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Formalizations and Models of a Basic Theory for the Foundations of Mathematics (***)

SUMMARY. — We give a formalization within the usual first order predicate calculus and we build an arithmetical model of some theories, named «Teorie Base» briefly B-Theories. These theories have been presented informally in [6] as a highly self-referential framework for the Foundations of Mathematics. The formalization given in the early sections is done with different symbols for predicates, and it seems «more natural» since it follows the exposition of [6]. In the appendix we give a «reformalization» using only one binary predicate.

Formalizzazioni e modelli di una Teoria Base per i fondamenti della matematica

RIASSUNTO. — Si dà una formalizzazione nell'usuale linguaggio del calcolo dei predicati del primo ordine di alcune Teorie Base esposte in modo informale in [6], e se ne costruisce un modello. Queste teorie sono state proposte come ambiente notevolmente autodescrittivo in cui sviluppare la riflessione sui fondamenti della matematica. La formalizzazione proposta nei primi paragrafi usa diversi simboli di predicato, e sembra «più naturale» seguendo più da vicino la linea espositiva di [6]. In appendice si dà una «riformalizzazione» facendo uso di un solo predicato binario.

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Introduction

Some basic theories of the Foundations of Mathematics have been proposed in [4], [2], [3], [5], [6]. The general idea is to reach a foundational theory with the maximum possible degree of self-reference, in accordance with the common usage of language (for example an English dictionary is written in English). The evolution of these researches led to prefer a presentation where a first core is given. Then, the various branches of Mathematics can be grafted on this simple nucleus ([2], [11], [12], [7]). This choice is also made to simplify the task of finding consistency proofs, allowing a step by step «construction» of models.

In [6] and [7] the initial theory is in fact called «Teoria Base», briefly B-Theory or also TB_ν, and it takes as primitive concepts the notions of system, natural number, collection, quality, operation and relation. B-Theory is quite naturally divided in a first essentially finitary trunk, and in a second one useful to obtain a good degree of self-description.

At the moment the relative consistency proofs for this kind of theories may be done with respect to a rather weak arithmetic, and this is shown in the last section of this work. On the other hand the relative consistency proofs for strong extensions of these theories use extensively Set Theory ([8], [9], [10]).

In [5], [6] and in [7] the presentation of the theories is given in a semiformal way. Hence the formalization is the first step in order to evaluate the strength of consistency of these theories.

So in this paper we give a rather natural formalization inspired at the presentations given in [6] and [7]. We choose First Order Predicate Calculus with identity as the formal frame (as a reference for the reader we hade found convenient [1], [13] and [14]).

In the first section we give a brief review of B-Theory. In the second we present our principal formalization useful for the first part of B-Theory using a language with only one symbol **P** for a quinary predicate and 33 symbols for constants.

Then, in the third section following the informal presentation given in [6] and [7], we give a natural infinite formalization of the stronger axioms of B-Theory. In the fourth section, we still use the quinary predicate **P** and a new binary predicate **Q**. This predicate allows for a finite formalization.

In the last section we give the construction of a model for B-Theory. The model is based on the standard concept of free structure, modified to accomply the pure relational presentation of the theory. In fact it is a sort of term model. The universe of the model is set of the natural numbers. All the codings and the interpretations of both **P** and **Q** are definable in Peano Recursive Arithmetic.

In the appendix we briefly sketch a finite formalization with only one symbol for a binary predicate, **R**, and the previously introduced symbols for constants. We find it useful to underline a precise notion of «reformalization» based on the classical definition of inner model.

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1. - A Brief Review of the Naïve Theory

In the informal presentation, all the things B-Theory talks about are called *objects* [1]. The notions of *quality* (or *property*), *collection*, *relation* (in particular *binary, ternary, quaternary relations* etc.), *operation* (in particular *simple, binary operations* etc.), *natural number*, *finite system* (or *finite indexed family*) are assumed as primitive.

The first qualities introduced are those corresponding to the different kinds of objects: the *quality of being a quality*, which enjoys itself, the *quality of being a binary relation*, the *quality of being a ternary relation*, the *quality of being a quaternary relation*, the *quality of being a collection*, etc.

After these *fundamental qualities*, three *fundamental relations* are introduced. They are the *fundamental relation of unary objects*, the *fundamental relation of binary objects* and the *fundamental relation of ternary objects*. The first one *rules* the unary objects: for example it relates a collection with any one of its members, and relates a quality to any object enjoying it. The second one rules the binary objects, e.g. relates a binary relation with the objects related by it. The third one rules the ternary objects, e.g. relates a binary operation with its two arguments and its value. So they respectively enjoy the qualities of being a binary relation, a ternary relation and a quaternary relation. The first simple operation that is introduced is the *identity*, the operation which transforms each object into itself.

The collections explicitly introduced are the *empty collection*, the *collection of all the objects* (of the theory) or universal collection and the *collection of all the collections*.

The *collection of natural numbers* is introduced. It contains the distinguished objects *zero* and *one*, and it is closed for the usual operations of *addition*, *multiplication* and *subtraction*. The natural numbers are ordered by the usual *order relation between natural numbers*. An induction principle was given in the following form: *every non empty collection of natural numbers has a least element*. Though this axiom seems weak, it gets more and more strength as further axioms on sets (to be viewed as particular collections) are grafted into the theory.

Another primitive kind of objects is introduced by means of the distinguished *collection of finite systems*. Its elements are binary objects, called finite systems

[1] In writing what follows we use *italic* characters for the main concepts, in particular for all the names of the distinguished objects introduced into the theory.

(or finite indexed families). Each system associates one or more *values* to every one of its *indexes*.

The *collection of univalent finite systems* contains all and only the *functional* systems: each of them associates to every index only one value. There are not two systems having the same behaviour. In other words these collections are both extensional.

Various other objects are introduced in order to have «many» finite systems. First of all there are the *operation generating the singular systems*, i.e. the univalent systems with only one index, and the binary operation of *union between finite systems*. By means of these objects one can get all the finite systems with two, three, ... indexes. In particular one gets all the ordered pairs, triplets ... identified with the univalent systems having as indexes the numbers one and two, the numbers one two and three etc.. For each natural number the *collection of the n-tuples* is introduced: it is the collection of the univalent systems having as indexes all and only the non zero numbers less than the given number. E.g. the collections of one-tuples has as elements all the singular systems having as only index the number one, the collection of zero-tuple has only an element, namely the *void system*, the system with no index neither value. The simple operation of *inversion* for systems together with the binary operation of *composition of systems* are introduced. Each system is the composition of a tuple with the inverse of a tuple. The *cardinality* of a system is the minimum natural number for which such a decomposition is possible.

At this point one introduces the *quality of being a (general) relation*, the *quality of being a (general) operation*, and the simple operation giving the *arity* of an object. For instance collections and qualities have arity equal to one: they are unary objects; systems and simple operations have arity equal to two, etc.

The final part of the presentation was concerned with the main objects useful to a good self description of the theory itself.

First there is the *operation generating fundamental relations*. It can transform a given nonzero natural number into a relation: this relates each object having the given arity with objects linked by it. For istance this operation applied to one gives the fundamental relation for unary objects described above. This conlcludes the basic part of the theory, which is used for various extensions, cfr. [7], [9].

An autonomous section is concerned with *universal relations*. They may be useful for a finitary self description, and they are introduced by means of an operation *generating universal relations*: namely *runiv*. When defined the relation $runiv(\nu)$ has arity equal to $\nu + 1$, and it describes the behaviour of all the objects of the theory. One can freely choose the least natural number ν for which $runiv(\nu)$ exists. For each choice one has a theory which is named TB_ν.

2. - The naïve formalization

Main notations and conventions.

The logical symbols used here are: $\wedge$ *conjunction*, $\vee$ *disjunction*, $\neg$ *negation*, $\rightarrow$ *implication*, $\leftrightarrow$ *double implication*, $\forall$ *universal quantifier*, $\exists$ *existential quantifier*, and $=$ *equality*. The usual axioms and rules of classical first order logic with equality are assumed, for example see [12] sec. 1.5 and sec. 2.8. Some lowercase latin letters, (possibly with indexes) $a, b, c, d, e, f, g, m, n, s, t, u, v, w, x, y, z, a^{(1)}, \ldots, a', \ldots,$ are used for variables of the formal language. The metavariables will be chosen among lowercase and uppercase greek $\alpha, \beta, \ldots,$ and i, j, q, r. Parentheses ((( ))) and comma , are also used.

It is worthwhile to notice that the only primitive terms of this formalization are variables and symbols for constants. This will remain true troughout the paper. In current usage symbols for functions stand for total defined functions, while a lot of symbols for constants here introduced stand for functions partially defined. Nevertheless some abbreviations, notations and defined terms will also be used. The usual conventions on replacing strings of symbols for variables in a formula are used here (see [13] page 16-17 and [14] page 67), in particular when substituting a string that is not free for a variable in a formula. In this case fresh variables are put in place of all the bound occurrences (in the formula) of the variables of the string. The usual notation for substitution of strings for variables in a formula, $\varphi[\alpha/\beta, \ldots, \gamma/\delta]$, is adopted. If Ψ stands for a formula with a distinguished free variable, say α, one writes also $\Psi(\alpha)$. The notation for existence and uniqueness is used, $\exists!\, \alpha\Phi(\alpha)$. For example $\exists!\, x\Phi(x)$ stands for $\exists x(\Phi(x) \wedge \forall y(\Phi[y/x] \rightarrow x = y))$.

We essentially follow the axioms given in [7], but we change something. The groups of axioms are labelled by Af, ..., Hf. The notations respective to each group of axioms are denoted by NA, ...NH.

2.1: Axioms on fundamental qualities and relation.

There is only one symbol **P** for predicates, and its arity is equal to five. The intended meaning of $\mathbf{P}(\alpha, \beta, \gamma, \delta, \varepsilon)$ is that α is a quaternary relation that holds among the objects $\beta, \gamma, \delta, \varepsilon$.

There are no symbols for functions.
The first symbols for constants introduced in this section are:

qqual: for the *quality of being a quality*;
rfun: for the *fundamental relation of unary objects*;
rbin: for the *fundamental relation of binary objects*;
rfter: for the *fundamental relation of ternary objects*;
qrelb: for the *quality of being a binary relation*;

qrelt: for the *quality of being a ternary relation*;
qrelq: for the *quality of being a quaternary relation*;
qops: for the *quality of being a simple operation*;
qopb: for the *quality of being a binary operation*;
id: for the operation of *identity*;

The first notation introduced is the following:

NA1. Assuming that **P(rfter, rfbin, rfun, qqual,** q):

$q\ \alpha$ stands for **P(rfter, rfbin, rfun,** q, α).

the intended meaning is *q is a quality* and the object α *enjoys it*.
The first axiom of the formal theory is the following:

Af1. **P(rfter, rfbin, rfun, qqual, qqual)**.

This axiom states that **qqual** *is a quality*, and is equivalent to **qqual qqual**.

Af2. **qqual qrelb** $\wedge$ **qqual qrelt** $\wedge$ **qqual qrelq** .

The axiom Af2 means that **qrelb**, **qrelt**, **qrelq** are qualities. The second notation is

NA2. Assuming respectively **qrelb**r, **qrelt**ϱ and **qrelq**τ, then:

1. $r\ \alpha\beta$ stands for **P(rfter, rfbin,** r, α, β);
2. $\varrho\ \alpha\beta\gamma$ stands for **P(rfter,** ϱ, α, β, γ);
3. $\tau\ \alpha\beta\gamma\delta$ stands for **P**($\tau, \alpha, \beta, \gamma, \delta$).

the intended meaning of NA2.1 is that *r is a binary relation and* α *and* β *are in the relation r*. NA2.2 and NA2.3 are the analogous notations in the case of ternary and quaternary relations.

Af3. **qrelb rfun** $\wedge$ **qrelt rfbin** $\wedge$ **qrelq rfter.**

Af3 means that **rfun**, **rfbin**, **rfter** are respectively a binary, ternary, quaternary relation.

Af4. **qqual qops** $\wedge$ **qqual qopb**

The axiom Af3 means that **qops**, **qopb** are qualities.

Af5. 1. $\forall u$(**qops** $u \rightarrow \forall x\ \forall y\ \forall z$(**rfbin** $uxy \wedge$ **rfbin** $uxz \rightarrow y = z$));

2. $\forall v$(**qopb** $v \rightarrow \forall x\ \forall y\ \forall t\ \forall z$(**rfter** $vxyt \wedge$ **rfter** $vxyz \rightarrow t = z$))

This last axiom states that simple and binary operations are functional.

Af6. **qops id** $\wedge \forall x\ \forall y$(**rfbin id**$xy \leftrightarrow x = y$) .

The axiom Af5 means that **id** is a simple operation and transforms each object into itself. These last axioms allow to use the following notation:

NA3. Assuming **qops** φ, and **qopb** ψ, for every formula $\Psi(z)$ of the language:

1. $\varphi(\alpha) = \beta$, $\beta = \varphi(\alpha)$ stand for **P(rfter, rfbin,** φ, α, β**)**;
2. $\psi(\alpha, \beta) = \gamma$, $\gamma = \psi(\alpha, \beta)$ stand for **P(rfter,** $\psi, \alpha, \beta, \gamma$**)**;
3. $\Psi(\varphi(\alpha))$, $\Psi(\psi(\alpha, \beta))$ stand, respectively, for

   $\exists\, z(\Psi(z) \wedge \varphi(\alpha) = z)$, $\exists\, z(\Psi(z) \wedge \psi(\alpha, \beta) = z)$.

Clauses like NA3.3 go hidden in the sequel.

The symbols for collections are:

qcoll: for the *quality of being a collection*;

V: for the *universal collection*;

coll: for the *collection of all collections*;

∅: for the *void collection*.

The first axiom on collections is:

Af7.
1. **qqual qcoll**;
2. **qcoll V** $\wedge$ **qcoll coll** $\wedge$ **qcoll ∅**;
3. $\forall x\big($**rfun V**$x \wedge \neg$ **rfun ∅**$x \wedge ($**rfun coll** $x \leftrightarrow$ **qcoll** $x)\big)$.

The axiom Af7.3 means that: each object belongs to the collection **V**, no object belongs to the empty collection ∅, and the objects belonging to the collection **coll** are all and only the collections, *i.e.* the objects enjoying the quality of being a collection.

NA4. Assuming, **qcoll** γ, **qcoll** δ:

1. $\alpha \in \beta$ stands for **qcoll** $\beta \wedge$ **rfun** $\beta\alpha$;

   $\alpha \notin \beta$ stands for **qcoll** $\beta \wedge \neg$ **rfun** $\beta\alpha$;
2. $\gamma \subseteq \delta$ stands for $\forall x(x \in \gamma \rightarrow x \in \delta)$

If $\alpha \in \beta$ one says that α *belongs to* the collection β, or α is an *element* of the collection β.

Af8. $\forall x\, \forall y($**qcoll** $x \wedge$ **qcoll** $y \wedge \forall t(t \in x \leftrightarrow t \in y) \rightarrow x = y)$.

This axiom states the *extensionality* of collections.

2.2: Natural numbers.

The symbols for arithmetic are:

N: for the *collection of natural numbers*;
nadd: for the binary operation of *addition* of natural numbers;
nmul: for the binary operation of *multiplication* of natural numbers;
nsub: for the binary operation of *subtraction* of natural numbers;
nord: for the binary relation of *order between natural numbers*;
0: for the number *zero*;
1: for the number *one*.

Bf1. **1.** $\mathbf{N} \in \mathbf{coll}$;

2. $\mathbf{qopb\ nadd} \wedge \mathbf{qopb\ nmul} \wedge \mathbf{qopb\ nsub}$;

3. $\mathbf{qrelb\ nord}$;

4. $\mathbf{0} \in \mathbf{N} \wedge \mathbf{1} \in \mathbf{N}$.

This first axiom simply states: **N** is a collection, **nadd**, **nmul** and **nsub** are binary operations, **nord** a binary relation and **0**, **1** belong to **N**.

Bf2. **1.** $\forall x\ \forall y(\mathbf{nord}\ xy \rightarrow x \in \mathbf{N} \wedge y \in \mathbf{N})$;

2. $\forall x\ \forall y(x \in \mathbf{N} \wedge y \in \mathbf{N} \rightarrow \mathbf{nord}\ xy \vee \mathbf{nord}\ yx)$;

3. $\forall x\ \forall y\ \forall z(\mathbf{nadd}(x,y) = z \rightarrow x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge z \in \mathbf{N})$;

4. $\forall x\ \forall y\big(x \in \mathbf{N} \wedge y \in \mathbf{N} \rightarrow \exists z(\mathbf{nadd}(x,y) = z)\big)$;

5. $\forall x\ \forall y\ \forall z(\mathbf{nmul}(x,y) = z \rightarrow x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge z \in \mathbf{N})$;

6. $\forall x\ \forall y\big(x \in \mathbf{N} \wedge y \in \mathbf{N} \rightarrow \exists z(\mathbf{nmul}(x,y) = z)\big)$;

7. $\forall x\ \forall y\ \forall z(\mathbf{nsub}(x,y) = z \rightarrow x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge z \in \mathbf{N} \wedge \mathbf{nord}\ yx)$;

8. $\forall x\ \forall y\big(x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge \mathbf{nord}\ yx \rightarrow \exists z(\mathbf{nsub}(x,y) = z)\big)$.

This last axiom specifies the domain and the range of **nadd**, **nmul**, **nsub** and **nord**. Bf2.1-2 means that **nord** is a linear relation between natural numbers. Bf3-4 means that **nadd** is a binary operation defined on all and only the natural numbers and its values are natural numbers. Bf5-6 means exactly the same for the operation **nmul**. Bf7-8 specifies the usual domain of the subtraction between natural numbers.

NB1. Assuming $\alpha \in \mathbf{N}$, $\beta \in \mathbf{N}$,

1. $\alpha \leqslant \beta$, $\beta \geqslant \alpha$ stand for $\mathbf{nord}\ \alpha\beta$;

2. $\alpha < \beta, \beta > \alpha$ stand for $\alpha \leq \beta \wedge \alpha \neq \beta$;

3. $\alpha + \beta$ stands for $\mathbf{nadd}(\alpha, \beta)$;

4. $\alpha \cdot \beta$, simply $\alpha\beta$ stand for $\mathbf{nmul}(\alpha, \beta)$;

5. $\alpha - \beta$ stands for $\mathbf{nsub}(\alpha, \beta)$.

As usual if $\alpha \leq \beta$ one says that α is less, or smaller than β.

Bf3. $\forall x\, \forall y(x \leq y \wedge y \leq x \rightarrow x = y)$.

The axiom Bf3 states that **nord** is an antisymmetric relation.

Bf4. 1. $\forall x\, \forall y(x \in \mathbf{N} \wedge y \in \mathbf{N} \rightarrow x + y = y + x)$;

2. $\forall x\, \forall y\, \forall z(x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge z \in \mathbf{N} \rightarrow (x + y) + z = x + (y + z))$;

3. $\forall x(x \in \mathbf{N} \rightarrow x + \mathbf{0} = x)$;

4. $\forall x\, \forall y\, \forall z(x - y = z \leftrightarrow x = y + z)$.

Bf5. 1. $\forall x\, \forall y(x \in \mathbf{N} \wedge y \in \mathbf{N} \rightarrow x \cdot y = y \cdot x)$;

2. $\forall x\, \forall y\, \forall z(x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge z \in \mathbf{N} \rightarrow (x \cdot y) \cdot z = x \cdot (y \cdot z))$;

3. $\forall x\, \forall y(x \in \mathbf{N} \rightarrow x \cdot \mathbf{1} = x)$;

4. $\forall x\, \forall y\, \forall z(x \in \mathbf{N} \wedge y \in \mathbf{N} \wedge z \in \mathbf{N} \rightarrow x \cdot (y + z) = (x \cdot y) + (x \cdot z))$;

These last two axioms state the usual properties of the operations of natural numbers. Namely the addition and the multiplication are both commutative, associatve, with respective unit element **0** and **1**, and the multiplication is distributive with respect to the addition. Bf4.4 establishes the relation between the addition and the subtraction.

Bf6. 1. $\forall x\, \forall y(x < y \leftrightarrow \exists n(n \neq \mathbf{0} \wedge y = x + n))$

2. $\mathbf{0} < \mathbf{1} \wedge \forall x(x > \mathbf{0} \rightarrow x \geq \mathbf{1})$

The axiom Bf6.1 specifies the relation between **nord** and **nadd**, and the axiom Bf6.2 states that **1** is the immediate successor of **0**.

NB2. Standard characters for 2, 3... are used as they are not symbols of constants of the theory like **0** and **1**.

Finally the induction principle for collections of natural numbers states that every non empty collection of natural numbers has a minimum element:

Bf7. $\forall x((x \in \mathbf{coll} \wedge x \subseteq \mathbf{N} \wedge x \neq \emptyset) \rightarrow \exists n(n \in x \wedge \forall t(t \in x \rightarrow n \leq t)))$.

This axiom may be rather weak if there are few collections.

2.3: Finite systems.

The notion of *finite system* is inspired to the usual n-tuples and more generally to indexed finite lists. In the sequel, finite systems are often called systems.

The symbols for finite systems are:

syf: for the *collection of finite systems*;
syuf: for the *collection of univalent finite systems*;
usys: for the operation of *binary union* between systems;
csys: for the operation of *composition* of systems;
isys: for the operation of *inversion* of systems;
$\emptyset_2$: for the *empty system*;
carsys: for the operation of *cardinality* for systems.

The first axiom on systems simply states that **syuf** is a subcollection of **syf**.

Cf1. 1. $\mathbf{syf} \in \mathbf{coll} \land \mathbf{syuf} \in \mathbf{coll}$;

2. $\mathbf{syuf} \subseteq \mathbf{syf}$.

NC1. When σ is a systems, and **rfbin** $\sigma\alpha\beta$, one says that α is an *index* of σ and that β is a *value* of σ associated to α.

Cf2. 1. $\forall x\ \forall y(x \in \mathbf{syf} \land y \in \mathbf{syf} \land\ \forall u\ \forall v(\mathbf{rfbin}\ xuv \leftrightarrow \mathbf{rfbin}\ yuv) \leftrightarrow x = y)$;

2. $\forall x(x \in \mathbf{syuf} \leftrightarrow x \in \mathbf{syf} \land \forall u\ \forall y\ \forall z(\mathbf{rfbin}\ xuy \land \mathbf{rfbin}\ xuz \rightarrow y = z))$.

Axiom Cf2.1 states that the collection of finite systems is *extensional*: there are no two systems that associate the same values to the same indexes. Axiom Cf2.2 states that univalent systems are *functional*: a univalent system associates exactly one value to each of its indexes.

Cf3. $\forall x\ \forall y \exists\ z(z \in \mathbf{syuf} \land \forall u\ \forall v(\mathbf{rfbin}\ zuv \leftrightarrow u = x \land v = y))$.

This axiom states that for every couple of objects α, β there exists a system that has α as its only index and that has β as its only value. These axioms allow to use the next notation

NC2. Assuming $\sigma \in \mathbf{syuf}$:

1. $\sigma(\alpha) = \beta,\ \beta = \sigma(\alpha),\ \sigma_\alpha = \beta,\ \beta = \sigma_\alpha$ stand for **rfbin** $\sigma\alpha\beta$;

2. $\binom{\alpha}{\beta} = \gamma,\ \gamma = \binom{\alpha}{\beta}$ stand for

$$\gamma \in \mathbf{syuf} \land \forall u\ \forall v(\mathbf{rfbin}\ \gamma uv \leftrightarrow u = \alpha \land v = \beta)).$$

CF4. 1. **qopb usys**;

2. $\forall z\ \forall x\ \forall y(\mathbf{usys}(x,y)=z \to x \in \mathbf{syf} \wedge y \in \mathbf{syf} \wedge z \in \mathbf{syf})$;

3. $\forall x\ \forall y(x \in \mathbf{syf} \wedge y \in \mathbf{syf} \to \exists\, z(\mathbf{usys}(x,y)=z))$;

4. $\forall x\ \forall y\ \forall u\ \forall v(\mathbf{rfbin}\ \mathbf{usys}(x,y)uv \leftrightarrow \mathbf{rfbin}\ xuv \vee \mathbf{rfbin}\ yuv)$.

Axiom Cf4.4 describes the action of the union of two systems. The union of two systems α and β is a system γ. This system associates to each index of α all its values in α, and it associates to each index of β all its values in β. Each index of γ is either an index of α or a index of β. In other words the «graph» of γ is the «union» of the graphs of α and β.

Cf5. 1. **qopb csys**;

2. $\forall z\ \forall x\ \forall y(\mathbf{csys}(x,y)=z \to x \in \mathbf{syf} \wedge y \in \mathbf{syf} \wedge z \in \mathbf{syf})$;

3. $\forall x\ \forall y(x \in \mathbf{syf} \wedge y \in \mathbf{syf} \to \exists\, z(\mathbf{csys}(x,y)=z))$;

4. $\forall x\ \forall y\ \forall u\ \forall v(\mathbf{rfbin}\ \mathbf{csys}(x,y)uv \leftrightarrow \exists\, w\ \mathbf{rfbin}\ xuw \wedge \mathbf{rfbin}\ ywv)$;

5. $\mathbf{csys}\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right) = \emptyset_2$.

Axiom Cf5.4 describes the action of the composition of two systems. Namely if γ is obtained by composing two systems α and β, then γ associates to any index μ of α all the values associated in β to the values of μ in α. Axiom Cf5.3 states that any two systems can be composed, so by composing the systems $\begin{pmatrix}0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ (axiom Cf5.5) one gets the *void system*: $\emptyset_2$. By the extensionality of **syf**, it is the only system without indexes and values.

Cf6. 1. **qops isys**;

2. $\forall z\ \forall x(\mathbf{isys}(x)=z \to x \in \mathbf{syf} \wedge z \in \mathbf{syf})$;

3. $\forall x(x \in \mathbf{syf} \to \exists\, z(\mathbf{isys}(x)=z))$;

4. $\forall x\ \forall u\ \forall v(\mathbf{rfbin}\ \mathbf{isys}(x)uv \leftrightarrow \mathbf{rfbin}\ xvu)$.

Axiom Cf6.4 describes the action of the inverse of a system. If a system associates to an index a value the inverse system associates to this value the given index. Hence the indexes of the inverse system are the values of the original system, and vice-versa.

NC3. Assuming $\sigma \in \mathbf{syuf}$, $\sigma' \in \mathbf{syuf}$, and given $\gamma \ldots \delta$ and $\gamma' \ldots \delta'$,

1. $\sigma \cup \sigma'$, $\sigma \circ \sigma'$, σ^{-1} stand respectively for

$\mathbf{usys}(\sigma,\sigma')$, $\mathbf{csys}(\sigma,\sigma')$, $\mathbf{isys}(\sigma)$;

2. $\begin{pmatrix} \gamma \dots \delta \\ \gamma' \dots \delta' \end{pmatrix}$ stands for $\begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} \cup \dots \cup \begin{pmatrix} \delta \\ \delta' \end{pmatrix}$

Cf7. $\forall n \big(n \in \mathbf{N} \rightarrow \exists\, v (v \in \mathbf{coll} \wedge$

$$\forall z (z \in v \leftrightarrow (z \in \mathbf{syuf} \wedge \forall x (\exists\, y (z_x = y) \leftrightarrow 1 \leqslant x \leqslant n)))\big).$$

If $\mu \in \mathbf{N}$, thanks to the extensionality of **coll**, this last axiom uniquely determinates a collection whose elements are the univalent systems having as indexes all the non zero numbers less than μ. The notation for this collections of tuples is

NC4. Assuming $\mu \in \mathbf{N}$:

$\mathbf{V}^{\mu} = \alpha$ stands for $\alpha \in \mathbf{coll} \wedge$

$$\forall x (x \in z \leftrightarrow x \in \mathbf{syuf} \wedge \forall u (\exists\, v (x_u = v) \leftrightarrow 1 \leqslant u \leqslant \mu))) .$$

Hence the collection having only the element $\emptyset_2$ is denoted with $\mathbf{V}^0$.

Cf8. $\forall n (n \in \mathbf{N} \rightarrow \exists\, z (z \in \mathbf{V}^n \wedge \forall m (1 \leqslant m \leqslant n \rightarrow z_m = m))).$

Before explaining the axiom it is useful to introduce the following notation for «tuples». It is possible thanks to extensionality and functionality. One has to consider the fact that the «elements of **N**» may be more than the concrete natural numbers. Hence two notations for tuples are adopted.

NC5. If μ stands for a concrete natural number, $\mu' > 1$, $\delta \in V^{\mu'}$

and ε is a symbol for variable:

1.1. $(\varepsilon^{(1)}, \dots, \varepsilon^{(\mu)}) = \beta$, $\beta = (\varepsilon^{(1)}, \dots, \varepsilon^{(\mu)})$ stand for

$$\mu \geqslant 1 \wedge \beta \in \mathbf{V}^{\mu} \wedge \beta_1 = \varepsilon^{(1)} \wedge \dots \wedge \beta_{\mu} = \varepsilon^{(\mu)};$$

1.2. $(\beta, \gamma) = \alpha$, $\alpha = (\beta, \gamma)$ stand for

$$\alpha \in \mathbf{V}^2 \wedge \alpha_1 = \beta \wedge \alpha_2 = \gamma;$$

1.3. $[\alpha]$ stands for (α);

2.1. $(1, \dots, \mu') = \alpha$, $\alpha = (1, \dots, \mu')$ stand for

$$\alpha \in \mathbf{V}^{\mu'} \wedge \forall m (1 \leqslant m \leqslant \mu' \rightarrow \alpha_m = m);$$

2.2. $(\delta_1 \dots \delta_{\mu'})$ or $(\delta_1 \dots \delta_i \dots \delta_{\mu'})$ stand for δ.

The axiom Cf8 states the existence of the μ-tuple $(1, \dots, \mu)$ for all $\mu \in \mathbf{N}$.

The next axiom deals with the operation giving the cardinality of a system.

Cf9. 1. **qops carsys**;

2. $\forall z(z \in \mathbf{syf} \rightarrow \exists n(n \in \mathrm{N} \wedge \mathbf{carsys}(z) = n))$;

3. $\forall z\ \forall n(z \in \mathbf{syf} \wedge \mathbf{carsys}(z) = n \rightarrow$

$$n \in \mathrm{N} \wedge \exists x \exists y(x \in \mathrm{V}^n \wedge y \in \mathrm{V}^n \wedge z = y \circ x^{-1}));$$

4. $\forall z\ \forall n((z \in \mathbf{syf} \wedge n \in \mathrm{N} \wedge$

$$\exists x \exists y(x \in \mathrm{V}^n \wedge y \in \mathrm{V}^n \wedge z = y \circ x^{-1})) \rightarrow \mathbf{carsys}(z) \leqslant n)$$

5. $\forall z\ \forall x\ \forall y\ \forall n\Big(n \in \mathrm{N} \wedge x \in \mathrm{V}^n \wedge y \in \mathrm{V}^n \wedge z = y \circ x^{-1} \rightarrow$

$$\Big(n = \mathbf{carsys}(z) \leftrightarrow \forall u\ \forall v(u \in \mathrm{N} \wedge v \in \mathrm{N} \wedge$$

$$1 \leqslant u \leqslant n \wedge 1 \leqslant v \leqslant n \wedge u \neq v \rightarrow \begin{pmatrix} x_u \\ y_u \end{pmatrix} \neq \begin{pmatrix} x_v \\ y_v \end{pmatrix}\Big)\Big)\Big).$$

The following notation is useful to explain the meaning of the axioms Cf9.

NC5. Assuming $\sigma \in \mathbf{syf}$, $\mathbf{carsys}(\sigma) = \mu$ and $\alpha \in V^\mu$, $\beta \in V^\mu$

$$\sigma = \begin{pmatrix} \alpha_1 \ldots \alpha_\mu \\ \beta_1 \ldots \beta_\mu \end{pmatrix} \text{ stands for } \sigma = \beta \circ \alpha^{-1}.$$

Axioms Cf9.2, Cf9.3 state in what acception systems can be considered finite. They state that **carsys** is a simple operation defined on all the finite systems and with values in N. The value of this operation on a system is called the *cardinality* of the system. The cardinality of a given system is the minimum element $n \in \mathrm{N}$ such that the system is the composition of a n-tuple and of the inverse of a n-tuple. Conversely, if a system is such a composition then n is its cardinality if and only if the singular systems, obtained by associating each value of the first n-tuple to the corresponding value of the second n-tuple, are all distinct. Hence using the introduced notations

$$n = \mathbf{carsys}(z) \wedge z = y \circ x^{-1} = \begin{pmatrix} x_1 \ldots x_n \\ y_1 \ldots y_n \end{pmatrix} \Rightarrow \forall i, j \begin{pmatrix} x_i \\ y_i \end{pmatrix} \neq \begin{pmatrix} x_j \\ y_j \end{pmatrix}.$$

2.4: Arity, relations and operations.

The symbols introduced in this section are:

ar: for the *operation giving the arity*;
qrel: for the *quality of being a relation*;
qop: for the *quality of being an operation*.

Df1. 1. $\mathbf{qqual\ qrel} \wedge \mathbf{qqual\ qop}$;

2. $\forall x(\mathbf{qrelb}\ x \vee \mathbf{qrelt}\ x \vee \mathbf{qrelq}\ x \rightarrow \mathbf{qrel}\ x)$;

3. $\forall x(\mathbf{qopb}\ x \vee \mathbf{qops}\ x \rightarrow \mathbf{qop}\ x)$.

The next axiom specifies the arity of principal kinds of objects. The domain and codomain of the operation **ar** are not specified by this axiom. Anyway the axiom excludes that an object may have different arities.

Df2. 1. $\mathbf{qops\ ar}$;

2. $\forall n(n \in \mathbf{N} \rightarrow \mathbf{ar}(n) = 0)$;

3. $\forall x(\mathbf{qqual}\ x \vee x \in \mathbf{coll} \rightarrow \mathbf{ar}(x) = 1)$;

4. $\forall x((\mathbf{qrelb}\ x \leftrightarrow \mathbf{qrel}\ x \wedge \mathbf{ar}(x) = 2) \wedge$
$(\mathbf{qops}\ x \leftrightarrow \mathbf{qop}\ x \wedge \mathbf{ar}(x) = 2) \wedge (x \in \mathbf{syf} \rightarrow \mathbf{ar}(x) = 2))$;

5. $\forall x((\mathbf{qrelt}\ x \leftrightarrow \mathbf{qrel}\ x \wedge \mathbf{ar}(x) = 3) \wedge$
$(\mathbf{qopb}\ x \leftrightarrow \mathbf{qop}\ x \wedge \mathbf{ar}(x) = 3))$;

6. $\forall x(\mathbf{qrelq}\ x \leftrightarrow \mathbf{qrel}\ x \wedge \mathbf{ar}(x) = 4)$.

3. - Intermezzo: a weaker theory with a natural infinite formalization

In this section μ, μ' stand for concrete natural numbers. The symbols here introduced are:

for any non zero μ a symbol for a $\mu + 1$-ary predicate $\mathbf{P}_\mu$;
rfond: for the *operation generating fundamental relations*;
runiv: for the *operation generating universal relations*.

Ef1. 1. $\forall x\ \forall y\ \forall z((\mathbf{P}_1(x, y) \leftrightarrow \mathbf{P}(\mathbf{rfter}, \mathbf{rfbin}, \mathbf{rfun}, x, y) \wedge$
$\mathbf{P}_2(x, y, z) \leftrightarrow \mathbf{P}(\mathbf{rfter}, \mathbf{rfbin}, x, y, z)))$;

2. $\forall x\ \forall y\ \forall z\ \forall v\ \forall w((\mathbf{P}_3(x, y, z, v) \leftrightarrow \mathbf{P}(\mathbf{rfter}, x, y, z, v) \wedge$
$\mathbf{P}_4(x, y, z, v, w) \leftrightarrow \mathbf{P}(x, y, z, v, w)))$.

This axiom relates $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$ with $\mathbf{P}$. The idea is that $\mathbf{P}_1$ is used to describe the graph of the first fundamental relation **rfun**, $\mathbf{P}_2$ and $\mathbf{P}_3$ do the same for **rfbin** and **rfter**, while $\mathbf{P}_\mu$, with $\mu \geq 4$, is used to decribe the fundamental relation of μ-ary objects $\mathbf{rfond}(\mu)$ to be introduced below with axiom Ef3.μ.

Ef2. 1. **qops rfond**;

2. $\mathbf{rfond}(1) = \mathbf{rfun} \wedge \mathbf{rfond}(2) = \mathbf{rfbin} \wedge \mathbf{rfond}(3) = \mathbf{rfter}$.

Ef3.μ. 1. $\exists x(\mathbf{rfond}(\mu) = x)$;

2. $\forall x(\mathbf{rfond}(\mu) = x \rightarrow \mathbf{ar}(x) = \mu + 1 \wedge \mathbf{qrel}\, x)$;

3. $\forall x\, \forall x^{(1)} \ldots \forall x^{(\mu)}(\mathbf{ar}(x) = \mu \rightarrow$

$$\mathbf{P}_\mu(x, x^{(1)}, \ldots, x^{(\mu)}) \leftrightarrow \mathbf{P}_{\mu+1}(\mathbf{rfond}(\mu), x, x^{(1)}, \ldots, x^{(\mu)})).$$

These axioms allow the self-descrption of the theory: for the *external* predicate $\mathbf{P}_\mu$ is described by means of the *inner* object $\mathbf{rfond}(\mu)$ and the predicate $\mathbf{P}_{\mu+1}$.
The next axiom deals with universal relations:

Ef4.μ. 1. **qops runiv**;

2. $\forall x(\mathbf{runiv}(\mu) = x \rightarrow \mathbf{ar}(x) = \mu + 1 \wedge \mathbf{qrel}\, x)$;

3. $\forall x\, \forall x^{(1)}, \ldots, x^{(\mu)} \forall y(\mathbf{runiv}(\mu) = x \wedge$

$$\mathbf{P}_{\mu+1}(x, x^{(1)}, \ldots, x^{(\mu)}, y) \rightarrow x^{(1)} = 1 \wedge, \ldots, x^{(\mu)} = \mu);$$

4. $\forall x\, \forall y(\mathbf{runiv}(\mu) = x \wedge \mathbf{P}_{\mu+1}(x, 1, \ldots, \mu, y) \rightarrow$

$$y \in \mathbf{V}^2 \wedge \exists\, n(n \in \mathbf{N} \wedge \mathbf{ar}(y_1) = n) \wedge y_2 \in \mathbf{V}^n);$$

5.μ'. $\forall x\, \forall x^{(1)}, \ldots, x^{(\mu')} \forall z(\mathbf{runiv}(\mu) = x \wedge \mathbf{ar}(z) = \mu' \rightarrow$

$$\mathbf{P}_{\mu+1}(x, 1, \ldots, \mu, (z, (x^{(1)}, \ldots, x^{(\mu')}))) \leftrightarrow \mathbf{P}_{\mu'}(z, x^{(1)}, \ldots, x^{(\mu')})).$$

The axioms Ef4.μ.5.μ' allow the description of *all* the predicates $\mathbf{P}_{\mu'}$ by means of the relation $\mathbf{runiv}(\mu)$ and the predicate $\mathbf{P}_{\mu+1}$. Less formally the axioms Ef4.μ.5.μ' can be written

$$\sigma \in V^{\mu'} \rightarrow (\mathbf{P}_{\mu+1}(\mathbf{runiv}(\mu), 1, \ldots, \mu, (z, \sigma)) \leftrightarrow \mathbf{P}_{\mu'}(z, \sigma_1 \ldots \sigma_{\mu'})).$$

Note that the axioms above do not guarantee the operation **runiv** be defined on natural number. Then fixed any natural number ν greater than four one assumes the axiom

Ef5.ν. $\quad \forall n(n \in \mathbf{N} \wedge n \geqslant \nu \rightarrow \exists x(\mathbf{runiv}(n) = x))$.

The group of axioms Af, Bf, Cf, Df, Ef are a formalization of a theory weaker than TB_{ν}. In fact there are fundamental and universal relations only for concrete natural numbers instead of for all the elements of **N**.

4. - Finite axiomatization of the self-descriptive axioms

4.1: A new symbol for predicate.

An object of the theory with low arity and describing the behaviour of all the others would put severe constraints on the possible extensions of the theory. Nevertheless this idea can be fruitfull for a finite axiomatization of the full theory. Hence a new symbol **Q** for binary predicate is introduced:

The intuitive meaning of $\mathbf{Q}(\alpha, \beta)$ is that β is a k-tuple, the object α has arity k and acts on the components of β. First one defines **Q** in terms of **P** on lower arity objects:

Ff1.

1. $\forall x\, \forall y(\mathbf{Q}(x, y) \rightarrow \exists n(n \in \mathbf{N} \wedge \mathbf{ar}(x) = n \wedge y \in \mathbf{V}^n));$

2. $\forall x\, \forall y\, \forall z((\mathbf{Q}(x, [y]) \leftrightarrow \mathbf{P}(\mathbf{rfter}, \mathbf{rfbin}, \mathbf{rfun}, x, y)) \wedge$

   $(\mathbf{Q}(x, (y, z)) \leftrightarrow \mathbf{P}(\mathbf{rfter}, \mathbf{rfbin}, x, y, z)));$

3. $\forall x\, \forall y\, \forall z\, \forall v\, \forall w(\,(\mathbf{Q}(x, (y, z, v)) \leftrightarrow \mathbf{P}(\mathbf{rfter}, x, y, z, v)) \wedge$

   $(\mathbf{Q}(x, (y, z, v, w)) \leftrightarrow \mathbf{P}(x, y, z, v, w)))\,).$

Observe that from axiom Ff1.1 it follows that the first argument of **Q** must have as arity an element of **N**. This implies that the formalization may have to be changed if one wants to extend the theory by introducing objects whose arity is not an element of **N**.

4.2: Fundamental and universal relations.

Recall the main symbols of this section:

rfond: for the *operation generating fundamental relations*;
runiv: for the *operation generating universal relations*.

Now it is possible to express the full strength of the axioms on **rfond** and **runiv**.

Gf1. 1. **qops rfond**;

2. $\forall n(n \in \mathbf{N} \wedge n \geqslant 1 \rightarrow \exists x\ \mathbf{rfond}(n) = x)$;

3. $\mathbf{rfond}(1) = \mathbf{rfun} \wedge \mathbf{rfond}(2) = \mathbf{rfbin} \wedge \mathbf{rfond}(3) = \mathbf{rfter}$;

4. $\forall x\ \forall y\ \forall z\ \forall v\ \forall w(\mathbf{Q}(\mathbf{rfond}(4), (x, y, z, v, w)) \leftrightarrow \mathbf{P}(x, y, z, v, w)))$;

5. $\forall x\ \forall n\ \forall y(\mathbf{ar}(x) = n \wedge n \in \mathbf{N} \rightarrow (\mathbf{Q}(x, y) \ \rightarrow$
$\exists z(z \in \mathbf{V}^{n+1} \wedge z_1 = x \wedge\ \forall m(m \in \mathbf{N} \wedge 1 \leqslant m \leqslant n \rightarrow$
$z_{m+1} = y_m)) \wedge \mathbf{Q}(\mathbf{rfond}(n), z)))$;

6. $\forall x\ \forall n\ \forall z\,(\mathbf{ar}(x) = n \wedge n \in \mathbf{N} \rightarrow (\mathbf{Q}(\mathbf{rfond}(n), z) \wedge z_1 = x \ \rightarrow$
$\exists y(y \in \mathbf{V}^n \wedge\ \forall m(m \in \mathbf{N} \wedge 1 \leqslant m \leqslant n \rightarrow z_{m+1} = y_m)) \wedge \mathbf{Q}(x, y)))$.

This axiom plays the role of the axiom Ef2 and of the infinite sequence of axioms Ef3.μ.

Gf2. 1. **qops runiv**;

2. $\forall x\ \forall n(n \in \mathbf{N} \wedge n \geqslant 1 \wedge \mathbf{runiv}(n) = x \rightarrow \mathbf{ar}(x) = n + 1 \wedge \mathbf{qrel}\ x)$;

3. $\forall n\ \forall x\ \forall y\,(\mathbf{runiv}(n) = x \rightarrow$
$(\mathbf{Q}(x, y) \ \rightarrow\ \forall m(m \in \mathbf{N} \wedge 1 \leqslant m \leqslant n \rightarrow y_m = m)))$;

4. $\forall n\ \forall x\ \forall y\,(\mathbf{runiv}(n) = x \rightarrow$
$(\mathbf{Q}(x, y) \ \rightarrow y_{n+1} \in \mathbf{V}^2 \wedge \mathbf{Q}((y_{n+1})_1, (y_{n+1})_2)))$;

5. $\forall n\ \forall x\ \forall u\ \forall v\,(\mathbf{runiv}(n) = x \rightarrow$
$(\mathbf{Q}(u, v) \ \rightarrow \exists y(\mathbf{Q}(x, y) \wedge (y_{n+1})_1 = u, (y_{n+1})_2 = v)))$.

This axiom plays the role of the infinite axioms scheme Ef4.μ. μ'.
Fix any natural number greater than four and denote it by ν. The new version of axiom Ef5.ν is then:

Gf3.ν. $\forall n(n \in \mathbf{N} \wedge n \geqslant \nu \rightarrow \exists x\ (\mathbf{runiv}(n) = x))$.

The groups of axioms Af, Bf, Cf, Df, Ff, Gf are a formalization of TB, in the language $\mathcal{L}$ with only **P** and **Q** as symbols for predicates and only the 33 symbols for constants introduced in section 2 and 4: this theory is named $\mathrm{TB}_\nu^{\mathbf{P},\mathbf{Q}}$.

5. - AN ARITHMETICAL MODEL OF $\mathrm{TB}_{\nu}^{P,Q}$

In this section we prove the consistency of $\mathrm{TB}_{\nu}^{P,Q}$ with respect to PA. This is done by making a model:

the universe of the model is the set of concrete natural numbers N.

the interpretation $\mathfrak{I}$ of the model is defined on symbols for constants as follows:

$\mathfrak{I}(\mathbf{qqual}) = 2,\ \mathfrak{I}(\mathbf{qrelb}) = 4,\ \mathfrak{I}(\mathbf{qrelt}) = 8,\ \mathfrak{I}(\mathbf{qrelq}) = 2^4,$

$\mathfrak{I}(\mathbf{qops}) = 2^5,\ \mathfrak{I}(\mathbf{qopb}) = 2^6,\ \mathfrak{I}(\mathbf{qcoll}) = 2^7,\ \mathfrak{I}(\mathbf{qrel}) = 2^8,\ \mathfrak{I}(\mathbf{qop}) = 2^9;$

$\mathfrak{I}(\mathbf{rfun}) = 3^3,\ \mathfrak{I}(\mathbf{nord}) = 3^{3^2},\ \mathfrak{I}(\mathbf{rfbin}) = 3^5,\ \mathfrak{I}(\mathbf{rfter}) = 3^7;$

$\mathfrak{I}(\mathbf{id}) = 5^3,\ \mathfrak{I}(\mathbf{isys}) = 5^{3^2},\ \mathfrak{I}(\mathbf{carsys}) = 5^{3^3},\ \mathfrak{I}(\mathbf{ar}) = 5^{3^4},\ \mathfrak{I}(\mathbf{rfond}) = 5^{3^5},$

$\mathfrak{I}(\mathbf{runiv}) = 5^{3^6},\ \mathfrak{I}(\mathbf{nadd}) = 5^5,\ \mathfrak{I}(\mathbf{nmul}) = 5^{5^2},$

$\mathfrak{I}(\mathbf{nsub}) = 5^{5^3},\ \mathfrak{I}(\mathbf{usys}) = 5^{5^4},\ \mathfrak{I}(\mathbf{csys}) = 5^{5^5};$

$\mathfrak{I}(\mathbf{V}) = 7^4,\ \mathfrak{I}(\mathbf{coll}) = 7^6,\ \mathfrak{I}(\boldsymbol{\emptyset}) = 7^8,\ \mathfrak{I}(\mathbf{N}) = 7^{10},\ \mathfrak{I}(\mathbf{syf}) = 7^{12},\ \mathfrak{I}(\mathbf{syuf}) = 7^{14};$

$\mathfrak{I}(\mathbf{0}) = 11,\ \mathfrak{I}(\mathbf{1}) = 11^2;$

$\mathfrak{I}(\boldsymbol{\emptyset}_2) = 13.$

This coding can be easily understood keeping in mind the following recipe: the different kinds of fundamental objects (qualities, relations, operations, collections, natural numbers and systems) correspond to powers of different primes: 2, 3, 5, 7, 11, 13. In general the interpretation of a relation or operation with arity α is of the kind $3^{\pi^{(\alpha)}}$ respectively $5^{\pi^{(\alpha)}}$, where $\pi^{(\alpha)}$ is the α^{th} prime number. In the sequel the name of a symbol for constant in italic characters (rather than boldface) is used to denote its interpretation by $\mathfrak{I}$. The only exceptions are the interpretation of $\mathbf{0}$ and $\mathbf{1}$. Moreover the usual notation is used for concrete tuples of natural numbers, even if it is the same one of the formal theory.

The interpretation of $\mathbf{P}$ by $\mathfrak{I}$ is a set of quintuples of natural numbers. Roughly speaking it is built with the labelled union of the graphs or extensions of all the inner objects having positive arity up to three. Namely those elements of N that are: inner qualities (collected in the set $Qual \subset$ N), inner collections (that are the elements of the set $Coll$), inner binary relations (the elements of Rel_2), inner ternary relations (the elements of Rel_3), inner relations (the elements of Rel), inner simple operations (the elements of Op_2), inner binary operations (the elements of Op_3), inner operations (the elements of Op), finite systems (the elements of Syf). The extensions, denoted with $Ext(\alpha)$, are introduced in the definitions 2-4.4: they are the graphs for operations and relations, while for qualities and collections are the subset of N corresponding, respectively, to all the objects enjoying the given quality or all the objects belonging to the given collection.

Hence $\mathfrak{I}(\mathbf{P})$ has the following form:

Def1.
$$\mathfrak{I}(\mathbf{P}) = \{(\mathit{rfter}, \mathit{rfbin}, \mathit{rfun}, \alpha, \beta) : (\alpha, \beta) \in \bigcup_{\alpha \in Qual \cup Coll} \{\alpha\} \times Ext(\alpha)\} \cup$$
$$\cup \{(\mathit{rfter}, \mathit{rfbin}, \alpha, \beta, \gamma) : (\alpha, (\beta, \gamma)) \in \bigcup_{\alpha \in Rel_2 \cup Op_2 \cup Syf} \{\alpha\} \times Ext(\alpha)\} \cup$$
$$\cup \{(\mathit{rfter}, \alpha, \beta, \gamma, \delta) : (\alpha, (\beta, \gamma, \delta)) \in \bigcup_{\alpha \in Rel_3 \cup Op_3} \{\alpha\} \times Ext(\alpha)\}.$$

We begin by introducing in definition 2 both the sets representing in the model the extensions of some qualities and collections (corresponding to the *principal kinds* of objects in the theory), and some other meaningful subsets of N. For example: *Nat* is the extension of the interpretation of **N**, in other words it is the set of the «inner natural numbers»: the number $11^{\mu+1}$ codifies μ as an «element of **N**». *Syf* is the extension of the interpretation of **syf**, in other words it is the set of «inner systems». The remaining extensions are introduced in definitions 4, after giving some useful notations in definition 3.

Def2.

1. $Ext(qqual) = Qual = \{2^i\}_{1 \leq i \leq 9}$;
2. $Rel_\alpha = \begin{cases} \{3^3, 3^9\} & \text{if } \alpha = 2, \\ \{3^{\pi^{(\alpha)}}\} & \text{if } 2 < \alpha \leq \nu, \\ \{3^{\pi^{(\alpha)}}, 3^{\pi^{(\alpha+1)}}\} & \text{if } \alpha > \nu, \end{cases}$

   Rel_α is the set of all the α – ary relations in the model;
3. $Ext(qrelb) = Rel_2$, $Ext(qrelt) = Rel_3$, $Ext(qrelq) = Rel_4$,

   $Ext(qrel) = Rel = \bigcup_{\alpha \geq 2} Rel_\alpha$;
4. $Ext(qops) = Op_2 = \{5^{3^i}\}_{1 \leq i \leq 6}$, $Ext(qopb) = Op_3 = \{5^{7^i}\}_{1 \leq i \leq 5}$,

   $Ext(qop) = Op = Op_2 \cup Op_3$,

   Op is the set of all the operations in the model
5. $Ext(coll) = Coll = \{7^{2i}\}_{2 \leq i \leq 7} \cup \{7^{\pi^{(\mu)}}\}_{\mu \in N}$;
6. $Ext(\mathfrak{I}(\mathbf{N})) = Nat = \{11^i\}_{1 \leq i}$;
7. $Ext(syf) = Syf = \{13^i\}_{1 \leq i}$.

Now we give some notations. In 3.1 we codify pairs, the number defined in 3.2 represents the system associating the value $\beta^{(i)}$ to the index $\alpha^{(i)}$. In 3.3 we give a notation for singular subsystems of a given system; in 3.4 we give a notation (resembling the usual one) for numbers representing «inner tuples».

Def3. 1. For all natural numbers α and β

$$[\alpha,\beta]=\frac{(\alpha+\beta)(\alpha+\beta+1)}{2}+\alpha\,;$$

2. For every finite set of pairs of natural numbers

$$\left\langle\begin{matrix}\alpha^{(1)}\dots\alpha^{(\mu)}\\ \beta^{(1)}\dots\beta^{(\mu)}\end{matrix}\right\rangle=13^{2^{[\alpha^{(1)},\beta^{(1)}]}+\dots+2^{[\alpha^{(\mu)},\beta^{(\mu)}]}+1}\,;$$

3. If $\sigma\in\mathrm{Syf}$, and $\sigma=\left\langle\begin{matrix}\alpha^{(1)}\dots\alpha^{(\mu)}\\ \beta^{(1)}\dots\beta^{(\mu)}\end{matrix}\right\rangle$ then

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\sqsubseteq\sigma \quad\text{stands for}\quad \exists i(\alpha=\alpha^{(i)}\wedge\beta=\beta^{(i)})\,.$$

4. For every finite sequence of natural numbers

$$\langle\alpha^{(1)}\dots\alpha^{(\mu)}\rangle \quad\text{stands for}\quad \left\langle\begin{matrix}11^{2}\dots11^{(\mu+1)}\\ \alpha^{(1)}\dots\alpha^{(\mu)}\end{matrix}\right\rangle.$$

The remaining extensions are given in the following definition 4. We divide the definition into four parts: 4.1 deals with the remaining inner objects having arity equal to one, while 4.2, 4.3, 4.4 deal with the extensions for objects having arities two, three, four respectively. For simplicity we make some preliminary remarks. The condition defining *Ext*(*syuf*) in 4.1 simply states that the system is univalent, *i.e.* it associates only one value to each of its indexes. The condition defining $Ext(7^{\pi^{(\mu)}})$ establishes that these numbers correspond in our model to collection of μ-tuples (namely $\mathbf{V}^{11^{\mu+1}}$). In 4.3.2, as *cys* is the interpretation of composition of systems, the clause defining its extension states that σ is the composition of $\sigma^{(1)}$ with $\sigma^{(2)}$.

Def4. 1. $Ext(V)=\mathrm{N}$, $Ext(\emptyset)=\emptyset$,

$$Ext(syuf)=\left\{\left\langle\begin{matrix}\alpha^{(1)}\dots\alpha^{(\mu)}\\ \beta^{(1)}\dots\beta^{(\mu)}\end{matrix}\right\rangle:\right.$$

$$\left.\forall i\,\forall j(1\leqslant i\leqslant\mu\wedge1\leqslant j\leqslant\mu\wedge\alpha^{(i)}=\alpha^{(j)}\rightarrow\beta^{(i)}=\beta^{(j)})\right\};$$

$$Ext(7^{\pi^{(\mu)}})=\{\langle\beta^{(1)}\dots\beta^{(\mu)}\rangle:\ \forall i(1\leqslant i\leqslant\mu\rightarrow\beta^{(i)}\in\Im(\mathbf{N})\}\,.$$

Now, thanks to definitions 2 and 4.1 we have defined all the extensions of inner objects in *Qual* ∪ *Coll*. Hence we can give the extension of the interpretation of the first fundamental relation **rfun**. In the same way we give the extensions for *rfbin* and *rfter* in 4.3.1 and 4.4 respectively.

2.1. $$Ext(rfun) = \bigcup_{\alpha \in Qual \cup Coll} \{\alpha\} \times Ext(\alpha);$$

2.2. $$Ext(nord) = \{(11^i, 11^j)\}_{1 \leq i \leq j};$$

2.3. $$Ext\left(\left\langle\begin{matrix}\alpha^{(1)} \dots \alpha^{(\mu)} \\ \beta^{(1)} \dots \beta^{(\mu)}\end{matrix}\right\rangle\right) = \{(\alpha^{(i)}, \beta^{(i)}) \colon 1 \leq i \leq \mu\};$$

2.4. $$Ext(id) = \{(\mu, \mu)\}_{\mu \in \mathbb{N}}, \quad Ext(isys) = \Bigg\{(\sigma^{(1)}, \sigma^{(2)}) \colon \sigma^{(1)} \in Syf \wedge \sigma^{(2)} \in Syf \wedge$$

$$\exists \alpha^{(1)} \dots \beta^{(\mu)} \left(\sigma^{(1)} = \left\langle\begin{matrix}\alpha^{(1)} \dots \alpha^{(\mu)} \\ \beta^{(1)} \dots \beta^{(\mu)}\end{matrix}\right\rangle \wedge \sigma^{(2)} = \left\langle\begin{matrix}\beta^{(1)} \dots \beta^{(\mu)} \\ \alpha^{(1)} \dots \alpha^{(\mu)}\end{matrix}\right\rangle\right)\Bigg\},$$

$$Ext(carsys) = \left\{(\sigma, 11^{\mu+1}) \colon \exists \alpha^{(1)} \dots \alpha^{(\mu)} \exists \beta^{(1)} \dots \beta^{(\mu)} \sigma = \left\langle\begin{matrix}\alpha^{(1)} \dots \alpha^{(\mu)} \\ \beta^{(1)} \dots \beta^{(\mu)}\end{matrix}\right\rangle\right\},$$

$$Ext(ar) = Nat \times \{11\} \cup (Qual \cup Coll) \times \{11^2\} \cup (Syf \cup Op_2) \times \{11^3\} \cup$$

$$Op_3 \times \{11^4\} \cup \bigcup_{\mu \geq 2} Rel_\mu \times \{11^{\mu+1}\},$$

2.5. $$Ext(rfond) = \{(11^{\mu+1}, 3^{\pi(\mu+1)})\}_{\mu \geq 1}, \quad Ext(runiv) = \{(11^{\mu+1}, 3^{\pi(\mu+1)})\}_{\mu \geq \nu};$$

3.1. $$Ext(rfbin) = \bigcup_{\alpha \in Rel_2 \cup Op_2 \cup Syf} \{\alpha\} \times Ext(\alpha);$$

3.2. $$Ext(nadd) = \{(11^{\alpha+1}, 11^{\beta+1}, 11^{\alpha+\beta+1})\}_{\substack{\alpha \in \mathbb{N} \\ \beta \in \mathbb{N}}}$$

$$Ext(nmul) = \{(11^{\alpha+1}, 11^{\beta+1}, 11^{\alpha\beta+1})\}_{\substack{\alpha \in \mathbb{N} \\ \beta \in \mathbb{N}}}$$

$$Ext(nsub) = \{(11^{\alpha+\beta+1}, 11^{\beta+1}, 11^{\alpha+1})\}_{\substack{\alpha \in \mathbb{N}, \\ \beta \in \mathbb{N}}}$$

$$Ext(csys) = \left\{ (\sigma^{(1)}, \sigma^{(2)}, \sigma) \colon \ \forall\alpha\forall\beta\left(\binom{\alpha}{\beta} \sqsubseteq \sigma \leftrightarrow \exists\gamma\left(\binom{\alpha}{\gamma} \sqsubseteq \sigma^{(2)} \wedge \binom{\gamma}{\beta} \sqsubseteq \sigma^{(1)}\right)\right)\right\},$$

$$Ext(usys) = \left\{ (\sigma^{(1)}, \sigma^{(2)}, \sigma) \colon \ \forall\alpha\forall\beta\left(\binom{\alpha}{\beta} \sqsubseteq \sigma \leftrightarrow \binom{\alpha}{\beta} \sqsubseteq \sigma^{(1)} \vee \binom{\alpha}{\beta} \sqsubseteq \sigma^{(2)}\right)\right\};$$

4. $$Ext(rfter) = \bigcup_{\alpha \in Rel_3 \cup Op_3} \{\alpha\} \times Ext(\alpha).$$

This concludes the definition of the interpretation of **P**. The partial structure with universe N and the given interpretation of the symbols for constants and of **P** verifies the groups of axioms from Af to Df given in section 2. This can be proven in a straightforward way.

At this point one can easily get a model of the natural but infinite theory given by the groups of axioms Af-Ef. Let

$$P_1 = \bigcup_{\alpha \in Qual \cup Coll} \{\alpha\} \times Ext(\alpha),$$

$$P_2 = \bigcup_{\alpha \in Rel_2 \cup Op_2 \cup Syf} \{\alpha\} \times Ext(\alpha),$$

$$P_3 = \bigcup_{\alpha \in Rel_3 \cup Op_3} \{\alpha\} \times Ext(\alpha),$$

then to get the model with domain N of the axioms from Af to Ef the interpretations of symbols for consatnts are the same given for $\mathfrak{I}$, while (up to an identification between $\mu + 1$-tuple and pairs with second component a μ-tuple) the interpretations of the symbols for predicates are $\mathfrak{I}'(\mathbf{P}_\mu) = P_\mu$, if $1 \leqslant \mu \leqslant 3$, and $\mathfrak{I}'(\mathbf{P}_{\mu+1}) = \{3^{\pi(\mu+1)}\} \times \mathfrak{I}'(\mathbf{P}_\mu)$, if $\mu \geqslant 3$.

Now we must define the interpretation for the predicate **Q**. This is the predicate that allows the self description of the theory. In fact the interpretation is quite similar to the weaker theory with an infinite numbers of symbols for predicate given in section 3. It is the union of a sequence of partial interpretations. These are recursively built (starting from P_1, P_2 and P_3) by adding at step μ the graphs, restricted to the previous levels, both of the μ^{th} fundamental relation **rfond**(μ) and (for $\mu \geqslant \nu$) of all the universal relations **runiv**(ν) … **runiv**(μ).

Then the interpretation of the symbol **Q** by $\mathfrak{I}$ is

Def5. 1. $\mathfrak{I}(\mathbf{Q}) = \bigcup_{\mu \in \mathbb{N}} Q_\mu$;

2. $Q_1 = P_1$, $Q_2 = \{ (\alpha, \langle \beta, \gamma \rangle) : (\alpha, (\beta, \gamma)) \in P_2 \}$,

$Q_3 = \{ (\alpha, \langle \beta, \gamma, \delta \rangle) : (\alpha, (\beta, \gamma, \delta)) \in P_3 \}$;

3. $Q_{\mu+1} = \{ (3^{\pi^{(\lambda)}}, \langle \alpha^{(1)}, \dots, \alpha^{(\lambda)} \rangle)$:

$3 < \lambda \leq \mu + 1,\ (\alpha^{(1)}, \langle \alpha^{(2)}, \dots, \alpha^{(\lambda)} \rangle) \in \bigcup_{\kappa \leq \mu} Q_\kappa \}$,

$3 \leq \mu < \nu$;

4. $Q_{\mu+1} = \{ (3^{\pi^{(\lambda)}}, \langle \alpha^{(1)}, \dots, \alpha^{(\lambda)} \rangle)$:

$3 < \lambda \leq \mu + 1,\ (\alpha^{(1)}, \langle \alpha^{(2)}, \dots, \alpha^{(\lambda)} \rangle) \in \bigcup_{\kappa \leq \mu} Q_\kappa \} \cup$

$\{ (3^{\pi^{(\lambda)}}, \langle 11^2 \dots 11^\lambda, \langle \alpha, \beta \rangle \rangle) : \nu < \lambda \leq \mu + 1,\ (\alpha, \beta) \in \bigcup_{\kappa \leq \mu} Q_\kappa \}$,

$\nu \leq \mu$.

This concludes the definition of the model. Note that

$$Q_{\mu+1} = \{ (3^{\pi^{(\mu+1)}}, \langle 3^{\pi^{(\mu)}}, \dots 3^{\pi^{(4)}}, \alpha, \beta, \gamma, \delta \rangle) : (\alpha, (\beta, \gamma, \delta)) \in P_3 \}, \quad 3 \leq \mu < \nu .$$

The first order structure $(\mathbb{N}, \mathfrak{I})$, that we denote in what follows with $\mathfrak{N}$, is indeed a model af the axioms of the groups Af-Df, Ff and Gf. We only sketch the proof that $\mathfrak{N}$ satisfies the crucial axioms Gf2.4 and Gf2.5. To this aim it is sufficient to show that for all $\lambda > \nu$ the following holds in $\mathfrak{N}$:

$$\mathbf{Q}(\mathbf{runiv}(\lambda - 1), (1, \dots, \lambda - 1, (\alpha, \beta))) \leftrightarrow \mathbf{Q}(\alpha, \beta) .$$

Now, for the very definition 5.4 of $\mathfrak{I}(\mathbf{Q})$, for every $\mu \geq \lambda - 1$ we have

if $(3^{\pi^{(\lambda)}}, \langle 11^2, \dots 11^\lambda, \langle \alpha, \beta \rangle \rangle) \in Q_{\mu+1}$ then there is $\kappa \in \mathbb{N}$ s.t. $(\alpha, \beta) \in Q_\kappa$.

whence the left to right implication holds good in $\mathfrak{N}$. The converse implication also holds, since: if $(\alpha, \beta) \in Q_\kappa$, for some $\kappa \in \mathbb{N}$, then taking any $\mu \geq \max\{\kappa, \lambda - 1\}$ we have that $(3^{\pi^{(\lambda)}}, \langle 11^2, \dots 11^\lambda, \langle \alpha, \beta \rangle \rangle) \in Q_{\mu+1}$.

It is worthwhile to notice that $\mathfrak{N}$ is minimal among the models of the theory having enough «atoms», in the sense that for every such other model there is a homomorphism from $\mathfrak{N}$ to it. In general the homomorphism is not injective: for example it is consistent to assume **qqual** = **coll**.

Appendix

A formalization with only one binary symbol for predicate.

Trying to get a finite formalization with less symbols for predicates, having also lower arity, it seems natural to define natural numbers other than 0 and 1. This allows a sort of coding of several predicates in terms of one. Here we show how to get a formalization with only one symbol, named **R**, for a binary predicate. Plainly the main problem was to define the predicate «$\alpha = (\beta, \gamma)$» identifying pairs, which is ternary, by means of a binary one. This is the reason why this machinery is set up. This predicate $R_{\mathbf{P},\mathbf{Q}}(\alpha, \beta)$ can be defined into the formal theory $\mathrm{TB}_v^{\mathbf{P},\mathbf{Q}}$ by a formula with the following intuitive meaning:

i) $R_{\mathbf{P},\mathbf{Q}}(\alpha, \mathbf{N})$ means that $\alpha \in \mathbf{N}$.

ii) If $\alpha \in \mathbf{N}$ and $\beta \in \mathbf{N}$, then $R_{\mathbf{P},\mathbf{Q}}(\alpha, \beta)$ means that $\alpha < \beta$.

In the following clauses suppose that $\alpha \notin \mathbf{N}$.

iii) $R_{\mathbf{P},\mathbf{Q}}(\alpha, 1)$ means that α is a one-tuple $[\beta]$. In this case $R_{\mathbf{P},\mathbf{Q}}(\beta, \alpha)$ holds.

iv) $R_{\mathbf{P},\mathbf{Q}}(\alpha, 2)$ means that α is the system $\binom{2}{\beta}$. In this case $R_{\mathbf{P},\mathbf{Q}}(\beta, \alpha)$ holds.

v) $R_{\mathbf{P},\mathbf{Q}}(\alpha, 3)$ means that α is a pair.

vi) $R_{\mathbf{P},\mathbf{Q}}(\alpha, (\beta, \gamma))$ means that α is either the system $\binom{1}{\beta}$ or the system $\binom{2}{\gamma}$.

vii) $R_{\mathbf{P},\mathbf{Q}}(\alpha, 4)$ means that α is a pair $(\delta_1, (\delta_2, (\delta_3, (i, j))))$ with $\mathbf{P}(\delta_1, \delta_2, \delta_3, i, j)$.

viii) $R_{\mathbf{P},\mathbf{Q}}(\alpha, 5)$ means that $a = (i, j)$ for some i, j such that $\mathbf{Q}(i, j)$.

Then the formal definition of $R_{\mathbf{P},\mathbf{Q}}$ in $\mathrm{TB}_v^{\mathbf{P},\mathbf{Q}}$ is:

DefH1. $\mathrm{R}_{\mathbf{P},\mathbf{Q}}(a, b)$ stands for

$$(a \in \mathbf{N} \wedge b = \mathbf{N}) \vee (a \in \mathbf{N} \wedge b \in \mathbf{N} \wedge a < b) \vee$$

$$\exists x(a = [x] \wedge b = 1) \vee \exists x\left(a = \binom{2}{x} \wedge b = 2\right) \vee$$

$$b = [a] \vee b = \binom{2}{a} \vee$$

$$\exists x\, \exists y(a = (x, y) \wedge b = 3) \vee$$

$$\exists x\, \exists y((a = [x] \wedge b = (x, y)) \vee \left(a = \binom{2}{y} \wedge b = (x, y)\right) \vee$$

$$\exists x\, \exists y\, \exists z\, \exists u\, \exists v\big(a = (x, (y, (z, (u, v))))\big) \wedge \mathbf{P}(x, y, z, u, v) \wedge b = 4) \vee$$

$$\exists x\, \exists y(a = (x, y) \wedge \mathbf{Q}(x, y) \wedge b = 5).$$

Conversely the intuitive meaning of this binary predicate allows the definition of both **P** and **Q** in terms of it, see definition H2. Hence the whole theory can be formalized with the previously introduced 33 symbols for constants and only one symbol for a binary predicate. This new language is named $\mathcal{L}'$.

R is a symbol for a binary predicate;

the symbols for constants are all and only those previously introduced.

The axioms from Hf1 to Hf4 give to **R** the structure to code ordered pairs and axiom Hf5 makes $R_{P,Q}$ a faithful translation of **R** in $\mathcal{L}$.
The first two axioms correspond to the clauses i) and ii). In the first axiom $\mathbf{R}(n, \mathbf{N})$ means that n is an element of **N**. Hence the axiom states that **0** and **1** are the first two natural numbers.

Hf1. 1. $\mathbf{R}(\mathbf{0}, \mathbf{N}) \land \forall n(\mathbf{R}(n, \mathbf{N}) \to \mathbf{R}(\mathbf{0}, n) \lor n = \mathbf{0})$;

2. $\mathbf{R}(\mathbf{1}, \mathbf{N}) \land \forall n(\mathbf{R}(n, \mathbf{N}) \land \mathbf{R}(\mathbf{0}, n) \to \mathbf{R}(\mathbf{1}, n) \lor n = \mathbf{1})$.

Keeping in mind the above observations we fix the following notations.

NH1. 1. $\alpha \in_{\mathbf{R}} \mathbf{N}$ stands for $\mathbf{R}(\alpha, \mathbf{N})$ and $\alpha \notin_{\mathbf{R}} \mathbf{N}$ stands for $\neg \mathbf{R}(\alpha, \mathbf{N})$

2 $\alpha <_{\mathbf{R}} \beta$ stands for $\mathbf{R}(\alpha, \beta) \land \mathbf{R}(\alpha, \mathbf{N}) \land \mathbf{R}(\beta, \mathbf{N})$.

The second axiom simply states that $<_{\mathbf{R}}$ is strict, linear ordering in **N** such that every element has an immediate successor.

Hf2. 1. $\forall n \, \forall m \, \forall n' \, (n \in_{\mathbf{R}} \mathbf{N} \land m \in_{\mathbf{R}} \mathbf{N} \land n' \in_{\mathbf{R}} \mathbf{N} \to \neg (n <_{\mathbf{R}} n) \land$
$(n <_{\mathbf{R}} m \land m <_{\mathbf{R}} n' \to n <_{\mathbf{R}} n') \land (n <_{\mathbf{R}} m \lor m <_{\mathbf{R}} n \lor m = n))$;

2. $\forall n(n \in_{\mathbf{R}} \mathbf{N} \to \exists n' \, (n <_{\mathbf{R}} n' \land$
$\forall m(n <_{\mathbf{R}} m \to n' <_{\mathbf{R}} m \lor n' = m).))$

It is useful to introduce four defined constants with the following notation

NH2. If $\mu <_{\mathbf{R}} \mu' \land \forall m(\mu <_{\mathbf{R}} m \to \mu' <_{\mathbf{R}} m \lor \mu' = m)$ then μ' is called *immediate successor* of μ.

The immediate successor of **1** will be denoted by $\mathbf{e}^{(2)}$.

Analogous meaning have $\mathbf{e}^{(3)}$, $\mathbf{e}^{(4)}$, $\mathbf{e}^{(5)}$.

The next axiom corresponds to the clauses iii) and iv). It states that for each object β there are the singular systems $\binom{1}{\beta}$ and $\binom{2}{\beta}$.

Hf3. 1. $\forall x \, \exists! \, y(\mathbf{R}(y, \mathbf{1}) \land y \notin_{\mathbf{R}} \mathbf{N} \land \mathbf{R}(x, y))$;

2. $\forall x \, \exists! \, y(\mathbf{R}(y, \mathbf{e}^{(2)}) \land y \notin_{\mathbf{R}} \mathbf{N} \land \mathbf{R}(x, y))$.

Then, keeping in mind the intended meaning of the axiom, a natural notation is

NH3. 1. $\alpha = \binom{1}{\beta}_{\mathbf{R}}$ stands for $\mathbf{R}(\alpha, 1) \wedge \alpha \notin_{\mathbf{R}} \mathbf{N} \wedge \mathbf{R}(\beta, \alpha)$;

2. $\alpha = \binom{2}{\beta}_{\mathbf{R}}$ stands for $\mathbf{R}(\alpha, \mathbf{c}^{(2)}) \wedge a \notin_{\mathbf{R}} \mathbf{N} \wedge \mathbf{R}(\beta, \alpha)$.

The fourth axiom corresponds to the clauses v) and vi). It states that there exist all the pairs of objects.

Hf4. $\forall x\, \forall y\, \exists !\, z\left(\mathbf{R}(z, \mathbf{c}^{(3)}) \wedge z \notin_{\mathbf{R}} \mathbf{N} \wedge \mathbf{R}\left(\binom{1}{x}_{\mathbf{R}}, z\right) \wedge \mathbf{R}\left(\binom{2}{y}_{\mathbf{R}}, z\right)\right)$.

The last axiom allows to identify two ordered objects by means of their pair, hence we put

NH4. $\alpha = (\beta, \gamma)_{\mathbf{R}}$ stands for $\mathbf{R}(\alpha, \mathbf{c}^{(3)}) \wedge \mathbf{R}\left(\binom{1}{\beta}_{\mathbf{R}}, \alpha\right) \wedge \mathbf{R}\left(\binom{2}{\gamma}_{\mathbf{R}}, \alpha\right)$.

At this point we can define in the language of **R** the translations of the predicates understood as **P** and **Q** by means of two formulae $P_{\mathbf{R}}$ and $Q_{\mathbf{R}}$, corresponding to the clauses vii) and viii).

DefH2. 1. $P_{\mathbf{R}}(a, b, c, d, e)$ stands for $\mathbf{R}\left(\left(a, (b, (c, (d, e)_{\mathbf{R}})_{\mathbf{R}})_{\mathbf{R}}\right)_{\mathbf{R}}, \mathbf{c}^{(4)}\right)$;

2. $Q_{\mathbf{R}}(a, b)$ stands for $\mathbf{R}((a, b)_{\mathbf{R}}, \mathbf{c}^{(5)})$.

The above definitions determine two correspondences between the formulae and the terms of the language $\mathcal{L}$ of $\mathrm{TB}_{\nu}^{\mathrm{P,Q}}$ and the formulae and the terms of the new language $\mathcal{L}'$. These correspondences keep fixed any symbol for constants and any variable. The first one transforms every formula Φ of the language $\mathcal{L}$ in

$$\Phi^{1} = \begin{cases} P_{\mathbf{R}}[\alpha/a, \beta/b, \gamma/c, \delta/d, \varepsilon/e] & \text{if } \Phi \equiv \mathbf{P}(\alpha, \beta, \gamma, \delta, \varepsilon) \text{ and } \alpha, \beta, \gamma, \delta, \varepsilon \\ & \text{are variables or symbols for constants}, \\ Q_{\mathbf{R}}[\alpha/a, \beta/b] & \text{if } \Phi \equiv \mathbf{Q}(\alpha, \beta) \text{ and } \alpha,\ \beta \\ & \text{are variables or symbols for constants}, \\ \neg \Psi^{1} & \text{if } \Phi \equiv \neg \Psi, \\ \Theta^{1} \wedge \Psi^{1} & \text{if } \Phi \equiv \Theta \wedge \Psi, \\ \Theta^{1} \vee \Psi^{1} & \text{if } \Phi \equiv \Theta \vee \Psi, \\ \forall x \Psi^{1} & \text{if } \Phi \equiv \forall x \Psi, \\ \exists x\ \Psi^{1} & \text{if } \Phi \equiv \exists x\ \Psi. \end{cases}$$

The second one transforms every formula Φ of the new language $\mathcal{L}'$ in

$$\Phi^{J} = \begin{cases} R_{P,Q}[\alpha/a, \beta/b] & \text{if } \Phi \equiv \mathbf{R}(\alpha, \beta) \text{ and } \alpha,\ \beta \\ & \text{are variables or symbols for constants,} \\ \neg \Psi^{J} & \text{if } \Phi \equiv \neg \Psi, \\ \Theta^{J} \wedge \Psi^{J} & \text{if } \Phi \equiv \Theta \wedge \Psi, \\ \Theta^{J} \vee \Psi^{J} & \text{if } \Phi \equiv \Theta \vee \Psi, \\ \forall_a \Psi^{J} & \text{if } \Phi \equiv \forall_a\ \Psi, \\ \exists_a \Psi^{J} & \text{if } \Phi \equiv \exists_a \Psi. \end{cases}$$

Then the last *structural* axiom for **R** is:

Hf5. $\forall x\, \forall y (\mathbf{R}(x,y) \leftrightarrow (\mathbf{R}(x,y)^{J})^{I})$.

According to these definitions for any set of formulae X in one of the two languages, X^{I} or X^{J} denotes the set of the translated formulae.

The axioms Hf1, Hf2, Hf3, Hf4, Hf5 give the *basic structure* of **R**. Their theory is named $S^{\mathbf{R}}$.

To complete the formalization with **R** of TB_{ν}, one needs to put as axioms also all the formulae

$$(TB_{\nu}^{P,Q})^{I}.$$

The theory $(TB_{\nu}^{P,Q})^{I} \cup S^{\mathbf{R}}$ is named $TB_{\nu}^{\mathbf{R}}$.

The following theorem states in what acception this theory can be considered a reformalization of $TB_{\nu}^{P,Q}$ in the language $\mathcal{L}'$.

THEOREM: For every formula Φ of $\mathcal{L}$, and for every formula Ψ of $\mathcal{L}'$, one has

1. $TB_{\nu}^{P,Q} \vdash (S^{\mathbf{R}})^{J}$;

2. $S^{\mathbf{R}} \vdash \Psi \leftrightarrow \Psi^{JI}$;

3. $TB_{\nu}^{P,Q} \vdash \Phi \leftrightarrow \Phi^{IJ}$.

In particular one has $TB_{\nu}^{\mathbf{R}} \vdash \Psi \leftrightarrow \Psi^{JI}$.

List of the constants

qqual: for the *quality of being a quality*;
rfun: for the *fundamental relation of the unary objects*;
rbin: for the *fundamental relation of the binary objects*;
rfter: for the *fundamental relation of the ternary objects*;
qrelb: for the *quality of being a binary relation*;
qrelt: for the *quality of being a ternary relation*;
qrelq: for the *quality of being a quaternary relation*;
qops: for the *quality of being a simple operation*;
qopb: for the *quality of being a binary operation*;
id: for the operation of *identity*;
qcoll: for the *quality of being a collection*;
V: for the *universal collection*;
coll: for the *collection of all collections*;
∅: for the *void collection*.
N: for the *collection of natural numbers*;
nadd: for the binary operation of *sum*;
nmul: for the binary operation of *multiplication*;
nsub: for the binary operation of *subtraction*;
nord: for the binary relation of *order between natural numbers*;
0: for the number *zero*;
1: for the number *one*.
syf: for the *collection of finite systems*;
syuf: for the *collection of univalent finite systems*;
usys: for the operation of *binary union* between systems;
csys: for the operation of *composition* of systems;
isys: for the operation of *inversion* of systems;
$\emptyset_2$: for the *empty system*;
carsys: for the operation of *cardinality* for systems.
ar: for the *operation giving the arity*
qrel: for the *quality of being a relation*
qop: for the *quality of being an operation*
rfond: for the *operation generating fundamental relations*;
runiv: for the *operation generating universal relations*.

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