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Nonexpansive Type Mappings and a Fixed Point Theorem in Convex Metric Spaces(**)

ABSTRACT. — Let K be a nonempty closed convex subset of a complete convex metric space (X, d) and let T be a mapping of K into itself. The main result of this paper is the following: if there are nonnegative real numbers a, b, c, with $0 < b < 1$ and $a + b + 2c \leq 1$, such that the inequality

$$d(Tx, Ty) \leq a\, d(x, y) + b \max\{ d(x, Tx),\ d(y, Ty) \} + c\left[d(x, Ty) + d(y, Tx)\right]$$

holds for all x, y in K, then T has a unique fixed point, and at this point T is continuous. This result generalizes and extends previuous results of Greguš [8], Delbosco et al. [3] and Li [9]. An example is given to show that our theorem is a strict generalization of many known results.

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Un teorema di punto fisso per applicazioni di tipo non espansivo in uno spazio metrico convesso

RIASSUNTO. — Sia K un insieme chiuso, convesso e non vuoto in uno spazio metrico completo (X, d) munito di una struttura convessa, e sia T un'applicazione di K in sé. Il risultato principale del presente articolo è il seguente: se esiste una terna a, b, c di numeri reali non negativi, con $0 < b < 1$, $a + b + 2c \leq 1$, tale che, per ogni coppia x, y di elementi di K, valga la diseguaglianza

$$d(Tx, Ty) \leq a\, d(x, y) + b \max\{ d(x, Tx),\ d(y, Ty) \} + c\left[d(x, Ty) + d(y, Tx)\right],$$

allora l'applicazione T possiede un unico punto fisso, ed è continua in questo punto. Si estendono così precedenti risultati ottenuti da Greguš [8], da Delbosco et al. [3] e da Li [9]. Si prova poi con un esempio che il nostro teorema è un'effettiva generalizzazione di molti risultati noti.

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0. - Introduction

Let (X, d) be a metric space, T a mapping of X into itself, and k a nonnegative real number such that the inequality $d(Tx, Ty) \leq k\, d(x, y)$ holds for all x, y in X. If $k < 1$, then T is said to be a *contraction mapping*; if $k = 1$, then T is said to be a *nonexpansive mapping*. The well known Banach's contraction principle — already obtained in particular situations by Liouville, Picard and Goursat — states that if X is complete, then every contraction mapping T has a unique fixed point, which is the limit of $T^n x$, the n-iteration of T applied to any point x of X. However, a nonexpansive mapping may not have fixed points. Yet these mappings have a fixed point when X has a convex structure. There exists a very abundant literature about contractive and nonexpansive type mappings, where the contractive and nonexpansive conditions are replaced by more general conditions.

Let X be a Banach space and C a nonempty closed convex subset of X. Generalizing the fixed point theorem of Greguš [8], Delbosco, Ferrero and Rossati proved the following result:

Theorem A (Delbosco et al. [3]): *Let $T\colon C \to C$ be a mapping satisfying*

$$(1) \qquad \|Tx - Ty\| \leq a\,\|x - y\| + b\left[\|Tx - x\| + \|Ty - y\|\right] + c\left[\|Tx - y\| + \|Ty - x\|\right]$$

for all x, y in C, where a, b, c are nonnegative real numbers such that

$$(1) \qquad 0 < a < 1, \qquad b \neq c, \qquad b \geq (1 - a^2)/(2 + 6a),$$

$$(2) \qquad a + 2b + 2c = 1.$$

Then T has a unique fixed point.

Many results which are closely related to the theorem of Greguš have been published recently ([2-5], [7-10]).

The purpose of this note is to introduce and investigate a class of mappings which are more general than those considered in Theorem A. Moreover, we shall replace the Banach space X by a convex metric space. In this more general context we shall prove a fixed point theorem, which extends Theorem A, as well as the theorems of Li [9] and Greguš [8]. We shall consider mappings T of a metric space (X, d) into itself (not necessarily continuous) satisfying the following contractive definition:

$$(4) \qquad d(Tx, Ty) \leq a\, d(x, y) + b \max\{\, d(x, Tx), d(y, Ty)\} + c\left[d(x, Ty) + d(y, Tx)\right],$$

where a, b, c are nonnegative real numbers such that

$$(5) \qquad 0 < b < 1,$$

$$(6) \qquad a + b + 2c \leq 1.$$

We point out that the inequality (1) implies the inequality (4) with b replaced by $2b$.

Moreover, (2) and (3) imply (5) and (6) (yet with b replaced by $2b$). So our result is a twofold generalization of Theorem A. An example is given to show that, in fact, our theorem strictly generalizes Theorem A as well as the results of Greguš [8] and Li [9].

1. - Main result

We shall use the following definition of a convex metric space.

Definition 1.1 (Takahashi [11]): Let (X, d) be a metric space and $I = [0, 1]$ the closed unit interval. A continuous mapping $W: X \times X \times I \to X$ is said to be a *convex structure* on X if the inequality

$$d[u, W(x, y, \lambda)] \le \lambda d(u, x) + (1 - \lambda) d(u, y)$$

holds for all x, y, u in X and λ in I. The metric space X together with a convex structure is called a *convex metric space*. A subset K of X is *convex* if $W(x, y, \lambda) \in K$ for all x, y in K and λ in I.

Clearly a Banach space, or any convex subset of it, is a convex metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda) y$. More generally, if X is a linear space with a translation invariant metric d satisfying $d(\lambda x + (1 - \lambda) y, 0) \le \lambda d(x, 0) + (1 - \lambda) d(y, 0)$, then X is a convex metric space. There are many other examples, but we consider these as paradigmatic.

Before stating and proving a fixed point theorem for mappings which satisfy (4), we shall prove the following Lemma, which is of interest also in its own right.

Lemma 1.1: *Let K be a nonempty convex subset of a convex metric space (X, d) and T a (not necessarily continuous) mapping of K into itself. If T satisfies the inequality (4) for all x, y in K, where the nonnegative coefficients a, b, c satisfy (5) and (6), then*

$$\inf \{d(x, Tx) : x \in K\} = 0 . \tag{7}$$

Proof: If (6) holds with the strict inequality, then (even without the condition (5) and the convexity assumptions concerning X and K) the statement of Lemma follows as a consequence of Theorem 1 of [1]. So we shall assume $a + b + 2c = 1$.

It suffices to show that for any point x_0 in K there exists a point y in K such that

$$d(y, Ty) \le \lambda d(x_0, Tx_0),$$

where λ is defined by

$$\lambda = \begin{cases} 1 - bc & \text{if } c > 0, \\ 1 - \dfrac{1}{4} ab & \text{if } c = 0. \end{cases}$$

Consider the sequence $\{x_n\}$ in K defined by $x_{n+1} = Tx_n$ (for $n = 0, 1, 2, \ldots$), and set

$$r_n = d(x_n, Tx_n), \qquad s_n = d(x_n, Tx_{n+1}).$$

From (4) we have

$$r_n \leq a\, r_{n-1} + b \max\{r_{n-1}, r_n\} + c\, s_{n-1}, \tag{8}$$

$$s_n \leq a\, s_{n-1} + b \max\{r_{n-1}, r_{n+1}\} + c\,[d(x_{n-1}, Tx_{n+1}) + r_n]. \tag{9}$$

Since we have, by the triangle inequality, $s_{n-1} \leq r_{n-1} + r_n$, we get, from (8),

$$r_n \leq a\, r_{n-1} + b \max\{r_{n-1}, r_n\} + c\,(r_{n-1} + r_n).$$

Hence it follows that if $r_{n-1} < r_n$, for some n, then we have

$$r_n < a\, r_n + b\, r_n + 2c\, r_n = r_n,$$

which is a contradiction. Therefore, $r_n \leq r_{n-1}$ for each n, which implies

$$r_n \leq r_0 = d(x_0, Tx_0) \qquad (\text{for } n = 1, 2, \ldots). \tag{10}$$

As we have, by the triangle inequality,

$$d(x_{n-1}, Tx_{n+1}) \leq s_{n-1} + r_{n+1},$$

we get from (9) and (10)

$$s_n \leq a\, s_{n-1} + b\, r_0 + c\,(s_{n-1} + r_{n+1} + r_n),$$

and hence, as $s_{n-1} \leq r_{n-1} + r_n \leq 2r_0$,

$$s_n \leq (2a + b + 4c)\, r_0 = (2 - b)\, r_0 \qquad (\text{for } n = 1, 2, \ldots). \tag{11}$$

We get from (8), (10) and (11)

$$r_n \leq [a + b + c(2 - b)]\, r_0 = (1 - bc)\, r_0 \qquad (\text{for } n = 2, 3, \ldots). \tag{12}$$

If $c > 0$, then we have from (12), for $n = 2$,

$$d(x_2, Tx_2) \leq (1 - bc)\, d(x_0, Tx_0) = \lambda\, d(x_0, Tx_0). \tag{13}$$

Consider now the case $c = 0$. In this case (6) reduces to $a + b = 1$. Set

$$z = W\left(Tx_1,\ Tx_2,\ \frac{1}{2}\right).$$

Since K is convex, $z \in K$. Definition 1.1 and inequalities (10), (11) imply

$$d(x_1, z) \leq \frac{1}{2} d(x_1, Tx_1) + \frac{1}{2} d(x_1, Tx_2) = \frac{1}{2} r_1 + \frac{1}{2} s_1 \leq \frac{1}{2} r_0 + \frac{1}{2}(2 - b)\, r_0, \tag{14}$$

$$d(x_2, z) \leq \frac{1}{2} d(x_2, Tx_2) = \frac{1}{2} r_2 \leq \frac{1}{2} r_0. \tag{15}$$

$$d(Tz, z) \leq \frac{1}{2}\, d(Tz, Tx_1) + \frac{1}{2}\, d(Tz, Tx_2). \tag{16}$$

On the other hand, using (4) (with $c = 0$) and (10), we obtain

$$d(Tz, Tx_j) \leq a\, d(z, x_j) + b \max\{d(z, Tz), r_0\} \qquad (\text{for } j = 1, 2). \tag{17}$$

So, by (16), (17), (14) and (15), we get

$$\begin{aligned} d(z, Tz) &\leq \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}(2-b) + \frac{1}{2}\right] r_0 + b \max\{d(z, Tz), r_0\} = \\ &= \left(a - \frac{1}{4}ab\right) r_0 + b \max\{d(z, Tz), r_0\} \leq \\ &\leq \left(a - \frac{1}{4}ab + b\right) \max\{d(z, Tz), r_0\} = \lambda \max\{d(z, Tz), r_0\}. \end{aligned}$$

Since $\lambda < 1$, this implies

$$d(z, Tz) \leq \lambda r_0 = \lambda\, d(x_0, Tx_0). \tag{18}$$

By (13) and (18) we conclude that in any case there exists a point y in K such that

$$d(y, Ty) \leq \lambda\, d(x_0, Tx_0). \tag{19}$$

This completes the proof.

Theorem 1.1: *Let K be a nonempty closed convex subset of a complete convex metric space (X, d) and T a mapping of K into itself. If a, b, c are nonnegative real numbers such that (5) and (6) hold, and if T satisfies (4) for any x, y in K, then T has a unique fixed point in K and at this point T is continuous. Moreover, if (6) holds with the strict inequality, then the convexity assumptions (for X and K), as well as the condition (5), can be omitted.*

Proof: If (6) holds with the strict inequality, then the statement follows from Theorem 1 of [1] and in this case the condition (5) and the convexity assumptions are superfluous. So we suppose $a + b + 2c = 1$. We shall show that the inequality

$$\max\{d(Tx, Ty), d(x, y)\} \leq \frac{1 + a + 2b}{b} \max\{d(x, Tx), d(y, Ty)\} \tag{20}$$

holds for all x, y in K. Write $M = \max\{d(x, Tx), d(y, Ty)\}$. By the triangle inequality we have

$$d(x, y) \leq d(x, Tx) + d(Tx, Ty) + d(y, Ty) \leq d(Tx, Ty) + 2M, \tag{21}$$

$$d(x, Ty) \leq d(x, Tx) + d(Tx, Ty) \leq M + d(Tx, Ty). \tag{22}$$

Using (4), from (21) and (22) we have

$$d(Tx, Ty) \leq a\,[d(Tx, Ty) + 2M] + bM + 2c\,[M + d(Tx, Ty)]$$

and hence, as (5) and (6) imply $a + 2c = 1 - b < 1$,

$$d(Tx, Ty) \leq \frac{2a + b + 2c}{b} M = \frac{1 + a}{b} M .$$

From this and (21) we get (20).

By Lemma 1.1 we can choose a sequence $\{x_n\}$ in K such that

$$\text{(23)} \qquad d(x_n, Tx_n) \leq 1/n \qquad (\text{for } n = 1, 2, \ldots).$$

We have, from (20) and (23),

$$\max\{d(Tx_m, Tx_n),\ d(x_m, x_n)\} \leq \frac{1 + a + 2b}{bn} \qquad \text{for } 1 \leq n \leq m .$$

Therefore, both $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences in K, and since K is closed and X complete, they converge in K. Moreover, by (23) they have a common limit, say u. From (4) we have

$$d(Tu, Tx_n) \leq a\,d(u, x_n) + b \max\{d(u, Tu),\ d(x_n, Tx_n)\} + c\,[d(u, Tx_n) + d(x_n, Tu)].$$

Passage to the limit as n tends to infinity yields

$$\text{(24)} \qquad d(u, Tu) \leq (b + c)\,d(u, Tu).$$

Since (5) and (6) imply $b + c = 1 - (a + c) < 1$, we have from (24) that $d(u, Tu) = 0$. Hence $Tu = u$. Let v be also fixed point of T. Then we obtain, from $d(u, v) = d(Tu, Tv)$ and (4),

$$d(u, v) \leq (a + 2c)\,d(u, v).$$

Since by (5) and (6) $a + 2c = 1 - b < 1$, we have $d(u, v) = 0$ and so T has a unique fixed point u.

Now let $\{u_n\}$ be a sequence in K with limit u. From (4) we have

$$\begin{aligned} d(u, Tu_n) = d(Tu, Tu_n) &\leq \\ &\leq a\,d(u, u_n) + b\,d(u_n, Tu_n) + c\,[d(u, Tu_n) + d(u_n, u)] \leq \\ &\leq a\,d(u, u_n) + b\,[d(u_n, u) + d(u, Tu_n)] + c\,[d(u, Tu_n) + d(u_n, u)] \end{aligned}$$

and hence, letting n go to infinity, we obtain

$$\limsup_{n\to\infty} d(u, Tu_n) \leq (b + c)\cdot \limsup_{n\to\infty} d(u, Tu_n).$$

As $b+c<1$, the last inequality implies

$$\limsup_{n\to\infty} d(u, Tu_n) = 0,$$

and this means that T is continuous at u. Thus, the proof is complete.

REMARK 1.1: If $c=0$, we obtain the result of Fisher [5]. This result also appears in [2], [4], [6] and [9] as a corollary of common fixed point theorems.

If $c>0$, then in Lemma 1.1, as well as in Theorem 1.1, a simple inspection of the proof suffices to show that the convexity assumptions concerning X and K are superfluous.

REMARK 1.2: If, in our Theorem 1.1, the inequality (6) is replaced by the equality $a+b+2c=1$, then the condition (5) (i.e. $0<b<1$) can not be omitted. Indeed, if $b=1$, then Example 2 of Greguš [8] shows that T may not have fixed points. The following example shows that T also may not have fixed points in the case $b=0$.

EXAMPLE 1.1: Let X be the set of reals (with Euclidean metric) and $K=X$. Define the mapping T of K into itself by $Tx=x+1$. We then have $d(Tx, Ty)=d(x,y)$ and

$$d(x,Ty)+d(y,Tx)=\begin{cases}2d(x,y) & \text{if } d(x,y)\geq 1,\\ 2 & \text{otherwise}.\end{cases}$$

Therefore T satisfies (4) if the nonnegative coefficients a, b, c satisfy $b=0$ and $a+2c=1$. Nevertheless T does not have a fixed point.

COROLLARY 1.1 (Li [9]): *Let K be a nonempty closed convex subset of a convex metric space X and let T be a mapping of K into itself satisfying the inequality*

(25) $d(Tx,Ty)\leq a\,d(x,y)+b\,[d(x,Tx)+d(y,Ty)]+c\,[d(x,Ty)+d(y,Tx)]$

for all x,y in K, where

(26) $0\leq a<1, \quad b\geq 0, \quad c\geq 0, \quad a+c>0,$

(27) $a+2b+3c\leq 1.$

If X has the property that every decreasing sequence of nonempty closed subsets of X with diameters tending to zero has a nonempty intersection, then T has a unique fixed point in K.

PROOF: It is easy to see that (25) implies (4) with b replaced by $2b$. Moreover, in the case $a+2b+2c<1$ our Theorem holds without the condition (5). So it remains to show that in the case $a+2b+2c=1$ we have $0<2b<1$. Since (27) implies that the equality $a+2b+2c=1$ is possible only if $c=0$, we have from (26) $0<a+c=$

$= a < 1$. This and the equality $a + 2b = 1$ imply $0 < 1 - a = 2b < 1$. Since the property of X imposed in Corollary 1.1 is equivalent to the completeness of X, we see that all the assumptions of Theorem 1.1 are satisfied.

Finally, we give a simple example which shows that our Theorem 1.1 is actually an improvement of the results of Delbosco, Ferrero and Rossati [3], Greguš [8], Fisher [6] and Li [9].

EXAMPLE 1.2: Let K be the closed convex subset $[-4, 4]$ of the real line and T the mapping of K into itself defined by

$$Tx = \begin{cases} \frac{1}{6}x & \text{if } x \in [-1, 4], \\ 4 & \text{otherwise}. \end{cases}$$

It is clear that if $x, y \in [-1, 4]$ or $x, y \in [-4, -1[$, then $d(Tx, Ty) \leq \frac{1}{6} d(x, y)$. Let now $x \in [-1, 4]$ and $y \in [-4, -1[$. Then we have

$$d(Tx, Ty) \leq 4 + \frac{1}{6} = \frac{5}{6} \cdot 5 \leq \frac{5}{6} \cdot \max\{d(y, Ty),\ d(x, Tx)\}.$$

Therefore, T satisfies the condition (4) with $a = \frac{1}{6}$, $b = \frac{5}{6}$ and $c = 0$. Since K is compact, hence complete, all the assumptions of Theorem 1.1 are satisfied and $u = 0$ is the unique fixed point of T. But T does not satisfy (1) with a, b, c satisfying (2) and (3). Indeed, for all x in $[-1, 0]$ and y in $[-2, -1[$ we have

$$a\,d(x, y) + b\left[d(x, Tx) + d(y, Ty)\right] + c\left[d(x, Ty) + d(y, Tx)\right] \leq$$

$$\leq (a + 2b + 2c)\max\left\{d(x, y),\ \frac{1}{2}[d(x, Tx) + d(y, Ty)],\ \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\} \leq$$

$$\leq 1 \cdot \max\left\{2,\ \frac{1}{2}\left(\frac{5}{6} + 6\right),\ \frac{1}{2}(5 + 2)\right\} = 3.5 < 4 \leq d(Tx, Ty).$$

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