_0-1, t_0)$ a point t_1 such that

$$\|w(t_1)\|_{V'}^2 \geq \beta\|w(t_0)\|_{V'}^2 , \quad \beta > 1 .$$

We can repeat this procedure for the interval $[t_1-1, t_1]$, and so on, constructing a decreasing sequence $\{t_k\}$ such that

$$\|w(t_{k+1})\|_{V'}^2 \geq \beta\|w(t_k)\|_{V'}^2 .$$

We have then, necessarily,

$$\max_{k \to +\infty} \lim \|w(t_k)\|_{V'} = +\infty .$$

This contradicts our assumption that u, v are bounded on $\mathbf{R}$. The uniqueness theorem is therefore proved. ■

8. - Existence and uniqueness of an almost-periodic solution

We now recall the definitions of (see [2]) almost-periodicity, weak almost-periodicity, almost-periodicity in the sense of Stepanov, and then we state some conditions that imply the existence of an almost-periodic solution and its uniqueness.

Let X be a Banach space, X^* the dual space and let $\langle\, , \rangle$ denote the duality between X and X^*.

Def. 8.1: Let f be a continuous function from $\mathbf{R}$ into X; f is said to be *almost-periodic* (*a.p.*) if to every $\varepsilon > 0$ there corresponds a relatively dense set $\{\tau\}_\varepsilon$ such that

$$\sup \|f(t+\tau) - f(t)\|_X \leq \varepsilon , \qquad \forall \tau \in \{\tau\}_\varepsilon . \quad ■$$

Def. 8.2: Let f be a function from $\mathbf{R}$ into X; f is said to be *weakly almost-periodic* (*w.a.p.*) if the function $\langle x^*, f\rangle$ is a.p., $\forall x^* \in X^*$. ■

Def. 8.3: Let $f \in L^p_{\mathrm{loc}}(\mathbf{R}; X)$, with $1 \leq p < +\infty$; the function f is said to be *almost-periodic in the sense of Stepanov* (S^p *a.p. or* $x - S^p$ *a.p.*) if to every $\varepsilon > 0$ there corresponds a relatively dense set $\{\tau\}_\varepsilon$ such that, $\forall \tau \in \{\tau\}_\varepsilon$, we have

$$\sup_{t \in \mathbf{R}} \left\{ \int_0^1 \|f(t+\tau+\eta) - f(t+\eta)\|_X^p \, d\eta \right\}^{1/p} \leq \varepsilon . \quad ■$$

We now turn to the almost-periodicity result.

Theorem 8.1: *Let us suppose the assumptions of Theorem* 6.1 *hold. Let us suppose moreover that f is $L^2 - S^2$ w.a.p. and f' is $V' - S^2$ w.a.p. There exists then one and only one $H - S^2$ a.p. solution of* (1.1), (1.2).

Proof: Since the assumptions of Theorem 6.1 hold, by Theorems 6.1 and 7.2, there exists a unique H-bounded weak solution u of (1.1), (1.2), which is H-continuous on $\mathbf{R}$ too. By the Bochner criterion [2], we must show that from every real sequence $\{l_n\}$, it is possible to select a subsequence $\{l_n'\}$ such that

$$\lim_{n \to +\infty} \int_{t-1}^{t} \|u(\eta - l_n') - z(\eta)\|_H^2 \, d\eta = 0 ,$$

uniformly on $\mathbf{R}$.

Assume this is not so; there exists then a sequence $\{\bar{l}_n\}$ with the following property. In correspondence to every subsequence $\{l_n\} \subseteq \{\bar{l}_n\}$ there exist two subsequences $\{\alpha'_n\} \subseteq \{l'_n\}$, $\{\alpha''_n\} \subseteq \{l'_n\}$, a sequence $\{t_n\}$ and a number $\chi > 0$ such that

$$(8.1) \qquad \int_{t_n-1}^{t_n} \|u(\eta - \alpha'_n) + u(\eta + \alpha''_n)\|_H^2 d\eta \geq \chi, \qquad \forall n.$$

We can obviously assume, since f is $L^2 - S^2$ w.a.p. and f' is $V' - S^2$ w.a.p., that

$$(8.2) \qquad \lim_{n \to +\infty} \sup_{\mathbf{R}} \int_{t-1}^{t} |\langle f(\eta + t_n + \alpha'_n) - f_l(\eta), h_1(\eta) \rangle| d\eta = 0,$$

$$(8.3) \qquad \lim_{n \to +\infty} \sup_{\mathbf{R}} \int_{t-1}^{t} |\langle f'(\eta + t_n + \alpha''_n) - f'_l(\eta), h_2(\eta) \rangle| d\eta = 0,$$

$\forall h_1 \in L^2(t-1, t; H)$, $h_2 \in L^2(t-1, t; V)$ and correspondingly

$$\lim_{n \to +\infty} u(t_n + \alpha'_n) = z_1, \qquad \lim_{n \to +\infty} u(t_n + \alpha''_n) = z_2,$$

in the weak topology of $H^1(t-1, t; H) \cap L^2(t-1, t; V) \cap L^5(t-1, t; (L^5(\Omega))^3)$, the weak* topology of $L^\infty(t-1, t; H)$, the strong topology of $L^2(t-1, t; H)$, for any fixed $t \in \mathbf{R}$. Moreover, z_1, z_2 are solutions, bounded on $\mathbf{R}$ corresponding to the function f_l defined by (8.2), (8.3).

By the uniqueness theorem of a bounded solution, it follows therefore that, if K is «sufficiently small», then $z_1 = z_2$. Hence,

$$(8.4) \qquad \lim_{n \to +\infty} \int_{t-1}^{t} \|u(\eta + t_n + \alpha'_n) - u(\eta + t_n + \alpha''_n)\|_H^2 d\eta = 0, \qquad \forall t \in \mathbf{R}.$$

Relation (8.4), written for $t = 0$, contradicts (8.1) and u is $H - S^2$ a.p. ∎

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