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A Nowhere Dense but not Porous Set in the Space of Convex Bodies (**)

Abstract. — We give an example of a subset which is nowhere dense but not porous in the space of all convex bodies.

Un insieme raro ma non poroso nello spazio dei corpi convessi

Riassunto. — La nozione di insieme poroso (σ-poroso) fu estesa ad uno spazio metrico qualsiasi nel 1976 da Zajíček. Tale nozione rappresenta un raffinamento della nozione di insieme raro (magro) in tutti quegli spazi in cui è possibile dare esempi di insiemi rari, ma non porosi (σ-porosi). Questo problema, risolto in uno spazio di Banach, è ancora aperto nello spazio C dei corpi convessi dotato della topologia indotta dalla metrica di Hausdorff. Nel presente lavoro si contribuisce alla soluzione di tale problema con un esempio in C di un insieme raro, ma non poroso.

The notion of porous set on the real line $\mathbb{R}$ was introduced by Dolženko in 1967 [2] and was generalized to a general metric space by Zajíček in 1976 [3]. In this paper we shall use the following definition of porous set [5].

Let (X, d) be a metric space and $B(x, \varepsilon)$ denotes the ball of center $x \in X$ and radius ε. A subset M of X is porous (with coefficient α) if there is a real number $\alpha > 0$ such that for each $x \in X$ and for each ball $B(x, \varepsilon)$ there exists an element $y \in B(x, \varepsilon)$ such that:

$$B(y, \alpha d(x, y)) \cap M = \emptyset .$$

A countable union of porous sets (all with coefficient α) is called σ-porous (with coefficient α).

It is obvious that a porous set is also nowhere dense and that a σ-porous set is mea-

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ger. The problem of finding nowhere dense sets which are not σ-porous is solved in the euclidean d-dimensional space E^d and more generally in a Banach space but «it is not known at which more general metric space» such a problem can be solved ([4], page 322).

In this paper we give a contribution to the solution of this problem by showing an example of a subset which is nowhere dense but not porous in the space $\mathbf{C}$ of all convex bodies endowed with the Hausdorff metric.

Eventually we recall that a convex body is a compact convex subset of E^d with nonempty interior and that if $C, D \in \mathbf{C}$ their Hausdorff distance $\delta(C, D)$ is defined in the following way:

$$\delta(C,D) = \max\left\{\sup_{x\in C}\inf_{y\in D} d(x,y)\,,\ \sup_{y\in D}\inf_{x\in C} d(x,y)\right\},$$

where d is the usual euclidean distance.

If for $F \in \mathbf{C}$ and for each positive real number ρ, we put

$$F_\rho = \{x \in E^d\colon d(x,F) \leqslant \rho\}$$

we have also that

$$\delta(C,D) = \inf\{\rho\colon C_\rho \supset D \text{ and } D_\rho \supset C\}\,.$$

Notations: The ball of E^d of center a point x and radius ε will be denoted by $B(x, \varepsilon)$ while the ball of $\mathbf{C}$ of center an element C of $\mathbf{C}$ and radius ε will be denoted by $\boldsymbol{B}\ (C, \varepsilon)$.

The abbreviations bd, int and conv stand for boundary, interior and convex hull.

The example

Choose on a straightline $\mathbf{R}$ of the euclidean d-dimensional space E^d a nowhere dense but not porous subset M of real numbers and define:

$$\boldsymbol{M} = \{C \in \mathbf{C}\colon (\mathrm{bd}\ C) \cap M \neq \emptyset\}\,.$$

1. - $\boldsymbol{M}$ is not porous in $\mathbf{C}$

Since M is not porous in $\mathbf{R}$, for each real number α, $0 < \alpha \leqslant 1$, there are an element $x \in \boldsymbol{R}$ and a positive ε such that, for each $z \in B(x, \varepsilon)$,

$$B(z, \alpha d(x,z)) \cap M \neq \emptyset\,.$$

Let C be a ball with center in a point c of R, radius ρ greater than $(5 + 3\sqrt{2})\,\varepsilon$ and such that $x \in \mathrm{bd}\ C$.

We shall show that, for each $D \in \boldsymbol{B}(C, \varepsilon)$,

$$B(D, \alpha\delta(D, C)) \cap \boldsymbol{M} \neq \emptyset . \tag{1}$$

Firstly we prove that, if z is the point of $(\operatorname{bd} D) \cap \mathbb{R}$ belonging also to the interval $(x-\varepsilon, x+\varepsilon)$ of $\mathbb{R}$, then

$$d(x, z) \leqslant \delta(C, D) . \tag{2}$$

Indeed, if $z \notin \operatorname{int} C$ we have easily:

$$d(z, x) = d(z, C) \leqslant \delta(D, C) .$$

If $z \in \operatorname{int} C$ we take a support hyperplane π to D at the point z. Afterwards we consider the hyperplane π' parallel to π, tangent to C such that the point v common to C and π belongs to the halfspace bounded by π and not containing c. Therefore:

$$d(z, x) \leqslant d(v, \pi) \leqslant d(v, D) \leqslant \delta(D, C)$$

and again (2) follows.

Now let q be an element of $B(z, \alpha d(x, z)) \cap M$.

a) If $q \notin \operatorname{int} D$, we set

$$F = \operatorname{conv}\{D \cup \{q\}\} .$$

Since $q \in (\operatorname{bd} F) \cap M$ it follows that $F \in \boldsymbol{M}$.

Moreover, using also (2):

$$\delta(F, D) \leqslant d(q, D) \leqslant d(q, z) < \alpha d(x, z) \leqslant \alpha\delta(D, C) .$$

Therefore $F \in \boldsymbol{B}(D, \alpha\delta(D, C))$ and (1) holds.

b) Let us assume now that $q \in \operatorname{int} D$. Then we can choose a point m of $\operatorname{bd} D$ such that $d(q, \operatorname{bd} D) = d(q, m)$. Afterwards we consider the hyperplane π through q and parallel to a support hyperplane π' to D at m. Then if P is the closed halfspace bounded by π and not containing m, we set

$$F = D \cap P .$$

We shall prove that

$$\delta(F, D) \leqslant d(q, m) . \tag{3}$$

If we put $d(q, m) = \gamma$, we have obviously that $D_\gamma \supset F$. So there is only to prove that a point a of D but not of F belongs also to F_γ.

Let a' be the point of π such that the straightline aa' is perpendicular to π. We claim that

$$a' \in (\operatorname{bd} F) \cap \pi . \tag{4}$$

Indeed, if otherwise $a' \notin (\operatorname{bd} F) \cap \pi$, we can choose a straightline S of π through a' which meets $(\operatorname{bd} F) \cap \pi$. We put $S \cap (\operatorname{bd} F) \cap \pi = [b, e]$ and we choose the points b and e in such a way that $d(a', b) < d(a', e)$.

Now we observe that from the definition of the points m, q and z we have that

$$d(\pi, \pi') \leqslant d(q, m) < d(q, z) < \alpha d(x, z) < \alpha\varepsilon < \varepsilon\,.$$

Therefore, since π' does not intersect the ball of center c and radius $\rho - \varepsilon$, the hyperplane π does not intersect the ball C' of center c and radius $\rho - 2\varepsilon$. Since $\rho > (5 + 3\sqrt{2})\,\varepsilon$, it follows then that the straightline through b and parallel to the straightline aa' cuts the boundary of C' in two points. If f is one of them,

$$\operatorname{conv}\{b, e, f\}$$

is a convex set of the plane spanned by the points a, a', b and its interior points are also interior points of D.

Then the straightline through the points a and b would contain the point b of $\operatorname{bd} D$, points of $\operatorname{int} D$ on one of the two half-lines bounded by b and points of D on the other one, which is a contradiction.

Then (4) holds.

Therefore:

$$d(a, F) \leqslant d(a, a') \leqslant d(\pi, \pi') \leqslant \gamma\,,$$

hence $a \in F_\gamma$ and also (3) holds.

Now, from (3) and (2), we can obtain that

$$\delta(D, F) \leqslant d(m, q) \leqslant d(q, z) < \alpha d(x, z) \leqslant \alpha\delta(D, C)$$

i.e.

$$F \in B(D, \alpha\delta(D, C))\,.$$

Since moreover $q \in (\operatorname{bd} F) \cap M$, we have that (1) is fulfilled and M is not porous in C.

2. - M is nowhere dense in C

We shall show that for each nonempty open set $B(D, \varepsilon)$ of $C(D \in C)$ it is possible to find another non empty open set $B(E, \eta)$ contained in $B(D, \varepsilon)$ and disjoint from M. There are three cases.

i) $(\operatorname{bd} D) \cap \mathbb{R} = \emptyset$.

Let α be a positive real number such that

$$2\alpha < \min(\varepsilon, d(D, \mathbb{R}))\,.$$

Since $D \subset D_\alpha \subset (D_\alpha)_\alpha$, from Lemma 12.9.13 of [1], vol. 3 page 139, there exists a positive η such that, if $S \in C$ and $\delta(D_\alpha, S) \leqslant \eta$, then $D \subset S \subset (D_\alpha)_\alpha$. Therefore

$(\mathrm{bd}\, S) \cap \mathbf{R} = \emptyset$ and

$$\boldsymbol{B}(D_\alpha, \eta)$$

is an open set contained in $\boldsymbol{B}(D, \varepsilon)$ and disjoint from $\boldsymbol{M}$.

ii) $(\mathrm{bd}\, D) \cap \mathbf{R} = \{x_1, x_2\}$ *i.e.* two points of $\mathbf{R}$.

Since $D \subset D_{\varepsilon/2}$ also $\mathrm{bd}\, D_{\varepsilon/2}$ intersects $\mathbf{R}$ in exactly two points y_1 and y_2. We can assume that the two open intervals of $\mathbf{R}$, (y_1, x_1) and (x_2, y_2), are disjoint from $\mathrm{int}\, D$.

Since M is nowhere dense there are two closed intervals $[b_1, a_1] \subset (y_1, x_1)$ and $[a_2, b_2] \subset (x_2, y_2)$ disjoint from M. We assume $b_1 \notin (a_1, x_1)$ and $b_2 \notin (x_2, a_2)$.

Afterwards we put

$$F = \mathrm{conv}\{D \cup \{a_1\} \cup \{a_2\}\}$$

and we choose a positive real number α such that

$$2\alpha < \min(d(a_1, b_1), d(a_2, b_2), d(b_1, F), d(b_2, F)).$$

Obviously $\alpha < \varepsilon/4$.

Moreover since $D \subset F \subset F_\alpha \subset (F_\alpha)_\alpha$; from the same Lemma used in (i) there is a positive η such that if $S \in \boldsymbol{C}$ and $\delta(S, F_\alpha) \leqslant \eta$, then $F \subset S \subset (F_\alpha)_\alpha$. We claim that

$$\boldsymbol{B}(F_\alpha, \eta)$$

is the open set we are looking for.

Firstly we observe that $b_1 \notin (F_\alpha)_\alpha$, since, otherwise, there would exist a point $y \in F$ such that $b_1 \in B(y, 2\alpha)$ and then $d(b_1, y) \leqslant 2\alpha < d(b_1, F)$, which is a contradiction.

Then $b_1 \notin (F_\alpha)_\alpha$, $a_1 \in \mathrm{bd}\, F$ and analogously for b_2 and a_2.

Since, for each $S \in \boldsymbol{B}(F_\alpha, \eta)$, we have that $F \subset S \subset (F_\alpha)_\alpha$, it follows that

$$(\mathrm{bd}\, S) \cap \mathbf{R} \subset [b_1, a_1] \cup [a_2, b_2]$$

and therefore $S \notin \boldsymbol{M}$.

In order to show that $\boldsymbol{B}(F_\alpha, \eta) \subset \boldsymbol{B}(D, \varepsilon)$ we put

$$\sigma = \max(d(a_1, D), d(a_2, D)).$$

Then, from $\sigma < \varepsilon/2$ and $F \subset D_\sigma$, it follows that

$$D \subset S \subset (F_\alpha)_\alpha \subset (D_\sigma)_{2\alpha} = D_{\sigma + 2\alpha} \subset D_\varepsilon.$$

iii) If $\mathbf{R}$ is a support straightline of D, i.e. $(\mathrm{bd}\, D) \cap \mathbf{R}$ is either a single point either a line segment, we can consider in the neighbourhood $\boldsymbol{B}(D, \varepsilon)$ the convex body $D_{\varepsilon/2}$. Then there is only to apply to the neighbourhood $\boldsymbol{B}(D_{\varepsilon/2}, \varepsilon/4)$ the procedure used in (ii).

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