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Existence and Uniqueness Results for the Dirichlet Problem in Unbounded Domains (**) (***)

SUMMARY. — In this paper we study the Dirichlet problem for a class of second order linear elliptic partial differential equations with discontinuous coefficients in unbounded domains of R^n. We obtain some existence and uniqueness results.

Teoremi di esistenza ed unicità per il problema di Dirichlet in aperti non limitati

RIASSUNTO. — In questo lavoro si studia il problema di Dirichlet per una classe di equazioni differenziali lineari ellittiche del secondo ordine a coefficienti discontinui in aperti non limitati di R^n. Si ottengono alcuni teoremi di esistenza ed unicità.

INTRODUCTION

We consider in an open subset Ω of R^n, $n \geq 2$, the uniformly elliptic linear differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au \tag{1}$$

with real coefficients.

Suitable regularity hypotheses and behaviour to the infinity of the coefficients a_i $(i = 1, \ldots, n)$ and a (see n. 3) are given, while

$$a_{ij} = a_{ji} \in L^\infty(\Omega)\,, \qquad i,j = 1, \ldots, n\,. \tag{2}$$

We are concerned with the problem

$$u \in W^2(\Omega) \cap \overset{\circ}{W}{}^1(\Omega)\,, \qquad Lu = f, \qquad f \in L^2(\Omega)\,. \tag{3}$$

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It is well known that, if $n>2$, condition (2) doesn't assure uniqueness for problem (3), whatever hypotheses on other data (see *e.g.* O. A. Ladyzhenskaja - N. N. Ural'tseva [8], D. Gilbarg - N. S. Trudinger [6]).

In some recent papers (see *e.g.* [10], [13], [14]) M. Troisi and one of the Authors studied problem (3) in an open unbounded and sufficiently regular Ω, with additional hypotheses on the a_{ij}. Some a priori bounds and existence and uniqueness results, extending to unbounded domains of R^n some classical results for the bounded domains (see e.g. [8], C. Miranda [9], M. Chicco [2], [3], [4], [5]), are given.

In particular in [10] hypotheses like those of C. Miranda [9] on the a_{ij} are given, namely

$$a_{ij} \in W^{1,s}_{\mathrm{loc}}(\Omega), \qquad i,j = 1, \ldots, n,$$

where $s>2$ if $n=2$ and $s=n$ if $n>2$, together with the condition to the infinity

$$\lim_{|x| \to +\infty} \|(a_{ij})_{x_h}\|_{L^s(\Omega \cap B(x,1))} = 0, \qquad i,j,h = 1, \ldots, n,$$

where $B(x,1) = \{y \in R^n \mid |y-x| < 1\}$. In [13], less restrictive regularity hypotheses on the a_{ij} are imposed, but together with the condition to the infinity

$$\lim_{|x| \to +\infty} a_{ij}(x) = a^0_{ij}.$$

In [14] the following case has been dealt with:

$$a_{ij} = a'_{ij} + a''_{ij}, \qquad a'_{ij} = a'_{ji} \in L^\infty(\Omega) \cap W^{1,s}_{\mathrm{loc}}(\overline{\Omega}), \qquad a''_{ij} = a''_{ji} \in C(\overline{\Omega})$$

and

$$\lim_{|x| \to +\infty} \|(a'_{ij})_{x_h}\|_{L^s(\Omega \cap B(x,1))} = 0, \qquad \lim_{|x| \to +\infty} a''_{ij}(x) = a''^0_{ij}$$

or

$$\lim_{|x| \to +\infty} a_{ij}(x) = a^0_{ij},$$

where $C(\overline{\Omega})$ is the space of uniformly continuous bounded functions on Ω.

Successively (see [15]) the same Authors studied problem (3) in more general hypotheses on the a_{ij}, and extended an a priori bound that in the previous papers is basic for some existence and uniqueness results.

In these hypotheses, like is pointed out by the Authors in [15], is not possible, with the methods of [10], [13], [14], to deduce from the a priori bound the theorems of existence and uniqueness.

In this paper we consider problem (3) in the hypotheses of [15], and in addition we suppose that $\partial\Omega$ is bounded. In such hypotheses we prove that problem (3) is uniquely solvable.

To obtain this result we use the a priori bound established in [15] and a new a priori bound proved in this paper for the solutions of the problem

$$u \in W^2(\Omega_r) \cap \overset{\circ}{W}{}^1(\Omega_r), \qquad Lu = f, \qquad f \in L^2(\Omega).$$

Here $\Omega_r = \{x \in \Omega \mid |x| < r\}$ and the constant of the bound doesn't depend on $r \in [r_0, + + \infty[$, where r_0 is a sufficiently large, fixed value.

1. - Some preliminary facts

We set

$$B(x, r) = \{y \in R^n \mid |y - x| < r\}, \qquad B_r = B(0, r),$$

$$|u|_{p,E} = \|u\|_{L^p(E)}, \qquad u_x = \left(\sum_{i=1}^{n} u_{x_i}^2\right)^{1/2}, \qquad u_{xx} = \left(\sum_{i,j=1}^{n} u_{x_i x_j}^2\right)^{1/2}.$$

Let E be an open subset of R^n, $n \geq 2$.

We denote by $W^m(E)$, $m \in N$, the usual Sobolev space $W^{m,2}(E)$ and by $\overset{\circ}{W}{}^m(E)$ the closure of $\mathcal{D}(E)$ in $W^m(E)$.

Moreover we set $W^0(E)$ for $L^2(E)$.

If E is unbounded, for any $p \in [1, +\infty[$ $M^p(E)$ denotes the space of all functions $f \in L^p_{\mathrm{loc}}(\overline{E})$ such that

$$\|f\|_{M^p(E)} = \sup_{x \in E} |f|_{p, E \cap B(x,1)} < +\infty, \tag{1.1}$$

normed by (1.1), and $M^p_0(E)$ denotes the subspace of $M^p(E)$ of all functions f such that

$$\lim_{|x| \to +\infty} |f|_{p, E \cap B(x,1)} = 0. \tag{1.2}$$

For some properties of the spaces $M^p(E)$ and $M^p_0(E)$ we refer to [10], [11].

Let s, t be two real numbers such that

$$s > 2 \text{ if } n = 2, \qquad s = n \text{ if } n > 2, \tag{1.3}$$

$$t = 2 \text{ if } 2 \leq n < 4, \qquad t > 2 \text{ if } n = 4, \qquad t = n/2 \text{ if } n > 4. \tag{1.4}$$

If $A \subset E$ is an open subset, $k \in N$ and $\nu \in R_+$, we write $E_k(\nu, A)$ for the class of the $k \times k$ square matrices $((e_{ij}))$ such that

$$e_{ij} = e_{ji} \in L^\infty(A) \cap W^{1,s}_{\mathrm{loc}}(\overline{A}), \qquad i, j = 1, \ldots, k, \tag{1.5}$$

$$\sum_{i,j=1}^{k} e_{ij} \xi_i \xi_j \geq \nu |\xi|^2 \quad \text{a.e. in } A, \qquad \forall \xi \in R^k. \tag{1.6}$$

Moreover, we denote by $G(A)$ the class of all functions $g \in L^\infty(A)$ such that $\operatorname*{ess\,inf}_A g > 0$.

We consider in E the second order linear differential operator L defined by (1) and write

$$L_0 u = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}. \tag{1.7}$$

We say that L_0 is of Chicco type in E (see [3]-[5] and [13]-[15]) if

$$a_{ij} = a_{ji} \in L^\infty(E), \quad i,j = 1, \dots, n, \tag{1.8}$$

and there exist $\nu \in R_+$, a matrix $((e_{ij})) \in E_n(\nu, E)$ and a function $g \in G(E)$ such that

$$\operatorname*{ess\,sup}_E \sum_{i,j=1}^{n} (e_{ij} - g a_{ij})^2 < \nu^2. \tag{1.9}$$

2. - Preliminary lemmas

Lemma 2.1: *Let $r_1 \in R_+$. For each $r \in [r_1, +\infty[$ there exists a continuous linear operator*

$$q_r \colon W^2(B_r) \to W^2(R^n)$$

such that

$$q_r u_{|B_r} = u, \tag{2.1}$$

$$\|q_r u\|_{W^k(R^n)} \leq c \|u\|_{W^k(B_r)}, \qquad k = 0, 1, 2, \tag{2.2}$$

where $c \in R_+$ is independent of r and u.

Proof: It is well known that there exists q_{r_1} satisfying these properties (see R. A. Adams [1]).

We note that, given $r \in [r_1, +\infty[$ and $u \in W^2(B_r)$ the function

$$v(y) = u\left(\frac{r}{r_1} y\right) \in W^2(B_{r_1})$$

and hence $q_{r_1} v \in W^2(R^n)$.

Write $w_r(x) = (q_{r_1} v)\left(\frac{r_1}{r} x\right)$, for every $x \in R^n$, then we define

$$q_r \colon u \in W^2(B_r) \to q_r u = w_r \in W^2(R^n).$$

If $x \in B_r$, we have

$$w_r(x) = v\left(\frac{r_1}{r} x\right) = u(x)$$

and then (2.1) is proved.

Moreover, we have that

$$(2.3)\qquad |w_r|_{2,R^n} = \left(\int_{R^n} \left((q_{r_1} v)\left(\frac{r_1}{r}x\right)\right)^2 dx\right)^{1/2} =$$

$$= \left(\int_{R^n} \frac{r^n}{r_1^n}((q_{r_1} v)(y))^2\, dy\right)^{1/2} \leq c\left(\int_{B_{r_1}} \frac{r^n}{r_1^n} v^2\, dy\right)^{1/2} = c|u|_{2,B_r},$$

with $c \in R_+$ independent of r and u, then (2.2) for $k=0$ is obtained.

On the other hand we get

$$(2.4)\qquad |(w_r)_x|_{2,R^n} \leq c(r^{-1}|u|_{2,B_r} + |u_x|_{2,B_r}),$$

$$(2.5)\qquad |(w_r)_{xx}|_{2,R^n} \leq c(r^{-2}|u|_{2,B_r} + r^{-1}|u_x|_{2,B_r} + |u_{xx}|_{2,B_r}),$$

with $c \in R_+$ independent of r and u.

From (2.3), (2.4) and (2.5) the bound (2.2) for $k=1, 2$ easily follows.

Lemma 2.2: *Let ε_1, $r_1 \in R_+$. Then there exists a constant $c \in R_+$ such that*

$$(2.6)\qquad \int_{B_r} u_x^2\, dx \leq c\left(\varepsilon \int_{B_r} u_{xx}^2\, dx + \frac{1}{\varepsilon}\int_{B_r} u^2\, dx\right)$$

$$\forall \varepsilon \in]0, \varepsilon_1],\ \forall r \in [r_1, +\infty[\ \textit{and}\ \forall u \in W^2(B_r).$$

Proof: We fix $\varepsilon \in]0, \varepsilon_1]$, $r \in [r_1, +\infty[$, $u \in W^2(B_r)$ and put

$$v(y) = u(ry).$$

Since $v \in W^2(B_1)$, for any $\varepsilon_0 \in \left]0, \dfrac{\varepsilon_1}{r_1^2}\right]$ we obtain

$$(2.7)\qquad \int_{B_1} v_y^2\, dy \leq c(B_1)\left(\varepsilon_0 \int_{B_1} v_{yy}^2\, dy + \frac{1}{\varepsilon_0}\int_{B_1} v^2\, dy\right),$$

where $c(B_1) \in R_+$ is independent of ε_0 and v (see *e.g.* [1], pag. 75).

(2.6) follows now from (2.7) when $y = \dfrac{x}{r}$ and $\varepsilon_0 = \dfrac{\varepsilon}{r^2}$.

Now we assume:

i$_1$) let Ω be an unbounded open subset of R^n, $n \geq 2$, of class C^2 with bounded boundary.

Put

$$\Omega_r = \Omega \cap B_r \qquad \forall r \in R_+.$$

Choose $r_1 \in R_+$ such that

$$\partial\Omega \subset B_{r_1}$$

and let ψ be a function in $\mathcal{D}(R^n)$ such that

$$\psi|_{R^n \setminus \Omega} = 1, \qquad \operatorname{supp}\psi \subset B_{r_1}.$$

Lemma 2.3: *For any $r \in [r_1, +\infty[$ there exists a continuous linear operator*

$$p_r \colon W^2(\Omega_r) \to W^2(\Omega)$$

such that

$$p_r u|_{\Omega_r} = u, \tag{2.8}$$

$$\|p_r u\|_{W^k(\Omega)} \leqslant c\|u\|_{W^k(\Omega_r)}, \qquad k = 0, 1, 2, \tag{2.9}$$

where $c \in R_+$ is independent of r and u.

Proof: Let $r \in [r_1, +\infty[$ and $u \in W^2(\Omega_r)$.
Define

$$v = \begin{cases} \psi u & \text{in } \Omega_r \\ 0 & \text{in } \Omega \setminus \Omega_r, \end{cases} \qquad w = \begin{cases} (1-\psi)\, u & \text{in } \Omega_r \\ 0 & \text{in } B_r \setminus \Omega_r, \end{cases} \qquad p_r u = v + (q_r w)_{|\Omega},$$

where q_r is the operator in Lemma 2.1.
From (2.1) and (2.2) we easily obtain (2.8) and (2.9), and the result follows.

Lemma 2.4: *For any $\varepsilon \in R_+$ there exists a constant $c(\varepsilon) \in R_+$ such that*

$$\int_{\Omega_r} u_x^2\, dx \leqslant \varepsilon \int_{\Omega_r} u_{xx}^2\, dx + c(\varepsilon) \int_{\Omega_r} u^2\, dx \qquad \forall r \in [r_1, +\infty[\text{ and } \forall u \in W^2(\Omega_r). \tag{2.10}$$

Proof: Let $r \in [r_1, +\infty[$ and $u \in W^2(\Omega_r)$.
If w is the map defined in the proof of Lemma 2.3, we have

$$\int_{\Omega_r} u_x^2\, dx \leqslant 2\left(\int_{\Omega_r} (\psi u)_x^2\, dx + \int_{\Omega_r} ((1-\psi)\, u)_x^2\, dx \right) = 2\left(\int_{\Omega_{r_1}} (\psi u)_x^2\, dx + \int_{B_r} w_x^2\, dx \right). \tag{2.11}$$

It is well known that

$$\int_{\Omega_{r_1}} (\psi u)_x^2 \leqslant \varepsilon_1 \int_{\Omega_{r_1}} u_{xx}^2\, dx + c_1(\varepsilon_1) \int_{\Omega_{r_1}} u^2\, dx, \tag{2.12}$$

with $c_1(\varepsilon_1) \in R_+$ independent of u.

Moreover, from Lemma 2.2 it follows that

$$\int_{B_r} w_x^2\,dx \leqslant \varepsilon_2 \int_{B_r} w_{xx}^2\,dx + c_2(\varepsilon_2)\int_{B_r} w^2\,dx, \tag{2.13}$$

with $c_2(\varepsilon_2) \in R_+$ independent of r and u.

From (2.11), (2.12) and (2.13) we obtain easily (2.10), as required.

3. - An a priori bound

We consider in Ω the second order linear differential operators L and L_0 defined respectively by (1) and (1.7).

Now we assume the following:

i$_2$) L_0 is of Chicco type in Ω; moreover

$$a_i \in M_0^s(\Omega), \qquad i = 1, \ldots, n, \tag{3.1}$$

$$a = a' + a'', \qquad a' \in M_0^t(\Omega), \qquad a'' \in M^t(\Omega); \tag{3.2}$$

i$_3$) there exist $\mu, \mu_0, r_0 \in R_+$, $((\alpha_{ij})) \in E_n(\mu, \Omega \setminus \overline{B}_{r_0})$, $((\alpha)) \in E_1(\mu_0, \Omega \setminus \overline{B}_{r_0})$, $\eta \in G(\Omega)$ such that

$$(\alpha_{ij})_{x_h}, \alpha_{x_h} \in M_0^s(\Omega \setminus \overline{B}_{r_0}) \qquad i, j, h = 1, \ldots, n, \tag{3.3}$$

$$\mu^{-2} \operatorname*{ess\,sup}_{\Omega \setminus \overline{B}_{r_0}} \sum_{i,j=1}^{n} (\alpha_{ij} - \eta a_{ij})^2 + \mu_0^{-2} \operatorname*{ess\,sup}_{\Omega \setminus \overline{B}_{r_0}} (\alpha - \eta a'')^2 < 1. \tag{3.4}$$

We note that the last condition is equivalent to condition i$_3$) defined in n. 3 of [15], where two examples of operators for which it holds are quoted (see examples 4 and 5 of [15]).

Next example contains the two quoted examples of [15].

Example 3.1: If we suppose:

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant a_0 |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in R^n,$$

$$a_{ij} = b_{ij} + c_{ij}, \qquad b_{ij} = b_{ji} \in L^\infty(\Omega), \qquad (b_{ij})_{x_h} \in M_0^s(\Omega),$$

$$\lim_{|x| \to +\infty} c_{ij}(x) = c_{ij}^0, \qquad c_{ij}^0 = c_{ji}^0, \qquad a'' \in G(\Omega),$$

then, clearly, i$_3$) holds with

$$\mu = \frac{a_0}{2}, \qquad \mu_0 = \operatorname*{ess\,sup}_{\Omega} a'', \qquad \alpha_{ij} = b_{ij} + c_{ij}^0, \qquad \alpha = \mu_0, \qquad \eta = 1$$

and $r_0 \in R_+$ such that

$$\sum_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j \geqslant \mu|\xi|^2 \quad \text{a.e. in } \Omega\setminus\bar{B}_{r_0},\ \forall\xi\varepsilon R^n,$$

$$\operatorname*{ess\,sup}_{\Omega\setminus\bar{B}_{r_0}} \sum_{i,j=1}^{n} (c_{ij}^0 - c_{ij})^2 < \mu^2 (\operatorname*{ess\,inf}_{\Omega} a''/\mu_0)^2 .$$

REMARK 3.1: From Example 3.1 we deduce that i$_2$) and i$_3$) are satisfied by the hypotheses, mentioned in the introduction, contained in the papers [10, 13, 14].

Let β denote a mapping $\Omega \to R_+$ such that

(3.5) $$\beta \in M^t(\Omega) \text{ and } \exists\delta \in M_0^s(\Omega) \quad \ni' \beta_x \leqslant \beta\delta,$$

(3.6) $$\beta, \beta^{-1} \in L^\infty_{\mathrm{loc}}(\bar{\Omega}).$$

Let $r_1 \in R_+$ such that

(3.7) $$r_1 > r_0, \quad \partial\Omega \subset B_{r_1},$$

where r_0 is the number defined in hypothesis i$_3$).

LEMMA 3.1: *If* i$_1$), i$_2$) *and* i$_3$) *hold, then there exist a constant* $c \in R_+$ *and a bounded open set* $\Omega_0 \subset \Omega$ *such that*:

(3.8) $$\|u\|_{W^2(\Omega_r)} \leqslant c(|Lu + \lambda\eta^{-1}\beta u|_{2,\Omega_r} + |u|_{2,\Omega_r\cap\Omega_0})$$

$$\forall\lambda \geqslant 0,\ \forall r \in [r_1, +\infty[\textit{ and } \forall u \in W^2(\Omega_r) \cap \overset{\circ}{W}{}^1(\Omega_r).$$

PROOF: Assume firstly that $L = L_0 + a''$, and fix $r_0' \in]r_0, r_1[$ and $\phi \in \mathcal{D}(R^n)$ such that $\partial\Omega \subset B_{r_0'}$, $\phi|_{B_{r_0'}} = 1$ and $\operatorname{supp}\phi \subset B_{r_1}$.

We first prove (3.8) for ϕu.

Since the operator $\eta\beta^{-1}L_0$ is of Chicco type in Ω_{r_1}, from [4] we obtain

(3.9) $$\|\phi u\|_{W^2(\Omega_r)} = \|\phi u\|_{W^2(\Omega_{r_1})} \leqslant c(r_1)(|(\eta\beta^{-1}(L_0 + a''))\,\phi u + \lambda\phi u|_{2,\Omega_{r_1}} + |\phi u|_{2,\Omega_{r_1}}) \leqslant$$
$$\leqslant c'(r_1)(|(L_0 + a'')\,\phi u + \lambda\eta^{-1}\beta\phi u|_{2,\Omega_{r_1}} + |\phi u|_{2,\Omega_{r_1}}) =$$
$$= c'(r_1)(|(L_0 + a'')\,\phi u + \lambda\eta^{-1}\beta\phi u|_{2,\Omega_r} + |\phi u|_{2,\Omega_r\cap\Omega_{r_1}}),$$

where $c(r_1), c'(r_1) \in R_+$ are independent of λ, r and u.

We prove now (3.8) for $v = (1 - \phi)\,u$.

Write $E_r = \Omega_r \setminus \bar{\Omega}_{r_0}$, and consider in E_r the operator

$$Av = -\sum_{i,j=1}^{n} \alpha_{ij} v_{x_i x_j}.$$

Note that $\operatorname{supp} v \subset \bar{\Omega}_r \setminus \Omega_{r_0'}$. Then, from well-known results (see [8], pag. 152, 162), using methods similar to those in [15] to establish (36), (37) and (38), it follows that

$$\mu^2 \int_{E_r} v_{xx}^2\,dx \leqslant \int_{E_r} (Av)^2\,dx + \varepsilon_1 \int_{E_r} v_{xx}^2\,dx + c(\varepsilon_1) \int_{E_r} \sum_{i,j=1}^{n} (\alpha_{ij})_x^2 v_x^2 , \tag{3.10}$$

where $c(\varepsilon_1) \in R_+$ is independent of r and v.

On the other hand we have:

$$\int_{E_r} (Av)^2\,dx \leqslant \int_{E_r} (Av + (\alpha + \lambda\beta)v)^2\,dx - \tag{3.11}$$

$$-(1-\varepsilon_2)\int_{E_r} (\alpha + \lambda\beta)^2 v^2\,dx + c(\varepsilon_2)\int_{E_r} \left(\sum_{i,j=1}^{n} (\alpha_{ij})_x^2 + \alpha_x^2 + \delta^2 \right) v_x^2\,dx ,$$

where $c(\varepsilon_2) \in R_+$ is independent of λ, r and v (see (41)-(43) of [15]).

Since

$$\operatorname*{ess\,inf}_{E_r} (\alpha + \lambda\beta) \geqslant \mu_0 \quad \forall \lambda \geqslant 0 ,$$

from (3.10) and (3.11) we obtain that

$$(\mu^2 - \varepsilon_1)\int_{E_r} v_{xx}^2\,dx + (\mu_0^2 - \varepsilon_3)\int_{E_r} v^2\,dx \leqslant \tag{3.12}$$

$$\leqslant \int_{E_r} (Av + (\alpha + \lambda\beta)\,v)^2\,dx + c(\varepsilon_1, \varepsilon_3)\int_{E_r} \left(\sum_{i,j=1}^{n} (\alpha_{ij})_x^2 + \alpha_x^2 + \delta^2 \right) v_x^2\,dx ,$$

where $c(\varepsilon_1, \varepsilon_3) \in R_+$ is independent of λ, r and v.

Put

$$g = \left(\sum_{i,j=1}^{n} (\alpha_{ij})_x^2 + \alpha_x^2 + \delta^2 \right)^{1/2} .$$

Since $g \in M_0^s(\Omega \setminus \bar{B}_{r_0})$, from Theorem 3.2 of [7] and from Lemma 2.3 we get

$$\int_{E_r} g^2 v_x^2\,dx \leqslant \int_{\Omega \setminus \bar{\Omega}_{r_0}} g^2 (p_r v)_x^2\,dx \leqslant \tag{3.13}$$

$$\leqslant \varepsilon_4 \int_{\Omega} (p_r v)_{xx}^2\,dx + c(\varepsilon_4) \int_{\Omega \cap \Omega(\varepsilon_4)} (p_r v)^2\,dx \leqslant \varepsilon_4 c \|v\|_{W^2(\Omega_r)} + c(\varepsilon_4) \int_{\Omega \cap \Omega(\varepsilon_4)} (p_r v)^2\,dx ,$$

where the constants c, $c(\varepsilon_4) \in R_+$ and the bounded open set $\Omega(\varepsilon_4) \subset \Omega$ are independent of r and v.

From (3.12), (3.13) and Lemma 2.4, we have that for any $\varepsilon' \in]0,\mu[$ and for any $\varepsilon'' \in]0,\mu_0[$ the following holds:

$$(3.14) \qquad (\mu-\varepsilon')^2 |v_{xx}|^2_{2,E_r} + (\mu_0-\varepsilon'')^2 |v|^2_{2,E_r} \leqslant$$

$$\leqslant |Av + (\alpha+\lambda\beta)\, v|^2_{2,E_r} + c(\varepsilon',\varepsilon'')|p_r v|^2_{2,\Omega\cap\Omega(\varepsilon',\varepsilon'')},$$

where the constant $c(\varepsilon',\varepsilon'') \in R_+$ and the bounded open set $\Omega(\varepsilon',\varepsilon'') \subset \Omega$ are independent of λ, r and v.

With the same proof of [15] to deduce (33) from (45), from (3.14) we have

$$(3.15) \qquad |v_{xx}|_{2,\Omega_r} + |v|_{2,\Omega_r} \leqslant c_1(|(L_0+a'')\, v + \lambda\eta^{-1}\beta v|_{2,\Omega_r} + |p_r v|_{2,\Omega\cap\Omega^*}),$$

where the constant $c_1 \in R_+$ and the bounded open set $\Omega^* \subset \Omega$ are independent of λ, r and v.

Now we choose $\sigma \in]r_1, +\infty[$ such that $\Omega^* \subset \Omega_\sigma$.

From Lemma 2.3 we have

$$(3.16) \qquad |p_r v|_{2,\Omega\cap\Omega_\sigma} \leqslant c|v|_{2,\Omega_r\cap\Omega_\sigma}.$$

(3.8) for v follows now from (3.15), (3.16) and Lemma 2.4.

Then we have

$$(3.17) \qquad \|u\|_{W^2(\Omega_r)} \leqslant \|\phi u\|_{W^2(\Omega_r)} + \|(1-\phi)\, u\|_{W^2(\Omega_r)} \leqslant$$

$$\leqslant c(|L_0(\phi u) + a''\phi u + \lambda\eta^{-1}\beta\phi u|_{2,\Omega_r} + |\phi u|_{2,\Omega_r\cap\Omega_0} +$$

$$+ |L_0((1-\phi)\, u) + a''(1-\phi)\, u + \lambda\eta^{-1}\beta(1-\phi)\, u|_{2,\Omega_r} + |(1-\phi)\, u|_{2,\Omega_r\cap\Omega_0}) \leqslant$$

$$\leqslant c_2(|L_0 u + a'' u + \lambda\eta^{-1}\beta u|_{2,\Omega_r} + |u|_{2,\Omega_r\cap\Omega_0} + |u_x|_{2,\Omega_{r_1}} + |u|_{2,\Omega_{r_1}}),$$

with $c_2 \in R_+$ independent of λ, r and u.

On the other hand, from Lemma 2.4 it follows:

$$(3.18) \qquad \int_{\Omega_{r_1}} u_x^2\, dx \leqslant \varepsilon \int_{\Omega_r} u_{xx}^2\, dx + c(\varepsilon) \int_{\Omega_{r_1}} u^2\, dx.$$

From (3.17) and (3.18) we deduce (3.8) for $L = L_0 + a''$.

In the general case for L we have then:

$$(3.19) \qquad \|u\|_{W^2(\Omega_r)} \leqslant c\left(|Lu + \lambda\eta^{-1}\beta u|_{2,\Omega_r} + |u|_{2,\Omega_r\cap\Omega_0} + \left|\sum_{i=1}^n a_i u_{x_i} + a' u\right|_{2,\Omega_r}\right).$$

On the other hand, since $a_i \in M_0^s(\Omega)$, $i = 1, \dots, n$, and $a' \in M_0^t(\Omega)$, from Theorem

3.2 in [7] and from Lemma 2.3 we get

$$\left|\sum_{i=1}^{n} a_i u_{x_i} + a' u\right|_{2,\Omega_r} \leqslant \varepsilon_5 \|p_r u\|_{W^2(\Omega)} + c(\varepsilon_5) \int_{\Omega \cap \Omega(\varepsilon_5)} (p_r u)^2 \, dx \leqslant \tag{3.20}$$

$$\leqslant \varepsilon_5 c \|u\|_{W^2(\Omega_r)} + c(\varepsilon_5) \int_{\Omega \cap \Omega(\varepsilon_5)} (p_r u)^2 \, dx ,$$

where the constant $c(\varepsilon_5) \in R_+$ and the bounded open set $\Omega(\varepsilon_5) \subset \Omega$ are independent of r and u.

From (3.19) and (3.20) we deduce that:

$$\|u\|_{W^2(\Omega_r)} \leqslant c_3 (|Lu + \lambda \eta^{-1} \beta u|_{2,\Omega_r} + |u|_{2,\Omega_r \cap \Omega_0} + |p_r u|_{2,\Omega \cap \Omega^{**}}) , \tag{3.21}$$

where the constant $c_3 \in R_+$ and the bounded open set $\Omega^{**} \subset \Omega$ are independent of λ, r and u.

Now we choose $\tau \in]r_1, +\infty[$ such that $\Omega^{**} \subset \Omega_\tau$; then

$$|p_r u|_{2.\Omega \cap \Omega_\tau} \leqslant c_4 |u|_{2,\Omega_r \cap \Omega_\tau} , \tag{3.22}$$

where $c_4 \in R_+$ is independent of r and u.

The result follows from (3.21) and (3.22).

4. - Existence results

Let G be a bounded open subset of class C^2 of R^n, $n \geqslant 2$.

We consider in G the second order linear differential operators L and L_0 defined in (1) and (1.7) respectively.

We consider the problem

$$u \in W^2(G) \cap \overset{\circ}{W}{}^1(G) , \qquad Lu = f, \qquad f \in L^2(G) . \tag{4.1}$$

Lemma 4.1: *If L_0 is of Chicco type in G, $a_i \in L^s(G)$ $(i = 1, \ldots, n)$, $a \in L^t(G)$ and*

$$\operatorname*{ess\,inf}_G a > 0 , \tag{4.2}$$

then the problem (4.1) *is uniquely solvable. Moreover, if $f \in L^\infty(G)$, then the solution $u \in L^\infty(G)$ and satisfies*

$$|u|_{\infty,G} \leqslant (\operatorname*{ess\,inf}_G (ga))^{-1} |gf|_{\infty,G} , \tag{4.3}$$

where g *is the function defined as in the hypothesis of Chicco type.*

Proof: The existence and uniqueness result for the problem (4.1) follows from a theorem of M. Chicco (see [5]). The last statement is proved, as in the proof of the theorem of [5], noting that, if $f \in L^\infty(G)$, the solution u is the weak limit in $W^2(G)$ of a sequence $(u_k)_{k\in N}$ of functions such that:

$$|u_k|_{\infty,G} \leqslant (\operatorname*{ess\,inf}_G (ga))^{-1} |gf|_{\infty,G} \quad \forall k \in N. \tag{4.5}$$

Concerning problem (3), we prove

THEOREM 4.1: *If* $i_1)$ $i_2)$ *and* $i_3)$ *hold, then* (3) *is an index problem with index zero.*

If moreover we suppose that

$$\operatorname*{ess\,inf}_\Omega a > 0, \tag{4.6}$$

then problem (3) *is uniquely solvable.*

PROOF: We suppose that $i_1)$, $i_2)$, $i_3)$ and (4.6) hold.

Fix a strictly increasing sequence $(r_k)_{k\in N}$ of positive real numbers, with r_1 satisfying (3.7).

Suppose firstly that $f \in L^2(\Omega) \cap L^\infty(\Omega)$ and consider, for every $k \in N$ the problem

$$u \in W^2(\Omega_{r_k}) \cap \overset{\circ}{W}{}^1(\Omega_{r_k}), \qquad Lu = f. \tag{4.7}$$

From Lemmas 3.1 and 4.1 it follows that the solution u_k, $k \in N$, of problem (4.7) belongs to $L^\infty(\Omega_{r_k})$ and satisfies

$$\|u_k\|_{W^2(\Omega_{r_k})} \leqslant c(|f|_{2,\Omega_{r_k}} + |u_k|_{2,\Omega_{r_k}\cap\Omega_0}), \tag{4.8}$$

$$|u_k|_{\infty,\Omega_{r_k}} \leqslant (\operatorname*{ess\,inf}_{\Omega_{r_k}} (ga))^{-1} |gf|_{\infty,\Omega_{r_k}}, \tag{4.9}$$

where the constant $c \in R_+$ and the bounded open set $\Omega_0 \subset \Omega$ are independent of k.

From (4.8) and (4.9) we get

$$\begin{aligned} \|u_k\|_{W^2(\Omega_{r_k})} &\leqslant c(|f|_{2,\Omega_{r_k}} + (\text{mis}\,(\Omega_{r_k}\cap\Omega_0))^{1/2} (\operatorname*{ess\,inf}_{\Omega_{r_k}} (ga))^{-1} |gf|_{\infty,\Omega_{r_k}}) \leqslant \\ &\leqslant c(|f|_{2,\Omega} + (\text{mis}\,\Omega_0)^{1/2} (\operatorname*{ess\,inf}_\Omega (ga))^{-1} |gf|_{\infty,\Omega}) \quad \forall k \in N. \end{aligned} \tag{4.10}$$

Write $w_k = p_{r_k} u_k$, $k \in N$, where p_{r_k} is the operator defined in Lemma 2.3.

From Lemma 2.3 and from (4.10) it follows that $w_k \in W^2(\Omega)$, $k \in N$, and there exists a constant $c_0 \in R_+$ such that

$$\|w_k\|_{W^2(\Omega)} \leqslant c_0 \quad \forall k \in N. \tag{4.11}$$

From (4.11) we deduce the existence of a subsequence of $(w_k)_{k\in N}$ weakly convergent in $W^2(\Omega)$ to a function $u \in W^2(\Omega) \cap \overset{\circ}{W}{}^1(\Omega)$. Since u_k is solution of (4.7) for every $k \in N$, with standard considerations we prove that u is solution of problem (3) with $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

From Theorem 3 of [15] and well-known results, it follows that the range $R(L)$ of the operator

$$L\colon u \in W^2(\Omega) \cap \overset{\circ}{W}{}^1(\Omega) \to Lu \in L^2(\Omega)$$

is a closed subspace of $L^2(\Omega)$. Moreover, from above result we have that $L^2(\Omega) \cap L^\infty(\Omega) \subset R(L)$.

On the other hand, $L^2(\Omega) \cap L^\infty(\Omega)$ is dense in $L^2(\Omega)$; then we have:

$$R(L) = L^2(\Omega)\,. \tag{4.12}$$

Suppose now that i$_1$), i$_2$), i$_3$) hold, but (4.6) doesn't hold.

Consider a function $\beta\colon \Omega \to R_+$ of class $M_0^t(\Omega)$ and satisfying (3.5) and (3.6).

For example

$$\beta\colon x \in \Omega \to \frac{1}{(1+|x|)^\tau}\,, \qquad \tau \in R_+\,.$$

We note now that hypothesis i$_3$) implies:

$$b_0 = \operatorname*{ess\,inf}_{\Omega \setminus \bar{B}_{r_0}} a'' > 0\,.$$

Fix $\zeta \in \mathcal{D}(R^n)$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta|_{B_{r_0}} = 1$, $\operatorname{supp} \zeta \subset B_{2r_0}$ and put:

$$b = \zeta b_0 + (1-\zeta)\, a''$$

Clearly we have

$$\operatorname*{ess\,inf}_{\Omega} b \geqslant b_0\,, \qquad a - b = a' + \zeta(a'' - b_0) \in M_0^t(\Omega)\,,$$

$$\mu^{-2} \operatorname*{ess\,sup}_{\Omega \setminus \bar{B}_{2r_0}} \sum_{i,j=1}^n (\alpha_{ij} - \eta a_{ij})^2 + \mu_0^{-2} \operatorname*{ess\,sup}_{\Omega \setminus \bar{B}_{2r_0}} (\alpha - \eta b)^2 < 1\,.$$

We consider the operator

$$A_\lambda\colon u \in W^2(\Omega) \cap \overset{\circ}{W}{}^1(\Omega) \to -\sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n a_i u_{x_i} + (b + \lambda \eta^{-1} \beta)\, u \in L^2(\Omega)\,.$$

From above results we have:

$$R(A_\lambda) = L^2(\Omega) \qquad \forall \lambda \geqslant 0\,. \tag{4.13}$$

On the other hand, with the same proof of [10] to deduce Corollary 4.2 from Theorem 4.4, from Theorem 3 of [15] it follows that there exists $\lambda_0 \in R_+$ such that

$$N(A_\lambda) = \{0\} \qquad \forall \lambda \geqslant \lambda_0\,, \tag{4.14}$$

where $N(A_\lambda)$ is the kernel of the operator A_λ.

From (4.13) and (4.14) it follows that for every $\lambda \geqslant \lambda_0$ A_λ is a bijective operator.

Since the operator:

$$u \in W^2(\Omega) \to (a - b - \lambda\eta^{-1}\beta)\, u \in L^2(\Omega)$$

is compact (see Lemma 3.4 of [10]) and:

$$Lu = A_\lambda u + (a - b - \lambda\eta^{-1}\beta)\, u\,,$$

we deduce, from well-known results, that (3) is an index problem with index zero.

If also (4.6) holds, from above and from (4.12) we obtain that (3) is uniquely solvable.

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