



Rendiconti

Accademia Nazionale delle Scienze detta dei XL

*Memorie di Matematica*

109° (1991), Vol. XV, fasc. 12, pagg. 203-211

MICHELANGELO FRANCIOSI (\*)

## Weak Weighted Reverse Integral Inequalities (\*\*) (\*\*\*)

SUMMARY. — In this paper a weak reverse integral inequality is considered. From this higher integrability is deduced using reduction to one dimension and Muckenhoupt lemma.

### Disuguaglianze integrali deboli alla rovescia pesate

RIASSUNTO. — In questo lavoro si prende in considerazione una disuguaglianza integrale alla rovescia di tipo debole. A partire da questa disuguaglianza si deduce un risultato di maggiore integrabilità passando al caso unidimensionale ed usando un classico lemma di Muckenhoupt.

#### 1. - INTRODUCTION

The aim of the paper is to consider a reverse integral inequality of weak type that generalize in some sense the classical one introduced by Gehring. In particular some asymptotic properties relative to the higher integrability deduced from this inequality are proved.

We consider the inequality:

$$\begin{aligned}
 \text{(WRH)} \quad \mu \left( \left\{ x \in Q_0 : \sup_{x \in Q} \left\{ \int_Q f^2 d\mu - \left( \int_Q f d\mu \right)^2 \right\} > \lambda^2 \right\} \right) &\leq \\
 &\leq \nu \left( \left\{ x \in Q_0 : (M, f)^2(x) > \frac{\lambda^2}{k^2 - 1} \right\} \right)
 \end{aligned}$$

$\lambda > 0, k > 1$  independent of  $\lambda$ . For the notations see section 2. We emphasize that con-

(\*) Indirizzo dell'Autore: Istituto di Matematica, Facoltà di Scienze, Università di Salerno, 84100 Salerno.

(\*\*) Work partially supported by Italian MURST.

(\*\*\*) Memoria presentata il 30 settembre 1991 da Mario Troisi, socio dell'Accademia.

dition (WRH) in the case  $\mu \equiv \nu$  is weaker than Gehring condition:

$$(G) \quad \int_Q f^2(x) d\mu \leq k^2 \left( \int_Q f(x) d\mu \right)^2.$$

The main result is that if  $f$  verifies (WRH), then  $f$  belongs to  $L^p(Q_0, \mu)$  for  $1 \leq p \leq \gamma(k)$  and if  $k$  tends to one, then  $\gamma(k)$  tends to infinity like  $c/\sqrt{k^2 - 1}$ , where  $c$  is a dimensional constant.

The main step in the proof is a bound for the decreasing rearrangement of  $f$  in terms of the decreasing rearrangement of the Fefferman-Stein sharp function of  $f$ . Using this bound, we prove that the weighted rearrangement of  $f$ , in the hypothesis (WRH), is an  $A_1$  weight of Muckenhoupt, and from this we deduce the higher integrability result.

## 2. - SOME NOTATIONS AND HYPOTHESES

Let  $Q_0$  be a cube of  $\mathbb{R}^n$ , parallel to the axes,  $w, v$  two nonnegative weights belonging to  $L^1(Q_0)$ . For any subset  $E$  of  $Q_0$  Lebesgue measurable, we set:

$$\mu(E) = \int_E w(x) dx; \quad \nu(E) = \int_E v(x) dx$$

and for any  $f \in L^1(Q_0, \mu) \cap L^1(Q_0, \nu) \equiv L^1$ :

$$\int_E f d\mu = \frac{1}{\mu(E)} \int_E f d\mu; \quad \int_E f d\nu = \frac{1}{\nu(E)} \int_E f d\nu.$$

We indicate with  $|E|$  the Lebesgue measure of  $E$  and suppose that a constant  $A > 1$  there exists such that, for any cube  $Q \subseteq Q_0$  and parallel to  $Q_0$  and for any  $E \subseteq Q$ ,

$$(2.1) \quad \max \left\{ \frac{\mu(Q)}{\mu(E)}, \frac{\nu(Q)}{\nu(E)} \right\} \leq A \frac{|Q|}{|E|},$$

$$(2.2) \quad \nu(E) \leq A\mu(E).$$

For any  $x \in Q_0$  we set:

$$Mf(x) = \sup_{x \in Q} \int_Q f d\mu; \quad M_\nu f(x) = \sup_{x \in Q} \int_Q f d\nu;$$

$$f_\mu^\#(x) = \sup_{x \in Q} \left\{ \int_Q \left| f - \int_Q f d\mu \right|^2 d\mu \right\}^{1/2},$$

where  $Q$  is the generic subcube of  $Q_0$  parallel to  $Q_0$ .



For a nonnegative  $f$  belonging to  $L^1$  the following condition is considered:

$$(WRH) \quad \mu \left( \left\{ x \in Q_0 : \sup_{x \in Q} \left\{ \int_Q f^2 d\mu - \left( \int_Q f d\mu \right)^2 \right\} > \lambda^2 \right\} \right) \leq \\ \leq \nu \left( \left\{ x \in Q_0 : (M_\nu f)^2(x) > \frac{\lambda^2}{k^2 - 1} \right\} \right),$$

where  $k > 1$  is a constant independent of  $\lambda > 0$ .

Moreover, we define, for  $t > 0$ , the weighted decreasing rearrangements:

$$f^{*,\mu}(t) = \inf \{ \lambda > 0 : \mu(\{x \in Q_0 : f(x) > \lambda\}) \leq t \},$$

$$f^{*,\nu}(t) = \inf \{ \lambda > 0 : \nu(\{x \in Q_0 : f(x) > \lambda\}) \leq t \}.$$

If  $\mu = \nu$  then (WRH) becomes:

$$(2.3) \quad \mu \left( \left\{ x \in Q_0 : \sup_{x \in Q} \left\{ \int_Q f^2 d\mu - \left( \int_Q f d\mu \right)^2 \right\} > \lambda^2 \right\} \right) \leq \\ \leq \mu \left( \left\{ x \in Q_0 : (M_\mu f)^2(x) > \frac{\lambda^2}{k^2 - 1} \right\} \right).$$

Condition (2.3) is obviously weaker than «Gehring condition»:

$$\int_Q f^2(x) d\mu \leq k^2 \left( \int_Q f(x) d\mu \right)^2,$$

$k$  independent of the cube  $Q \subseteq Q_0$ .

### 3. - A BOUND FOR THE DECREASING WEIGHTED REARRANGEMENT

We prove first the following:

LEMMA 3.1: Let  $f$  belongs to  $L^1$ ,  $f \geq 0$ . If  $f$  verifies (WRH) then, for any  $\lambda > 0$ ,

$$\mu(\{x \in Q_0 : f_\mu^\#(x) > \lambda\}) \leq \nu \left( \left\{ x \in Q_0 : M_\nu f(x) > \frac{\lambda}{\sqrt{k^2 - 1}} \right\} \right).$$

PROOF: We note that:

$$\int_Q \left| f - \int_Q f d\mu \right|^2 d\mu = \int_Q f^2 d\mu - \left( \int_Q f d\mu \right)^2$$

and then:

$$(f_\mu^\#(x))^2 = \sup_{x \in Q} \left\{ \int_Q f^2 d\mu - \left( \int_Q f d\mu \right)^2 \right\}$$

for any  $x \in Q_0$ .

From this we have for any  $\lambda > 0$  such that  $f_\mu^\#(x) > \lambda$ , using condition (WRH)

$$\begin{aligned} \mu(\{x \in Q_0 : f_\mu^\#(x) > \lambda\}) &\leq \mu \left( \left\{ x \in Q_0 : \sup_{x \in Q} \left\{ \int_Q f^2 d\mu - \left( \int_Q f d\mu \right)^2 \right\} > \lambda^2 \right\} \right) \leq \\ &\leq \nu \left( \left\{ x \in Q_0 : (M_\nu f)^2(x) > \frac{\lambda^2}{k^2 - 1} \right\} \right) \end{aligned}$$

for any  $\lambda > 0$ . ■

From Lemma 3.1. and the definition of rearrangement we obtain:

LEMMA 3.2: Let  $f$  be a nonnegative function belonging to  $L^1$ . If  $f$  verifies (WRH), then:

$$(f_\mu^\#)^{*,\mu}(t) \leq \sqrt{k^2 - 1} (M_\nu f)^{*,\nu}(t),$$

for any  $t > 0$ .

Later on, we shall also need the following lemma, which is a weighted version of a covering lemma in [DBS].

LEMMA 3.3: Let  $G \subseteq Q_0$  be an open subset such that:

$$\mu(G) \leq \frac{1}{2^{n+1} A^2} \mu(Q_0).$$



Then a sequence of cubes  $(Q_j)_{j \in \mathbb{N}}$  there exists, with pairwise disjoint interiors such that:

$$(3.1) \quad \begin{aligned} \mu(Q_j) &< 2A\mu(G^c \cap Q_j), \\ \sum_j \mu(Q_j) &\leq 2^{n+1}A^2\mu(G), \\ G &\subseteq \bigcup_j Q_j. \end{aligned}$$

PROOF: The proof is like that in [BDS], and is obtained using property (2.1) of  $\mu$ .

Now we can prove:

THEOREM 3.1: For any  $0 < t < \mu(Q_0)/32^{n+1}A^2$ , we have:

$$\frac{1}{t} \int_0^t f^{*,\mu}(s) ds - f^{*,\mu}(t) \leq 32^{n+1}A^2(2A+1)(f_\mu^\#)^{*,\mu}(t).$$

PROOF: For  $0 < t < \mu(Q_0)/32^{n+1}A^2$ , we set:

$$E = \{x \in Q_0 : f(x) > f^{*,\mu}(t)\}, \quad F = \{x \in Q_0 : f_\mu^\#(x) > (f_\mu^\#)^{*,\mu}(t)\}.$$

By well-known properties of decreasing rearrangements ([S], [FM]), we have  $\mu(E \cup F) \leq 2t$ , and then an open subset  $G$  of  $Q_0$  there exists, such that  $\mu(G) \leq 3t$ ,  $E \cup F \subset G \subset Q_0$ .

Since  $\mu(G) < \mu(Q_0)/2^{n+1}A^2$ , Lemma 3.3 implies the existence of a covering  $(Q_j)_{j \in \mathbb{N}}$  of  $G$  verifying (3.1.).

We have:

$$(3.2) \quad \begin{aligned} \int_0^t (f^{*,\mu}(s) - f^{*,\mu}(t)) ds &\leq \int_E (f(x) - f^{*,\mu}(t)) d\mu = \\ &= \sum_j \int_{E \cap Q_j} (f(x) - f^{*,\mu}(t)) d\mu \leq \sum_j \int_{Q_j} \left| f(x) - \int_{Q_j} f d\mu \right| d\mu + \\ &\quad + \sum \mu(E \cap Q_j) \left( \int_{Q_j} f d\mu - f^{*,\mu}(t) \right). \end{aligned}$$

Obviously:

$$\sum_j \mu(E \cap Q_j) \left( \int_{Q_j} f d\mu - f^{*,\mu}(t) \right) \leq \sum_j \mu(Q_j) \left( \int_{Q_j} f d\mu - f^{*,\mu}(t) \right),$$

where the summation  $\sum'$  extends to the  $j$  such that  $\int_{Q_j} f d\mu - f^{*,\mu}(t) \geq 0$ . Then by (3.1), since  $G^C \cap Q_j \cap E$  is empty,

$$(3.3) \quad \sum'_j \mu(E \cap Q_j) \left( \int_{Q_j} f d\mu - f^{*,\mu}(t) \right) \leq \\ \leq 2A \sum'_j \int_{G \cap Q_j} \left( \int_{Q_j} f d\mu - f^{*,\mu}(t) \right) d\mu \leq 2A \sum_j \int_Q \left| f(x) - \int_{Q_j} f d\mu \right| d\mu.$$

From (3.2) and (3.3) we can deduce:

$$\int_0^t |f^{*,\mu}(s) - f^{*,\mu}(t)| ds \leq (2A + 1) \sum_j \int_{Q_j} \left| f(x) - \int_{Q_j} f d\mu \right| d\mu.$$

But  $\mu(Q_j) \neq 0$  for any  $j$ , and then, by (3.1),  $F^C \cap Q_j$  is not empty for any  $j$ . Let  $x_j$  belong to  $F^C \cap Q_j$ . Then:

$$(f_\mu^\#)^{*,\mu}(t) \geq f_\mu^\#(x_j)$$

and using Hölder inequality:

$$\sum_j \int_{Q_j} \left| f(x) - \int_{Q_j} f d\mu \right| d\mu \leq \sum_j \mu(Q_j) f_\mu^\#(x_j) \leq \sum_j \mu(Q_j) (f_\mu^\#)^{*,\mu}(t).$$

From this:

$$\int_0^t (f^{*,\mu}(s) - f^{*,\mu}(t)) ds \leq (2A + 1)(2^{n+1} A^2) 3t (f_\mu^\#)^{*,\mu}(t).$$

Now we state the following useful result due to Herz ([H]):

LEMMA 3.4 ([H], [S]): Let  $f$  be a nonnegative function belonging to  $L^1(Q_0, \mu)$ . Then, for any  $0 < t < \nu(Q_0)$ ,

$$c_1 (M_\nu f)^{*,\nu}(t) \leq \frac{1}{t} \int_0^t f^{*,\nu}(s) ds \leq c_2 (M_\nu f)^{*,\nu}(t),$$

where  $c_1, c_2$  are constants depending only on  $\nu$  and  $n$ .

PROOF: See [S].

From Theorem 3.1., using Lemmas 3.1 and 3.4. we are now able to state the following:



THEOREM 3.2: Let  $f$  be nonnegative and belonging to  $L^1$ . Then, if  $f$  verifies (WRH), we have:

$$\frac{1}{t} \int_0^t f^{*,\mu}(s) ds - f^{*,\mu}(t) \leq \frac{3}{c_1} (2A+1)(2^{n+1}A^2) \sqrt{k^2-1} \frac{1}{t} \int_0^t f^{*,\nu}(s) ds$$

for any  $0 < t < \mu(Q_0)/32^{n+1}A^2$ .

#### 4. - HIGHER INTEGRABILITY FROM CONDITION (WRH)

We use the following famous Muckenhoupt lemma:

LEMMA 4.1 ([M], [BSW]): Let  $b$  be a real function defined in the interval  $]0, a[$ , nonnegative and decreasing. If:

$$\frac{1}{t} \int_0^t b(s) ds \leq D b(t)$$

for  $0 < t < a/2$ , where  $D > 1$  is independent of  $t$ , then, for any  $1 \leq r \leq D/(D-1)$ , we have:

$$\frac{1}{a} \int_0^a b^r(s) ds \leq \frac{1}{D^{r-1}(D+r-rD)} \left( \frac{1}{a} \int_0^a b(s) ds \right)^r.$$

We note that, starting from the definition of weighted decreasing rearrangement, and using (2.2), we can obtain from Theorem 3.2.:

THEOREM 4.1: Let  $f$  be nonnegative and belonging to  $L^1$ . If  $f$  verifies (WRH), with:

$$(4.1) \quad 1 < k < \sqrt{1 + \frac{c_1}{32^{n+1}(2A+1)A^3}}$$

then, for any  $0 < t < \mu(Q_0)/32^{n+1}A^2$  we have:

$$\frac{1}{t} \int_0^t f^{*,\mu}(s) ds \leq \frac{c_1}{c_1 - 3(2A+1)A^3 2^{n+1} \sqrt{k^2-1}} f^{*,\mu}(t).$$

Using Lemma 4.1. and Theorem 4.1. a higher integrability result for  $f$  is easily deduced if (WRH) and (4.1) hold.

THEOREM 4.2: Let  $f$  be nonnegative and belonging to  $L^1$ . We suppose that (WRH) and (4.1) are satisfied. Then  $f \in L^1(Q_0, \mu)$  for any:

$$(4.2) \quad 1 \leq p < \frac{c_1}{3(2A+1)A^3 2^{n+1} \sqrt{k^2-1}}$$

and:

$$(4.3) \quad \int_{Q_0} f^p d\mu \leq \left( \frac{c_1 32^n A^2}{c_1 - 3(2A+1)A^3 2^{n+1} \sqrt{k^2-1}} \right)^{p-1} \cdot \left( \frac{c_1 - 3p(2A+1)A^3 2^{n+1} \sqrt{k^2-1}}{c_1 - 3(2A+1)A^3 2^{n+1} \sqrt{k^2-1}} \right) \left( \int_{Q_0} f d\mu \right)^p.$$

REMARK 4.1: If  $k$  tends to one, the higher integrability exponent in (4.2) tends to infinity like  $\gamma/\sqrt{k^2-1}$ , where  $\gamma$  is a constant depending only on  $\mu, \nu$  and  $n$ .

It is easy to prove (see e.g. [B]) that the result is optimal. Moreover the majorita-tion constant in (4.3) tends to one if  $f$  tends to one.

REMARK 4.2: Condition (WRH) is not puntual and is not a condition on every subcube  $Q$  of  $Q_0$ , like condition (G), but is a condition on the measure of the level sets of the maximal function.

PROOF OF THEOREM 4.2: We note that in our hypotheses, Theorem 4.1. works and then we can apply Lemma 4.1 to  $f^{*,\mu}$ .

REMARK 4.3: In the above hypotheses, a more general condition can be consid-ered. Namely, if we set:

$$\hat{f}_\mu^\#(x) = \sup_Q \int_Q \left| f - \int_Q f d\mu \right| d\mu$$

then, from the condition:

$$\mu(\{x \in Q_0 : \hat{f}_\mu^\#(x) > \lambda\}) \leq \nu\left(\left\{x \in Q_0 : M_\nu f(x) > \frac{\lambda}{\varepsilon}\right\}\right)$$

for any  $\lambda > 0$ ,  $0 < \varepsilon < \sqrt{c_1/(32^{n+1}(2A+1)A^3)}$ , higher integrability is deduced for  $f$ .

## REFERENCES

- [BDS] C. BENNETT - R. DE VORE - R. SHARPLEY, *Weak- $L^\infty$  and BMO*, *Ann. Math.*, (2), 113 (1981), 601-611.
- [B] B. BOJARSKI, *Remarks on the stability of reverse Hölder inequalities and quasiconformal mappings*, *Ann. Acad. Sci. Fenn. Ser. AI Math.*, 10 (1985), 291-296.



- [BSW] B. BOJARSKI - C. SBORDONE - I. WIK, *The Muckenhoupt class  $A_1(R)$* , to appear.
- [FM] M. FRANCIOSI - G. MOSCARIELLO, *Higher integrability results*, Manuscripta Math., 52 (1985), 151-171.
- [G] F. GEHRING, *The  $L^p$ -integrability of the partial derivatives of a quasiconformal mapping*, Acta Math., 130 (1973), 265-277.
- [H] C. HERZ, *The Hardy-Littlewood maximal theorem*, in, *Symposium on Harmonic Analysis*, Univ. of Warwick (1968).
- [M] B. MUCKENHOUT, *Weighted inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., 165 (1972), 207-226.
- [S] C. SBORDONE, *Some reverse integral inequalities*, Atti Accad. Pontaniana, Napoli (1984).
- [W] I. WIK, *Reverse Hölder inequalities with constant close to 1*, Ricerche Mat., XXXIX (1), (1990), 151-157.