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MARIA LAVINIA RICCI (*)

A Non Linear Model of the Motion of Vibrating and not Tensioned String (**) (***)

SUMMARY. — A non linear model of a not tensioned string is studied with the assumption that the motion is trasversal. The model is constituted by a variational partial differential equation obtained as limit of a sequence of discrete models corresponding to ordinary differential equations. An existence theorem for the solution of the Cauchy-Dirichlet problem is proved.

Un modello non lineare di corda vibrante e non tesa

RIASSUNTO. — Si studia un modello non lineare di corda non completamente tesa, soggetta a moto trasversale. Il modello è costituito dalla equazione variazionale alle derivate parziali ottenuta come limite di una successione di modelli discreti corrispondenti a equazioni differenziali ordinarie. Si dimostra un teorema di esistenza delle soluzioni del problema di Cauchy-Dirichlet.

1. - THE PHYSICAL AND THE MATHEMATICAL MODELS

Starting from the classical model corresponding to the equation of D'Alembert, many models have, in the past, been proposed for the study of the transversal motion of an elastic string (see, for instance [1], [2], [3], [4]).

If the string is stretched in its rest position on the x axis between the points $x = 0$ and $x = L$, and if its unstretched length is Λ, a basic assumption that has always been made is that L must be $\geq \Lambda$.

Purpose of this note is to study the case $L < \Lambda$. Precisely, we shall assume that the string is homogeneous and fixed at the points $x = 0$ and $x = L$ of the x axis and that no

(*) Indirizzo dell'Autore: Dipartimento di Matematica del Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano.

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tension is exercised until, during its motion, the length of the string is $\leqslant \Lambda$; moreover, the external force $f(x, t)$ is normal to the x axis and exercised in the (x, y) plane, the motion occurs on the (x, y) plane and is trasversal.

We shall study the Cauchy-Dirichlet problem: denoting by $y(x, t)$ the elongation of the point x at the time t and setting

$$y(x, 0) = \bar{y}(x), \quad y_t(x, 0) = \bar{y}'(x), \quad y(0, t) = y(L, t) = 0,$$

we shall prove an existence theorem of the solution in a suitable functional class.

Precisely, § 1 will be devoted to the construction of a finite dimensional «approximate» model and the deduction of the «limit» equation, which we shall assume as governing the motion of the string. In the subsequent § §, using an existence and uniqueness theorem for the solutions of the «approximate» models, we shall prove that these solutions converge to a solution of the problem we are considering.

Taking as a basis the classical procedure followed in [5] and [6], we obtain the mathematical model of the problem substituting the string by a sistem of n rigid elements, every element is constituted by a small rod of length L/n, with telescopic springs, every spring is elastic and reacts when the length of the element is $> \Lambda/n$. These elements are connected by hinges at the points $p_i^{(n)}\,(iL/n, y_i^{(n)})$, sliding orthogonally to the x axis. Moreover we shall suppose that the external force $f(x, t)$ is concentrated on the hinges.

We introduce the following notations, where $n \in \mathrm{N}$, $i = -2, -1, \dots, n+1, n+2$;

$$\begin{aligned}
&b^{(n)} = L/n, \qquad n \in \mathrm{N};\\
&\Delta = \{x\colon 0 \leqslant x \leqslant L\};\\
&\Delta_i^{(n)} = \{x\colon (i - 1/2)b^{(n)} \leqslant x \leqslant (i + 1/2)b^{(n)}\};\\
&\bar{\Delta}_i^{(n)} = \{x\colon ib^{(n)} \leqslant x \leqslant (i+1)b^{(n)}\};\\
&Q = \{(x, t)\colon x \in \Delta, t \in [0, T]\};\\
&\{f_i^{(n)}\} = \{f_{n-2}, f_{n-1}, \dots, f_{n+1}, f_{n+2}\};\\
&\delta f_i = f_{i+1} - f_i;
\end{aligned}$$

obviously, for the sake of simplicity we shall write $\Delta_i, \bar{\Delta}_i, \dots$ instead of $\Delta_i^{(n)}, \bar{\Delta}_i^{(n)}, \dots$ when no confusion will be possible.

Owing to the nature of the problem, many functions are defined only on Δ or on Q; we shall extend them, if necessary, as follows

$$\text{(1.1)} \quad f(-x, t) = -f(x, t), \qquad f(2L - x, t) = -f(x, t), \qquad (x, t) \in \bigcup_{i=1}^{n} \Delta_i^{(n)} \times [0, T].$$

We obtain the system of ordinary differential equations associated to the discrete model as in [6].

In accordance with the physical properties of the model, we suppose that there is a reaction of the springs only if the elements have length $> \Lambda / n$, and that the reaction encreases linearly; then the tension $T^{(n)}$ of the generic element is connected to its length ζ by the relationship

$$T^{(n)} = \psi^{(n)}(\zeta) = \begin{cases} K^{(n)}(\zeta - \Lambda / n), & \zeta > \Lambda / n , \\ 0, & \zeta \leq \Lambda / n , \end{cases}$$

We suppose that the elements have similar properties, because the string is homogeneous, then we can calculate $K^{(n)}$ when ζ is the length of the system (or the string) and ζ / n is the length of every element. In this case, indicating by

$$T = \psi(\zeta) = \begin{cases} K(\zeta - \Lambda), & \zeta > \Lambda , \\ 0, & \zeta \leq \Lambda , \end{cases}$$

the connection between the tension of the string and its length, we have $T = T^{(n)}$, and finally,

$$\psi^{(n)}(\zeta / n) = \psi(\zeta), \qquad K^{(n)}\left(\frac{\zeta}{n} - \frac{\Lambda}{n}\right) = K(\zeta - \Lambda),$$

that is

$$K^{(n)} = K / n ,$$

$$T^{(n)} = \psi^{(n)}(\zeta) = \begin{cases} \dfrac{K}{n}(\zeta - \Lambda / n), & \zeta > \Lambda / n , \\ 0, & \zeta \leq \Lambda / n , \end{cases}$$

where K is obviously a constant of the problem.

Indicating by

$$f_i(t) = \frac{1}{b^{(n)}} \int_{\Delta_i^{(n)}} f(x, t)\, dx \tag{1.2}$$

the external force acting at the point $P_i^{(n)}(ib^{(n)}, y_i^{(n)})$, writing kinetic and potential energy of the system (E_c and E_p respectively)

$$E_c = \sum_{i=1}^{n-1} \frac{M}{2n} (y_i'^{(n)})^2 ,$$

$$E_p = \sum_{i=1}^{n} \int_{\zeta_i^{(n)}(0)}^{\zeta_i^{(n)}(t)} \psi^{(n)}(\zeta)\, d\zeta - b \sum_{i=1}^{n} \int_0^t f_i(\tau) y_i(\tau)\, d\tau ,$$

using Hamilton's principle, we obtain the following system of $n - 1$ ordinary differential equations on $n-1$ unknowns $y_i(t)$,

(1.3) $$y_i'' - \frac{\delta}{h}\varphi\left(\frac{\delta}{h}y_{i-1}\right) - f_i(t) = 0, \qquad i = 1, 2, \ldots, n-1.$$

where we have set

$$\varphi(a) = \begin{cases} L\left\{\sqrt{1+a^2} - \frac{\Lambda}{L}\right\}\frac{a}{\sqrt{1+a^2}}, & |a| \geq \sqrt{(\Lambda/L)^2 - 1}, \\ 0, & |a| < \sqrt{(\Lambda/L)^2 - 1}, \end{cases}$$

and, for the sake of simplicity, $K = 1$, $M/L = 1$.

We add to (1.3) the initial conditions

(1.4) $$y_i(0) = \frac{1}{h}\int_{\Delta_i} y(x, 0)\,dx = \bar{y}_i, \qquad y_i'(0) = \frac{1}{h}\int_{\Delta_i} y'(x, 0)\,dx = \bar{y}_i',$$

and, taking in account that the ends are fixed, the «boundary» conditions

(1.5) $$y_0(t) = y_n(t) = 0.$$

Equations (1.3)-(1.5) represent a discrete model of the problem; passing formally to the limit with $h \to 0$, we then obtain the corresponding continuous model, represented by a partial differential equation, with initial and boundary conditions

(1.6) $$\frac{\partial^2 y(x,t)}{\partial t^2} - \frac{\partial[\varphi(\partial y(x,t)/\partial x)]}{\partial x} - f(x,t) = 0,$$

(1.7) $$y(x, 0) = \bar{y}(x), \qquad y_t(x, 0) = \bar{y}'(x),$$

(1.8) $$y(0, t) = y(L, t) = 0.$$

In what follows when no doubt will be possible, we shall set $y(t) = \{y(x,t); x \in \Delta\}$, indicating by «'» the derivative with respect to t, and by «D» the derivative with respect to x, moreover we shall write $L^2, H^2, \ldots$ instead of $L^2(\Delta)$, $H^2(\Delta)$.

Instead of (1.6)-(1.8) we consider the following equivalent variational equation

(1.9) $$\int_0^t \left\{(y'(\tau), u'(\tau))_{L^2} - (\varphi(Dy(\tau)), Du(\tau))_{L^2} - (f(\tau), u(\tau))_{L^2}\right\} d\tau +$$
$$+ ((y'(t), u(t))_{L^2} - (y'(0), u(0))_{L^2} = 0,$$

(1.10) $$y(x, 0) = \bar{y}(x),$$

and we shall say that $y(x,t)$ is a solution of the problem above considered if

i) $y(t) \in L^2(0, T; H_0^1) \cap H^1(0, T; L^2)$;

ii) (1.9) holds a.e. in $(0, T)$, and $\forall u(t) \in L^2(0, T; H_0^1) \cap H^1(0, T; L^2)$;

iii) (1.10) holds a.e. in Δ.

The physical meaning of (1.3)-(1.5) would suggest us to assume the poligonal line connecting the points $P_i^{(n)}(t)$ as approximate solution of the problem, at the time t. Unfortunately we meet too many difficulties when we try to obtain a solution of (1.9)-(1.10) letting $n \to \infty$ in the solution of (1.3)-(1.5).

In fact the solutions of (1.3)-(1.5) are not regular enough, when $n \to \infty$, to allow such a procedure, because (1.3) is not linear.

Therefore we substitute (1.3)-(1.5) with the following regularized problem

$$(1.11) \quad z_i'' - \frac{\delta}{b}\varphi\left(\frac{\delta}{b} z_i\right) + \varepsilon(b)\frac{\delta^4}{b^4} z_{i-2} - f_i(t) = 0, \qquad i = 1, 2, \ldots, n-1,$$

$$(1.12) \quad z_0(t) - z_n(t) = 0, \quad z_{-1}(t) = -z_1(t), \quad z_{n+1}(t) = -z_{n-1}(t),$$

$$(1.13) \quad z_i(0) = \bar{y}_i, \quad z_i'(0) = \bar{y}_i'.$$

(Let's observe that (1.1)-(1.2) give

$$f_{-1} = -f_1, \quad f_{n+1} = -f_{n-1}).$$

In (1.11) we shall suppose $\varepsilon(b) \to 0$ for $b \to 0$, in order to obtain (1.9) for the continuous model.

Precisely, in § 2 we shall introduce, for every function $f(x) \in L^1(\Delta)$, a family of n-ples $\{f_i^{(n)}\}$ then for every n-ple $\{f_i^{(n)}\}$ we shall introduce a family of functions defined on Δ.

Finally we shall recall some properties which hold for the functions and the n-uples associated to them.

In § 3 we consider the continuous problem (1.9). Firtly, using n-ples connected with data, we construct a sequence of discrete problems

$$(1.14) \quad \begin{cases} z_i(t) = \bar{y}_i + \displaystyle\int_0^t v_i(\tau)\, d\tau, \\ v_i(t) = \bar{y}_i' + \displaystyle\int_0^t \left\{ \frac{\delta}{b}\varphi\left(\frac{\delta}{b} z_{i-1}(\tau)\right) + \varepsilon(b)\frac{\delta^4}{b^4} z_{i-2}(\tau) - f_i(\tau) \right\} d\tau, \\ \hfill i = 1, \ldots, (n-1); \\ z_0 = z_n = 0, \quad z_{-1} = -z_1, \quad z_{n+1} = -z_{n-1}, \end{cases}$$

equivalent to (1.11)-(1.13); then $\forall t \in (0, T)$ we consider the sequence of the poligonal lines connecting the points $P_i^{(n)}(t) = \{ib^{(n)}, z_i^{(n)}(t)\}$; supposing the data are regular enough, we prove that it is possible to choose a subsequence of poligonal lines weakly convergent to a solution of (1.9).

The uniqueness of the solution of (1.9) is a difficult problem, because the function $\varphi(\alpha)$ is not strictly monotone; this problem is still open.

Finally in § 4 we recall some classical results on problem (1.14), which have been utilized in the preceding sections.

2. - Connection between functions and n-ples

Let $f \in L^2(\Delta)$, extended by (1.1); then, $\forall n$, (1.3) define $n+5$ numbers

$$f_i = \frac{1}{b}\int_{\Delta_i} f(x)\,dx, \qquad i = -2, -1, \ldots, n+1, n+2, \tag{2.1}$$

which satisfy the following conditions

$$f_0 = f_n = 0, \quad f_{-2} = -f_2, \quad f_{-1} = -f_1, \quad f_{n+1} = -f_{n-1}, \quad f_{n+2} = -f_{n-2}, \ldots \tag{2.2}$$

and then

$$\delta f_{-1} = \delta f_0, \qquad \delta^2 f_{-1} = 0, \qquad \delta f_{n-1} = d f_n, \qquad \delta^2 f_{n-1} = 0, \ldots . \tag{2.3}$$

In what follows we shall indicate by S the class of n-ples satisfying the symmetricity conditions (2.2), (2.3).

Moreover the following lemma holds:

Lemma 2.1: *Let $f \in H^s$; then there exists a constant $\mathcal{C}_1$ such that*

$$\sum_{i=1}^{n}\left\{\frac{\delta^2}{b^s} f_{i-s}\right\}^2 \leqslant \frac{\mathcal{C}_1}{b}\|D^s f\|^2_{L^2}, \qquad (s \in \mathrm{N}); \tag{2.4}$$

let $f \in H^{s+\sigma,\infty}$, then there exists a constant $\mathcal{C}_1$ such that

$$\sum_{i=1}^{n}\left\{\frac{\delta^{s+1}}{b^{s+1}} f_{i-s}\right\}^2 \leqslant \frac{\mathcal{C}_1}{b} b^{2\sigma-2}\|D^s f\|^2_{H^{\sigma,\infty}}, \qquad (s \in \mathrm{N},\ 0 < \sigma < 1).$$

Moreover to every $\{z_i^{(n)}\}$ $(i = 1, 2, \ldots, n)$ it will be useful to associate the following functions

$$z^{(n)}(x) = z_i^{(n)}, \qquad x \in \Delta_i^{(n)}, \tag{2.5}$$

$$\mathfrak{z}^{(n)}(x) = z_i + \frac{\delta}{b} z_i (x - ib), \qquad x \in \bar{\Delta}_i^{(n)}, \tag{2.6}$$

$$\mathfrak{Z}^{(n)}(x) = \int_{-b/2}^{x} \eta^{(n)}(\xi)\,d\xi - \frac{L-x}{L}\int_{-b/2}^{0} \eta(\xi)\,d\xi - \frac{x}{L}\int_{-b/2}^{L} \eta(\xi)\,d\xi = \tag{2.7}$$

$$= \frac{1}{2}\left(z_i^{(n)} + z_{i+1}^{(n)}\right) + \frac{\delta}{b} z_{i-1}^{(n)}\left\{x - \left(i - \frac{1}{2}\right)b\right\} + \frac{\delta^2}{b^2} z_{i-1}^{(n)}\left\{x - \left(i - \frac{1}{2}\right)b\right\}^2, \quad x \in \Delta_i^{(n)}.$$

having set

$$\eta^{(n)}(x) = \frac{\delta}{b} z_{i-1}^{(n)} + \frac{\delta^2}{b^2} z_{i-1}^{(n)} \left\{ x - \left(i - \frac{1}{2} \right) b \right\}, \qquad x \in \Delta_i^{(n)}.$$

The function (2.5) is a step function, (2.6) is a poligonal line connecting the points $P_i^{(n)}$, whose derivative is a step function, function (2.7) has the first derivative continuous while the second is a step function.

Let now $\{z_i\} \in \mathcal{S}$, then the functions above defined satisfy the following boundary and symmetric conditions

$$z^{(n)}(0) = z^{(n)}(L) = 0, \qquad \mathfrak{z}^{(n)}(0) = \mathfrak{z}^{(n)}(L) = 0, \qquad \mathfrak{Z}^{(n)}(0) = \mathfrak{Z}^{(n)}(L) = 0;$$

$$z^{(n)}(x) = -z^{(n)}(-x), \qquad z^{(n)}(2L - x) = -z^{(n)}(x),$$

$$\mathfrak{z}^{(n)}(x) = -\mathfrak{z}^{(n)}(-x), \qquad \mathfrak{z}^{(n)}(2L - x) = -\mathfrak{z}^{(n)}(x),$$

moreover the following lemma holds:

Lemma 2.2: *Let $\{z_i\}$ be any n-ple; there exists then a constant $\mathcal{C}_2$ (independent of n) such that the following relations hold*

$$(2.8) \quad \begin{cases} \|z^{(n)}\|^2_{L^2} \leq b\mathcal{C}_2 \sum_{i=-1}^{n+1} |z_i^{(n)}|^2, \\ \|\mathfrak{z}^{(n)}\|^2_{L^2} \leq b\mathcal{C}_2 \sum_{i=-1}^{n} |z_i^{(n)}|^2, \qquad \|\mathfrak{z}^{(n)}\|^2_{H_0^1} \leq b\mathcal{C}_2 \sum_{i=-1}^{n} \left| \frac{\delta}{b} z_i^{(n)} \right|^2, \\ \|\mathfrak{Z}^{(n)}\|^2_{L^2} \leq b\mathcal{C}_2 \sum_{i=-1}^{n} |z_i^{(n)}|^2, \qquad \|\mathfrak{Z}^{(n)}\|^2_{H_0^1} \leq b\mathcal{C}_2 \sum_{i=-1}^{n} \left| \frac{\delta}{b} z_i^{(n)} \right|^2, \\ \qquad\qquad \|\mathfrak{Z}^{(n)}\|^2_{H_0^2} \leq b\mathcal{C}_2 \sum_{i=-1}^{n} \left| \frac{\delta^2}{b^2} z_i^{(n)} \right|^2, \end{cases}$$

$$(2.9) \quad \|D\mathfrak{Z}^{(n)} - D\mathfrak{z}^{(n)}\|^2_{L^2} \leq b^3 \mathcal{C}_2 \sum_{i=-1}^{n} \left\{ \left| \frac{\delta^2}{b^2} z_i^{(n)} \right|^2 + \left| \frac{\delta}{b} z_i^{(n)} \right|^2 \right\}.$$

Finally, the following lemma holds:

Lemma 2.3: *Let $f \in H^{s+1}$; then, constructing the successive n-ples $\{f_i^{(n)}\}$, $\{\delta f_i^{(n)}/b\}$, $\{\delta^2 f_i^{(n)}/b^2\}, \ldots, \{\delta^s f_i^{(n)}/b^s\}$, and denoting by $\mathcal{F}_s^{(n)}(x)$ the step function*

$$\mathcal{F}_s^{(n)}(x) = \delta^s f_i^{(n)} / b^s, \qquad x \in \Delta_i,$$

there exists a constant $\mathcal{C}_3$ such that

$$\|\mathcal{F}_s^{(n)} - D^s f\|^2_{L^2} \leq b^2 \mathcal{C}_3 \|D^{s+1} f\|^2_{L^2}.$$

3. - The continuous problem

We can state the following Existence theorem for the continuous problem:

Theorem 3 (Existence): *Let the following hypotheses hold:*

$$f \in L^2(Q); \quad \bar{y} \in H^{1+\sigma,*} \cap H_0^1, \ (\sigma > 0); \quad \bar{y}' \in L^2; \tag{3.1}$$

Then there exists at least a solution $y(t) \in H^1(0, T; L^2(\Delta)) \cap L^2(0, T; H_0^1(\Delta))$, *of* (1.9), (1.10) *in the sense indicated above* (*see* § 1).

We can divide the proof in the following steps:

a) *Let* (3.1) *hold and set* (*according* (2.1))

$$\bar{y}_i = \frac{1}{b} \int_{\Delta_i} \bar{y}(x)\,dx, \quad \bar{y}_i' = \frac{1}{b} \int_{\Delta_i} \bar{y}'(x)\,dx, \quad f_i(t) = \frac{1}{b} \int_{\Delta_i} f(x, t)\,dx, \tag{3.2}$$

we have then

$$\{\bar{y}_i\} \in \mathcal{S}, \quad \{\bar{y}_i'\} \in \mathcal{S}; \quad \{f_i(t)\} \in \mathcal{S}, \quad \forall t \in [0, T];$$

moreover, setting

$$\varepsilon(b) = b^{\mu}, \qquad 2 - 2\sigma < \mu < 2, \tag{3.3}$$

there exists a constant $\mathcal{C}_5$ *such that*

$$\sum_{i=1}^{n} f_i^2(t)\,d\tau \leqslant \frac{\mathcal{C}_5}{b} \|f(t)\|_{L^2}^2, \quad \sum_{i=1}^{n} \bar{y}_i'^2 \leqslant \frac{\mathcal{C}_5}{b} \|\bar{y}'\|_{L^2}^2,$$

$$\sum_{i=1}^{n} \left(\frac{\delta}{b}\bar{y}_{i-1}\right)^2 \leqslant \frac{\mathcal{C}_5}{b} \|D\bar{y}\|_{L^2}^2, \quad \varepsilon(b) \sum_{i=1}^{n} \left(\frac{\delta^2}{b^2}\bar{y}_{i-1}\right)^2 \leqslant b^{\mu-2+2\sigma} \frac{\mathcal{C}_5}{b} \|D\bar{y}\|_{H^{\sigma,*}}^2.$$

b) *Substituting* (3.2) *and* (3.3) *into* (5.2), *we have, by step a*), *Theorem* 5.1, *and Lemma* 5.1, *an unique solution* $\{z_i^{(n)}(t)\}$, *which satisfies* (5.5), $\forall n$ *and* $\forall t \in [0, T]$

c) *Setting*

$$\mathfrak{z}^{(n)}(x, t) = z_i^{(n)}(t) + \frac{\delta}{b} z_i^{(n)}(t) \qquad x \in \bar{\Delta}_i^{(n)},$$

it is possible to select a subsequence (*indicated by* $\{\mathfrak{z}^{(n)}\}$) *such that*

$$\lim_{n \to \infty} \mathfrak{z}^{(n)}(t) = y(t) \tag{3.4}$$

strongly in $L^2(Q)$, *weakly* in* $L^\infty(0, T; H^1)$,

$$\lim_{n \to \infty} \mathfrak{z}^{(n)}(0) = \bar{y}, \tag{3.5}$$

strongly in L^2, *weakly in* H^1, *and*

(3.6) $$\lim_{n\to\infty} \mathfrak{z}'^{(n)}(t) = y'(t), \qquad \lim_{n\to\infty} \varphi(D\mathfrak{z}^{(n)}(t)) = \phi(t),$$

weakly in* $L^\infty(0, T; L^2)$.

d) There exist three sequences $\{\mathfrak{A}^{(n)}(t)\}$, $\{\mathfrak{B}^{(n)}(t)\}$, $\{\mathfrak{S}^{(n)}(t)\}$, *with*

(3.7) $$\begin{cases} \lim\limits_{n\to\infty} \|\mathfrak{A}^{(n)}(t) - Du(t)\|_{L^\infty(0,T;L^2)} = 0\,, \\ \lim\limits_{n\to\infty} \|\mathfrak{B}^{(n)}(t) - u(t)\|_{L^\infty(0,T;L^2)} = 0\,, \\ \lim\limits_{n\to\infty} \int_0^t \|\mathfrak{S}^{(n)}(\tau)\|_{L^2} d\tau = 0\,, \end{cases}$$

such that the following equation holds:

(3.8) $$\int_0^t \{(\mathfrak{z}'^{(n)}(\tau), u'(\tau))_{L^2} - (\varphi(D\mathfrak{z}^{(n)}(\tau)), \mathfrak{A}^{(n)}(\tau))_{L^2} - (f(\tau), \mathfrak{B}^{(n)}(\tau))_{L^2}\} d\tau +$$
$$+ (\mathfrak{z}'^{(n)}(t), \mathfrak{B}^{(n)}(t))_{L^2} - (\mathfrak{z}'^{(n)}(0), \mathfrak{B}^{(n)}(0))_{L^2} + \int_0^t \mathfrak{S}^{(n)}(\tau) d\tau = 0\,,$$

a.e. *in* $[0, T]$, *and* $\forall u \in L^2(0, T; H_0^1) \cap H^1(0, T; L^2)$.

e) In (3.8) *we have*

(3.9) $$\lim_{n\to\infty} \varphi(D\mathfrak{z}^{(n)}(t)) = \phi(t) = \varphi(Dy(t))\,,$$

in the weak topology of $L^2(0, T; L^2)$.

Letting in (3.8) $n \to \infty$, the theorem is proved by (3.4)-(3.7), (3.9).

4. - Proof of the existence theorem

Step *a*). Follows obviously from Lemma 5.1.
Step *b*). Follows by step *a*), from Lemma 5.1 and Theorem 5.1.
Step *c*). Follows by the definition of $\mathfrak{z}^{(n)}(x, t)$, because from

$$\mathfrak{z}^{(n)}(x, t) = z_i^{(n)}(t) + \frac{\delta}{b} z_i^{(n)}, \qquad x \in \bar{\Delta}_i^{(n)},$$

we obtain

$$\mathfrak{z}'^{(n)}(t) = z_i'^{(n)}(t) + \frac{\delta}{b} z_i'^{(n)}(t), \qquad x \in \bar{\Delta}_i^{(n)},$$

$$D\mathfrak{z}^{(n)}(t) = \frac{\delta}{b} z_i(t), \qquad x \in \bar{\Delta}_i^{(n)},$$

and subsequently

$$\|\mathfrak{z}^{(n)}\|_{H_0^1(Q)} = \left\{ \int_Q [(D\mathfrak{z}^{(n)})^2 + (\mathfrak{z}'^{(n)})^2]\, dQ \right\}^{1/2} =$$

$$= \left\{ \int_0^T \left\{ \sum_{i=0}^{n-1} \int_{\Delta_i} \left[\left(\frac{\delta}{b} z_i(t) \right)^2 + \left(z_i'(t) + \frac{\delta}{b} z_i'(t)(x+ib) \right)^2 \right] dx \right\} dt \right\}^{1/2} \leq$$

$$\leq \mathcal{C}_5 \left\{ \int_0^T \left\{ \sum_{i=0}^{n-1} \left[\left(\frac{\delta}{b} z_i(t) \right)^2 + z_i'^2(t) + \frac{1}{2} (\delta z_i'(t))^2 \right] b \right\} dt \right\}^{1/2};$$

finally (5.5) gives, $\forall n$,

$$\|\mathfrak{z}^{(n)}\|_{H_0^1(Q)} \leq \mathcal{C}_6, \qquad \|\mathfrak{z}^{(n)}\|_{L^\infty(0,T;H^1)} \leq \mathcal{C}_6; \qquad \|\mathfrak{z}'^{(n)}\|_{L^\infty(0,T;L^2)} \leq \mathcal{C}_6 .$$

for suitable positive constants $\mathcal{C}_5$ and $\mathcal{C}_6$.

Step *d*). Let us calculate at first the first addendum of (3.8) supposing $u(t) \in H^{1,\infty}(0,T;L^2) \cap L^\infty(0,T;H^2 \cap H_0^1)$.

Setting, for the sake of simplicity,

$$m(x,t) = \int_0^x u(\xi,t)\, d\xi, \qquad m'(x,t) = \int_0^x u'(\xi,t)\, d\xi,$$

$$m_i(t) = \frac{1}{b} \int_{\Delta_i} m(x,t)\, dx, \quad m_i'(t) = \frac{1}{b} \int_{\Delta_i} m'(x,t)\, dx,$$

we have

$$(4.1) \qquad (\mathfrak{z}'^{(n)}(t), u'(t))_{L^2} = \int_\Delta \mathfrak{z}'^{(n)}(x,t)\, u'(x,t)\, dx =$$

$$= [\mathfrak{z}'^{(n)}(x,t)\, m'(x,t)]_0^L - \int_\Delta D\mathfrak{z}'^{(n)}(x,t)\, m'(x,t)\, dx =$$

$$= -\sum_0^{n-1} \int_{\Delta_i} \frac{\delta}{b} z_i'(t)\, m'(x,t)\, dx = -\sum_0^{n-1} \delta z_i'(t)\, m_i'(t) =$$

$$= -z_n'(t)\, m_{n-1}'(t) + z_0'(t)\, m_{-1}'(t) + \sum_0^{n-1} z_i'(t)\, \delta m_{i-1}'(t) = b \sum_0^{n-1} z_i'(t) \frac{\delta}{b} m_{i-1}'(t).$$

Easy calculations prove that the n-ple

$$(4.2)\qquad g_i(t) = \frac{\delta}{b} m_{i-1}(t) \in S, \quad \forall t \in [0, T],$$

In fact $u(x,t)$ and $u'(x,t)$ are odd functions with respect to $x = 0$ and $x = L$; then $m(x,t)$ and $m'(x,t)$ are even function with respect to $x = 0$ and $x = L$, $\forall t \in \in [0, T]$.

Moreover we can verify that

$$(4.3)\qquad g_i \in H^1(0, T), \quad i = 1, \dots, n.$$

In fact, we have

$$(4.4)\qquad \{g_i'(t)\}^2 = \frac{1}{b^4}\left\{ \int_{\lambda_{i-1}} dx \int_x^{x+b} u'(\xi, t)\, d\xi \right\}^2 \leqslant$$

$$\leqslant \frac{1}{b^2} \int_{\lambda_{i-1}} dx \int_x^{x+b} |u'(\xi, t)|^2 d\xi \leqslant \frac{1}{b} \int_{(i-1)b}^{(i+1)b} |u'(\xi, t)|^2 d\xi .$$

By (4.4) there exists a suitable $\mathcal{C}_7 > 0$ such that

$$(4.5)\qquad \begin{cases} \displaystyle\sum_{i=0}^{n-1} g_i'^2(t) \leqslant \frac{\mathcal{C}_7}{b} \|u'(t)\|^2_{L^2}, \quad t \in [0, T]; \\ \displaystyle\|g_i'\|^2_{L^\infty(0,T)} \leqslant \frac{\mathcal{C}_7}{b} \|u'\|^2_{L^\infty(0,T;L^2)} ; \\ \displaystyle\sum_{i=0}^{n-1} \|g_i'\|^2_{L^2(0,T)} \leqslant \frac{\mathcal{C}_7}{b} \|u'(t)\|^2_{L^2(Q)} ; \end{cases}$$

(4.5) proves (4.3).

Now (5.4), (4.1) and (4.2) give

$$\int_0^t (g'(\tau), u'(\tau))_{L^2(\Lambda)} d\tau = \int_0^t b \sum_{i=0}^{n-1} z_i'(\tau) g_i'(\tau)\, d\tau =$$

$$= -b \int_0^t \left\{ \sum_{i=1}^{n} \varphi\left(\frac{\delta}{b} z_{i-1}(\tau)\right) \frac{\delta}{b} g_{i-1}(\tau) + \varepsilon(b) \frac{\delta^2}{b^2} z_{i-2}(\tau) \frac{\delta^2}{b^2} g_{i-2}(\tau) - f_i(\tau) g_i(\tau) \right\} d\tau +$$

$$+ b \sum_{i=0}^{n} z_i'(t) g_i(t) - b \sum_{i=0}^{n} z_i'(0) g_i(0) .$$

Setting

$$\mathcal{A}^{(n)}(t) = \frac{\delta}{b} g_i(t), \qquad \mathcal{B}^{(n)}(t) = g_i(t),$$

$$\mathcal{G}^{(n)}(t) = \varepsilon(b)\frac{\delta^2}{b^2} z_{i-1}(t)\frac{\delta^2}{b^2} g_{i-1}(t), \qquad x \in \bar{\Delta}_i,$$

(2.15) gives finally (3.8).

Step *e*). In order to calculate the limit in (3.9), it is useful to set

$$\text{(4.6)} \qquad b(\alpha) = \varphi(\alpha) + \alpha, \qquad \varphi(\alpha) = b(\alpha) - \alpha,$$

subdividing $\varphi(\alpha)$ in a function $b(\alpha)$ strictly monotone and another linear. Then we have

$$\text{(4.7)} \qquad \Psi(t) = \lim_{n\to\infty} \varphi(D\mathfrak{z}^{(n)}(t)) = \lim_{n\to\infty} \{b(D\mathfrak{z}^{(n)}(t)) - D\mathfrak{z}^{(n)}(t)\} =$$

$$= \lim_{n\to\infty} b(D\mathfrak{z}^{(n)}(t)) - Dy(t) = \chi(t) - Dy(t),$$

in the weak topology of $L^2(0, T; L^2)$.

Let us now introduce the functions $\mathcal{Z}^{(n)}(x, t)$ given by (2.7), and observe that by (2.9) and (5.5) we have

$$\|D\mathcal{Z}^{(n)}(t) - D\mathfrak{z}^{(n)}(t)\|_{L^2} \leqslant b^3 \mathcal{C}_8 \sum_{i=-1}^{n} \left\{ \left(\frac{\delta}{b} z_i^{(n)}(t)\right)^2 + \left(\frac{\delta^2}{b^2} z_i^{(n)}\right)^2 \right\} \leqslant$$

$$\leqslant b^3 \mathcal{C}_7 \frac{1}{b\varepsilon(b)} = \frac{b^2}{\varepsilon(b)} \mathcal{C}_7 \qquad t \in [0, T],$$

and then, by (3.3),

$$\text{(4.8)} \qquad \lim_{n\to\infty} \|D\mathcal{Z}^{(n)}(t) - D\mathfrak{z}^{(n)}(t)\|_{L^2} = \lim_{n\to\infty} \frac{b^2}{\varepsilon(b)} = b^{2-\mu} = 0.$$

Finally (4.6) and (4.8) give

$$\lim_{n\to\infty} \|b(D\mathcal{Z}^{(n)}(t)) - b(D\mathcal{Z}^{(n)}(t))\|_{L^2} = 0, \qquad t \in [0, T],$$

and, by (4.7),

$$\text{(4.9)} \qquad \chi(t) = \lim_{n\to\infty} b(D\mathcal{Z}^{(n)}(t)).$$

Now we have to calculate $\chi(t)$ in (4.9); we shall obtain this following the same procedure as in [7].

In fact there exists a constant $\mathcal{C}_8$ such that the following inequalities hold

$$\|\mathcal{Z}'^{(n)}(t)\|_{L^2} \leqslant \mathcal{C}_8, \qquad \|\mathcal{Z}'^{(n)}(t)\|_{H_0^1} \leqslant \mathcal{C}_8, \qquad t \in [0, T], \tag{4.10}$$

$$\|\mathcal{Z}'^{(n)}\|_{H^1(0,T;(H^2 \cap H_0^1)')} \leqslant \mathcal{C}_8, \qquad \|\mathcal{Z}^{(n)}(t)\|_{H^2 \cap H_0^1} \leqslant \frac{\mathcal{C}_9}{\sqrt{\varepsilon}}, \qquad t \in [0, T]. \tag{4.11}$$

(4.10) and the second of (4.11) can easily be deduced from (2.8) and (5.5). We can obtain the first of (4.11) in the following way.

Assume that

$$u(t) \in L^2(0, T; H^2 \cap H_0^1),$$

and set

$$D^2 \Upsilon(x, t) = u(x, t), \qquad \Upsilon(0, t) = \Upsilon(L, t) = 0, \qquad t \in [0, T],$$

then $\Upsilon(t) \in L^2(0, T; H^4 \cap H_0^1)$, and moreover

$$\int_0^T dt \int_\Delta \mathcal{Z}''^{(n)}(x, t)\, u(x, t)\, dx = \int_0^T \left\{ [\mathcal{Z}''^{(n)}(x, t)\, D\Upsilon(x, t)]_0^L - \int_\Delta D\mathcal{Z}''^{(n)}(x, t)\, D\Upsilon(x, t)\, dx \right\} dt =$$

$$= \int_0^T \left\{ [-D\mathcal{Z}''^{(n)}(x, t)\, \Upsilon(x, t)]_0^L + \int_\Delta D^2 \mathcal{Z}''^{(n)}(x, t)\, \Upsilon(x, t)\, dx \right] dt,$$

and, by (2.7),

$$\int_0^T dt \int_\Delta \mathcal{Z}''^{(n)}(x, t)\, u(x, t)\, dx = \tag{4.12}$$

$$= \int_0^T \left\{ \sum_{i=0}^{n} \int_{\Delta_i} \frac{\delta^2}{b^2} z''_{i-1}(t)\, \Upsilon(x,t)\, dx - \int_{-b/2}^{0} \frac{\delta^2}{b^2} z''_{-1}(t)\, \Upsilon(x,t)\, dx - \int_L^{L+b/2} \frac{\delta^2}{b^2} z''_{n-1}(t)\, \Upsilon(x,t)\, dx \right\} dt =$$

$$= \int_0^T dt \sum_{i=0}^{n} \int_{\Delta_i} \frac{\delta^2}{b^2} z''_{i-1}(t)\, \Upsilon(x, t)\, dx.$$

Setting now

$$\Upsilon_i(t) = \frac{1}{b} \int_{\Delta_i} \Upsilon(x, t)\, dx, \qquad \mathcal{U}_i(t) = \frac{\delta^2}{b^2} \Upsilon_{i-1}, \tag{4.13}$$

$\{\Upsilon_i(t)\}, \{\mathcal{U}_i(t)\} \in S$, $\forall t \in [0, T]$ and moreover $\Upsilon_i, \mathcal{U}_i \in L^2(0, T)$, we have then successively, by (4.12) and (5.3),

$$\int_0^T dt \int_{\mathcal{A}} \mathcal{Z}^{\pi(n)}(x,t)\, u(x,t)\, dx = h \int_0^T dt \left\{ \sum_{i=0}^{n} \frac{1}{h} \int_{\mathcal{A}_i} \frac{\delta^2}{h^2} z_{i-1}^{n}(t)\, \Upsilon(x,t)\, dx \right\} =$$

$$= h \int_0^T dt \left\{ \sum_{i=0}^{n} z_i^{n}(t) \frac{\delta^2}{h^2} \Upsilon_{i-1}(t) \right\} = h \int_0^T dt \left\{ \sum_{i=0}^{n} z_i^{n}(t)\, \mathcal{U}_i(t) \right\} =$$

$$= \int_0^T \sum_{i=1}^{n} \left\{ \varphi\left(\frac{\delta}{h} z_{i-1}(t)\right) \frac{\delta}{h} \mathcal{U}_{i-1}(t) - \varepsilon(h) \frac{\delta^2}{h^2} z_{i-2}(t) \frac{\delta^2}{h^2} \mathcal{U}_{i-2}(t) - f_i(t)\, \mathcal{U}_i(t) \right\} dt =$$

$$= \int_0^T \sum_{i=1}^{n} \left\{ \varphi\left(\frac{\delta}{h} z_{i-1}(t)\right) \frac{\delta^3}{h^3} \Upsilon_{i-2}(t) - \varepsilon(h) \frac{\delta^2}{h^2} z_{i-2}(t) \frac{\delta^4}{h^4} \Upsilon_{i-3}(t) - f_i(t) \frac{\delta^2}{h^2} \Upsilon_{i-1}(t) \right\} dt .$$

Finally (2.4), (5.5), and (4.13) give, for h small enough and for suitable positive constants $\mathcal{C}_9$, $\mathcal{C}_{10}$

$$\left| \int_0^T dt \int_{\mathcal{A}} \mathcal{Z}^{\pi(n)}(x,t)\, u(x,t)\, dx \right|^2 \leqslant \mathcal{C}_9 + \mathcal{C}_{10} \int_0^T \{ \|D^4 \Upsilon(t)\|_{L^2}^2 + \|D^3 \Upsilon(t)\|_{L^2}^2 \}\, dt \leqslant$$

$$\leqslant + \mathcal{C}_9 + \mathcal{C}_{10} \|u(t)\|_{L^2(H^2 \cap H_0^1)}^2 ,$$

that is the first of (4.11).

Following now the same procedure as in [7], using the strict monotonicity of $b(\alpha)$, interpolating (4.10) and (4.11), and bearing in mind (4.8), we can prove that

$$\chi = \lim_{n \to \infty} b(D\mathcal{Z}^{(n)}) = b(Dy),$$

in the weak topology of $L^2(Q)$.

Bearing in mind (4.6) we have

$$\lim_{n \to \infty} \varphi(D\mathfrak{z}^{(n)}) = \lim_{n \to \infty} \{ b(D\mathfrak{z}^{(n)}) - D\mathfrak{z}^{(n)} \} = \lim_{n \to \infty} b(D\mathcal{Z}^{(n)}) - Dy = \varphi(Dy),$$

in the weak topology of $L^2(Q)$.

5. - The discrete problem

Theorem 5.1 (Existence and Uniqueness): *Let*

$$(5.1) \qquad \begin{cases} \{\bar{y}_i\}, \quad \{\bar{y}_i'\} \in \mathcal{S}; \\ \{f_i(t)\} \in \mathcal{S} \text{ a.e. in } (0,T), \quad f_i(t) \in L^2(0,T); \end{cases}$$

then the system (1.14)

$$(5.2)\quad \begin{cases} z_i(t) = \bar{y}_i + \int_0^t v_i(\tau)\,d\tau\,, \\ v_i(t) = \bar{y}_i' + \int_0^t \left\{ \frac{\delta}{b}\varphi\left(\frac{\delta}{b} z_{i-1}(\tau)\right) - \varepsilon(b)\frac{\delta^4}{b^4} z_{i-2}(\tau) + f_i(\tau) \right\} d\tau, \quad i = 1, \ldots, n-1\,, \\ z_0 = z_n = 0, \quad z_{-i} = -z_1, \quad z_{-2} = -z_2, \quad z_{n+1} = -z_{n-1}, \quad z_{n+2} = -z_{n-2} \end{cases}$$

has a unique solution $C^0[0, T] \cap H^1(0, T)$, *moreover* $z_i(t),\ z_i'(t) \in \mathcal{S}$ $\forall t \in [0, T]$.

Theorem 5.2 (Conservation of energy): *Let* (5.1) *hold.*

Then the solution $\{z_i(t), z_i'(t)\}$ *of* (5.2) *satisfies* $\forall t \in [0, T]$ *the «energy» equation*

$$\sum_{i=1}^{n} \left\{ \frac{1}{2} z_i'^2(t) + \frac{1}{2}\varepsilon(b)\left(\frac{\delta^2}{b^2} z_{i-1}(\tau)\right)^2 + \Phi\left(\frac{\delta}{b} z_{i-1}(\tau)\right) \right\} =$$

$$= \sum_{i=1}^{n} \left\{ \int_0^t f_i(\tau) z_i'(\tau)\,d\tau + \frac{1}{2}\bar{y}_i'^2 + \frac{1}{2}\varepsilon(b)\left(\frac{\delta^2}{b^2}\bar{y}_{i-1}\right)^2 + \Phi\left(\frac{\delta}{b}\bar{y}_{i-1}\right) \right\},$$

where we have set

$$\Phi(\alpha) = \int_0^\alpha \varphi(\beta)\,d\beta\,.$$

Observation 5.1 (The variational equations): Let us observe that it is possible to give to the problem (1.11)-(1.13) a variational form. Precisely *let* (5.1) *hold, then the solution* $\{z_i(t), z_1'(t)\}$ *of* (5.2) *satisfies the variational equations*

$$(5.3)\quad \int_0^t \sum_{i=1}^{n} \left\{ z_i'' u_i + \varphi\left(\frac{\delta}{b} z_{i-1}\right)\frac{\delta}{b} u_{i-1} + \varepsilon(b)\frac{\delta^2}{b^2} z_{i-2}\frac{\delta^2}{b^2} u_{i-2} + f_i u_i \right\} d\tau\,,$$

$\forall \{u_i(t)\} \in \mathcal{S}$, $\forall t$ *in* $[0, T]$, *and* $u_i \in L^2(0, T)$;

$$(5.4)\quad \int_0^t \sum_{i=1}^{n} \left\{ -z_i' u_i' + \varphi\left(\frac{\delta}{b} z_{i-1}\right)\frac{\delta}{b} u_{i-1} + \varepsilon(b)\frac{\delta^2}{b^2} z_{i-2}\frac{\delta^2}{b^2} u_{i-2} - f_i u_i \right\} d\tau +$$

$$- \left\{ \sum_{i=1}^{n} z_i(t) u_i(t) - z_i'(0) u_i(0) \right\} = 0\,,$$

$\forall \{u_i(t)\} \in \mathcal{S}$, $\forall t$ *in* $[0, T]$, *and* $u_i \in H^1(0, T)$.

Moreover the following lemma holds:

LEMMA 5.1: *Let (5.1) hold, and suppose that there exists a constant $\mathcal{C}_3$ such that*

$$\sum_{i=1}^{n} \int_0^t f_i(\tau)\,d\tau \leq \frac{\mathcal{C}_3}{b}, \quad \sum_{i=1}^{n} \bar{y}_i'^2 \leq \frac{\mathcal{C}_3}{b};$$

$$\sum_{i=1}^{n} \left(\frac{\delta}{b}\bar{y}_{i-1}\right)^2 \leq \frac{\mathcal{C}_3}{b}, \quad \varepsilon(b)\sum_{i=1}^{n}\left(\frac{\delta^2}{b^2}\bar{y}_{i-1}\right)^2 \leq \frac{\mathcal{C}_3}{b}.$$

Then there exists a constant $\mathcal{C}_4$ such that the following relations hold

$$\text{(5.5)} \quad \begin{cases} \displaystyle\sum_{i=1}^{n} z_i'^2(t) \leq \frac{\mathcal{C}_4}{b}, \quad \sum_{i=1}^{n}\left(\frac{\delta}{b} z_{i-1}(t)\right)^2 \leq \frac{\mathcal{C}_4}{b}, \\ \displaystyle\varepsilon(b)\sum_{i=1}^{n}\left(\frac{\delta^2}{b^2} z_{i-1}(t)\right)^2 \leq \frac{\mathcal{C}_4}{b}, \quad t \in [0, T]. \end{cases}$$

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Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna