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Regularization for Some Boundary Integral Equations of the First Kind in Mechanics ^(**)(***)

SUMMARY. — In this paper it is shown how to regularize some boundary integral equations of the first kind in Mechanics. This is done by means of the study of singular integral systems in which the unknown is a vector whose components are scalar functions while the data is a vector whose components are differential forms of degree one.

Regularizzazione di alcune equazioni integrali di prima specie in Meccanica

SUNTO. — In questo lavoro si mostra come regolarizzare alcune equazioni integrali di prima specie che si presentano in Meccanica. Ciò si ottiene mediante lo studio di sistemi di equazioni integrali singolari nelle quali l'incognita è un vettore a componenti funzioni scalari mentre il dato è un vettore a componenti forme differenziali di grado uno.

1. - PRELIMINARY

Let B and B' be Banach spaces. We say that a linear and continuous operator $S: B \rightarrow B'$ can be reduced on the left (on the right) if there exists a linear and continuous operator $S': B' \rightarrow B$ such that $S'S = I + T$ ($SS' = I + T$), where I is the identity and T is a completely continuous operator from B into itself (from B' into itself). One of the main properties of such operators is that the equation $S\phi = \psi$ has a solution if and only if the given data ψ satisfies the compatibility conditions (there may be infinitely many of them) : $\langle \gamma, \psi \rangle = 0$,

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$\forall \gamma \in B'^*$, $S^* \gamma = 0$, B'^* being the topological dual space of B' and $S^*: B'^* \rightarrow B^*$ the adjoint of S (for this and operators alike, see, e.g., [6], [9], [15]).

In what follows, let Ω be a bounded domain in \mathbb{R}^n such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. We shall suppose that $\Sigma = \partial\Omega$ is a Lyapunov boundary, i.e., Σ has a uniformly Hölder continuous normal field, and we denote by ν the outward unit normal. For later use, we need some basic notations for differential forms in \mathbb{R}^n . Following [7], we write

$$u = \frac{1}{k!} u_{i_1 \dots i_k} dx^{i_1} \dots dx^{i_k},$$

for a k -form in \mathbb{R}^n with the coefficient $u_{i_1 \dots i_k}$, and denote by du the differential of u ,

$$du = \frac{1}{k!} \frac{\partial}{\partial x_j} u_{i_1 \dots i_k} dx^j dx^{i_1} \dots dx^{i_k}.$$

(Here and in the sequel, the summation convention is employed). The adjoint of u is designated by $*u$ and is defined by

$$*u = \frac{1}{(n-k)!} u_{i_1 \dots i_{n-k}}^* dx^{i_1} \dots dx^{i_{n-k}},$$

where

$$u_{i_1 \dots i_{n-k}}^* = \frac{1}{k!} \delta_{s_1 \dots s_k i_1 \dots i_{n-k}}^1 \dots^n u_{s_1 \dots s_k}.$$

We also need δu , the co-differential of u ,

$$\delta u = (-1)^{n(k+1)+1} * d * u.$$

We now introduce J , the operator given by the differential of a simple layer potential on Σ :

$$J\phi(x) = \int_{\Sigma} \phi(y) d_x [S(x, y)] d\sigma_y, \quad x \in \Sigma,$$

where $S(x, y)$ is the fundamental solution of the n -dimensional Laplacian Δ_n ,

$$S(x, y) = \frac{1}{(2-n)c_n} |x-y|^{2-n}, \quad n \geq 3$$

with c_n being the hypersurface measure of the unit sphere in \mathbb{R}^n . Then, it can be shown that J is a linear and continuous operator from $L^p(\Sigma)$ into $L_1^p(\Sigma)$ (i.e. the space of differential forms of degree 1 on Σ such that their coefficients are integrable functions belonging to L^p in any admissible local system of coordinates) ($1 < p < \infty$). Moreover, it has been shown in [1] that J can be reduced on the left; that is,

$$(1.1) \quad J' J \phi = -\frac{1}{4} \phi + K^2 \phi,$$

where

$$J' \psi(x) = \underset{\Sigma}{*} \int_{\Sigma} \psi(x) \wedge d_z [S_{n-2}(z, x)], \quad x \in \Sigma,$$

$$S_{n-2}(z, x) = \sum_{j_1 < \dots < j_{n-2}} S(z, x) dz^{j_1} \dots dz^{j_{n-2}} dx^{j_1} \dots dx^{j_{n-2}},$$

$$K\phi(x) = \int_{\Sigma} \phi(y) \frac{\partial}{\partial \nu_x} S(x, y) d\sigma_y, \quad x \in \Sigma.$$

Here $\underset{\Sigma}{*}$ has the following meaning: if λ is a $(n-1)$ -form on Σ , say $\lambda = \lambda_0 d\sigma$ for some scalar function λ_0 , then $\underset{\Sigma}{*} \lambda = \lambda_0$.

REMARK 1: In [3, pp. 253-254] it is shown that

$$(1.2) \quad \mathcal{N}(J'J) = \mathcal{N}(J)$$

(\mathcal{N} = kernel). This condition does not give the so-called *equivalent reduction* (i.e., $\mathcal{N}(J') = \{0\}$), which would imply: $J\phi = \psi$ if and only if $J'J\phi = J'\psi$. Nevertheless, (1.2) still assures a kind of equivalence. In fact, (1.2) implies that: *if ψ is such that there exists at least a solution of $J\phi = \psi$ (and in our case, since J can be reduced, this happens if and only if ψ satisfies the compatibility conditions), then $J\phi = \psi$ if and only if $J'J\phi = J'\psi$* . The proof of this last statement is trivial.

2. - SINGLE- AND DOUBLE-LAYER POTENTIALS

One may apply the operators J and J' to the solutions of boundary integral equations of the first kind arising from boundary-value problems. To illustrate the idea, let us begin with the Dirichlet problem:

$$\Delta_n u = 0 \quad \text{in } \Omega,$$

$$u|_{\Sigma} = f \quad \text{on } \Sigma$$

for given $f \in W^{1,p}(\Sigma)$. If we seek a solution by means of a single layer potential, we then obtain the integral equation of the first kind

$$(2.1) \quad \int_{\Sigma} \phi(y) S(x, y) d\sigma_y = f(x), \quad x \in \Sigma.$$

Taking the differential of both sides of (2.1), we get the following singular integral equation:

$$\int_{\Sigma} \phi(y) d_x [S(x, y)] d\sigma_y = df(x), \quad x \in \Sigma$$

(in which the unknown is a scalar function $\phi \in L^p(\Sigma)$, while the data is a differential form of degree 1, $df \in L^p_1(\Sigma)$). That is, we obtain the equation:

$$(2.2) \quad J\phi = df.$$

Because of (1.1), the operator J can be reduced on the left, and therefore there exists a solution of (2.2) if and only if the data satisfy the relevant compatibility conditions. In [1], it is shown that, if $f \in W^{1,p}(\Sigma)$, then df satisfies these compatibility conditions and therefore there exists a solution $\phi \in L^p(\Sigma)$ of equation (2.2). Because of Remark 1, we see that (2.2) is completely equivalent to the Fredholm integral equation of the second kind:

$$-\frac{1}{4}\phi + K^2\phi = J'(df).$$

Of course, the solution of (2.2) is only uniquely determined up to a function ϕ_0 such that

$$\int_{\Sigma} \phi_0(y) S(x, y) d\sigma_y = \text{const} \quad (x \in \overline{\Omega}).$$

The latter can then be determined by (2.1).

Let us consider now the Neumann problem:

$$\begin{aligned} \Delta_n u &= 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= f & \text{on } \Sigma, \end{aligned}$$

where $f \in L^p(\Sigma)$ satisfies the condition $\int_{\Sigma} f d\sigma = 0$. We want to represent the solution by means of a double layer potential:

$$(2.3) \quad u(x) = \int_{\Sigma} \psi(y) \frac{\partial}{\partial \nu_y} S(x, y) d\sigma_y, \quad x \in \Omega$$

with ψ being sought in $W^{1,p}(\Sigma)$.

First, let us remark that $(\partial u / \partial \nu) d\sigma$ can be considered as the restriction of $\delta * u$ on Σ . On the other hand, we have

$$(2.4) \quad \delta * \int_{\Sigma} \psi(y) \frac{\partial}{\partial \nu_y} S(x, y) d\sigma_y = -d_x \int_{\Sigma} d\psi(y) \wedge S_{n-2}(x, y), \quad x \in \Omega,$$

(see [1][p. 187]). Therefore the boundary condition $\partial u / \partial \nu = f$ can be written as

$$\int_{\Sigma} d\psi(y) \wedge d_x [S_{n-2}(x, y)] = -f d\sigma,$$

i.e.,

$$(2.5) \quad J'(d\psi) = -f$$

(where J' is the operator we have previously introduced in (1.1)). Now if we write

$$(2.6) \quad \psi(x) = \int_{\Sigma} \phi(y) S(x, y) d\sigma_y, \quad x \in \Sigma,$$

then $d\psi = J\phi$ and, keeping in mind (1.1),

$$J'(d\psi) = J'J\phi = -\frac{1}{4}\phi + K^2\phi.$$

This shows that the operator $J' \circ d: W^{1,p}(\Sigma) \rightarrow L^p(\Sigma)$ can be reduced on the right. Then there exists a solution of (2.5) if and only if f satisfies the relevant compatibility conditions.

However, in this case, we can prove that there exists a solution of (2.5) in the following probably easier way: because of the aforementioned results of [1], any function $\psi \in W^{1,p}(\Sigma)$ can be written in the form of (2.6). Then there exists a solution $\psi \in W^{1,p}(\Sigma)$ of (2.5), if and only if there exists a solution $\phi \in L^p(\Sigma)$ of the equation:

$$(2.7) \quad -\frac{1}{4}\phi + K^2\phi = -f.$$

If, as we suppose, $\int_{\Sigma} f d\sigma = 0$, then it is well known that there exists $\gamma \in L^p(\Sigma)$ such that

$$(2.8) \quad -\frac{1}{2}\gamma + K\gamma = -f$$

and moreover, there exists $\phi \in L^p(\Sigma)$ satisfying the equation

$$(2.9) \quad \frac{1}{2}\phi + K\phi = \gamma.$$

Consequently, (2.8) and (2.9) imply (2.7).

We summarize these results in the following theorem:

THEOREM 1: *For any $f \in L^p(\Sigma)$, $\int_{\Sigma} f d\sigma = 0$, the solution of the Neumann problem can be represented in the form of a double layer potential (2.3). Moreover (2.3) is a solution of the Neumann problem if and only if its density ψ is given by (2.6) with ϕ being a solution of the Fredholm equation (2.7).*

3. - THE STOKES SYSTEM

In this section we shall consider the classical Stokes system for the viscous fluid flow:

$$\begin{cases} \mu \Delta_3 u - \text{grad } p = 0 & (\mu > 0), \\ \text{div } u = 0, \end{cases} \quad \text{in } \Omega.$$

Here the unknowns, $u = (u_1(x), u_2(x), u_3(x))$ and $p = p(x)$, are respectively the velocity and pressure field and the kinematic viscosity⁽¹⁾ μ is assumed to be constant. Our aim is to extend the results for the Laplacian to this system. In particular, we are interested in representing the solution of the Dirichlet problem for the Stokes system by a simple-layer potential.

A fundamental solution for this system is given by the pair of the fundamental velocity tensor and the pressure vector:

$$\begin{aligned} \gamma_{ij}(x, y) &= -\frac{1}{4\pi\mu} \left[\frac{\delta_{ij}}{|x-y|} - \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} |x-y| \right], \\ \varepsilon_j(x, y) &= \frac{1}{4\pi} \frac{\partial}{\partial x_j} \frac{1}{|x-y|}. \end{aligned}$$

We shall consider also the classical boundary operators:

$$T_j u = \left[-\delta_{ij} p + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] v_i, \quad T'_j u = \left[\delta_{ij} p + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] v_i$$

for the solution pair (u, p) of the Stokes system, and the corresponding double layer potential

$$(3.1) \quad \begin{cases} w_j(x) = \int_{\Sigma} u_b(y) T'_{jy} [\gamma^b(x, y)] d\sigma_y, \\ q(x) = 2\mu \int_{\Sigma} \frac{\partial}{\partial v_y} [\varepsilon_b(x, y)] u_b(y) d\sigma_y \end{cases}$$

for $x \notin \Sigma$. In (3.1), $\gamma^b(x, y) = (\gamma_{ib}(x, y))$ denotes the b^{th} column vector of the fundamental tensor $((\gamma_{ij}(x, y)))$. Here and in the sequel, it is understood that for fixed b , the fundamental solution pair $(\gamma^b, \varepsilon_b)$ plays the same role as that of the solution pair (u, p) .

Throughout this and the next sections, for $\psi \in L_1^p(\Sigma)$, we define the operator Θ , by

⁽¹⁾ In order to distinguish it from the unit normal, we deliberately denote it by μ instead of a more standard notation ν .

the relation,

$$\Theta_s(\psi) = * \int_{\Sigma} d_x [S_1(x, y)] \wedge \psi(y) \wedge dx^s,$$

where $S_1(x, y)$ is the Hodge parametrix considered in Section 1. This operator satisfies the equation

$$(3.2) \quad \frac{\partial}{\partial x_s} \int_{\Sigma} u(y) \frac{\partial}{\partial v_y} S(x, y) d\sigma_y = -\Theta_s(du), \quad x \in \Omega$$

for any $u \in W^{1,p}(\Sigma)$. This follows immediately from (2.4). Furthermore, we introduce the operators \mathcal{H}_{j_s} :

$$\mathcal{H}_{j_s} \psi = \Theta_s(\psi_j) - \delta_{ibp}^{123} \int_{\Sigma} \frac{\partial}{\partial x_s} [H_{ij}(x, y)] \wedge \psi_b(y) \wedge dy^p$$

for vector-valued function $\psi = (\psi_1, \psi_2, \psi_3) \in [L_1^p(\Sigma)]^3$, where

$$H_{ij}(x, y) = \frac{1}{4\pi} \frac{1}{|x - y|} \frac{\partial}{\partial y_i} |x - y| \frac{\partial}{\partial y_j} |x - y|.$$

In terms of the operators Θ_s and \mathcal{H}_{j_s} , we now establish some relevant properties for the double layer potential:

LEMMA 1: Let $u \in W^{1,p}(\Sigma)$. Then for $x \notin \Sigma$,

$$\frac{\partial w_j}{\partial x_s} = \mathcal{H}_{j_s}(du),$$

$$q(x) = -2\mu \Theta_b(du_b),$$

where $w_j(x)$ is the double-layer potential (3.1) and $du = (du_1, du_2, du_3)$.

PROOF: We have (see, e.g., [14, p. 55])

$$w_j(x) = -\frac{3}{4\pi} \int_{\Sigma} u_b(y) \frac{(x_b - y_b)(x_j - y_j)(x_i - y_i)}{|x - y|^5} v_i(y) d\sigma_y, \quad x \in \Omega,$$

which can be rewritten in the form

$$w_j(x) = \frac{1}{4\pi} \int_{\Sigma} u_j(y) \frac{\partial}{\partial v_y} \frac{1}{|x - y|} d\sigma_y + \frac{1}{4\pi} \int_{\Sigma} u_b(y) M_y^{ib} \left[\frac{(y_i - x_i)(y_j - x_j)}{|y - x|^3} \right] d\sigma_y.$$

Here we have adopted the notation:

$$M^{ib}v = \left(v_i \frac{\partial}{\partial x_b} - v_b \frac{\partial}{\partial x_i} \right) v$$

and note that

$$-3 \frac{x_b x_j x_i v_i}{|x|^5} = M^{ib} \left(\frac{x_i x_j}{|x|^3} \right) - \delta_{jb} \frac{x_i v_i}{|x|^3}.$$

Since $x \notin \Sigma$ and M^{ib} are tangential operators, we see that

$$w_j(x) = - \int_{\Sigma} u_j(y) \frac{\partial}{\partial v_y} S(x, y) d\sigma_y - \int_{\Sigma} H_{ij}(x, y) M^{ib} [u_b(y)] d\sigma_y.$$

Therefore, from (3.2), we obtain

$$\frac{\partial w_j}{\partial x_s} = \Theta_s(du) - \delta_{ibp}^{123} \int_{\Sigma} \frac{\partial}{\partial x_s} [H_{ij}(x, y)] \wedge du_b(y) \wedge dy^p = \mathcal{C}_{js}(du)$$

and

$$q(x) = 2\mu \frac{\partial}{\partial x_b} \int_{\Sigma} \frac{\partial}{\partial v_y} S(x, y) u_b(y) d\sigma_y = -2\mu \Theta_b(du_b).$$

This completes the proof of the lemma.

For the reducing operator, let us define the following singular integral operator:

$$(3.3) \quad \begin{cases} F' : [L_1^p(\Sigma)]^3 \rightarrow [L^p(\Sigma)]^3, \\ F'_j \psi = \mu [2\delta_{ij} \Theta_b(\psi_b) + \mathcal{C}_{ij} \psi + \mathcal{C}_{ji} \psi] v_i. \end{cases}$$

As will be seen, this will be the appropriate reducing operator for the Stokes system, and it will play the same role as J' for the Laplacian in Section 1. The following lemma shows that $F'_i \psi$ can be considered as the restriction on Σ of a particular differential form:

LEMMA 2: Let $\psi = (\psi_1, \psi_2, \psi_3) \in [L_1^p(\Sigma)]^3$ and ϑ_i be the following form:

$$\vartheta_j(x) = \mu [2\delta_{ij} \Theta_b(\psi_b) + \mathcal{C}_{ij} \psi + \mathcal{C}_{ji} \psi] dx^i, \quad x \notin \Sigma.$$

Then the restriction of $*\vartheta_j(x)$ on Σ is $F'_j \psi$.

PROOF: First, we note that from general theorems (see [4]), it can be shown that there exist Hölder continuous functions a_{ijb} such that

$$(3.4) \quad \lim_{x \rightarrow x_0^+} * \vartheta_j(x) = \pm a_{jsb}(x_0) \psi_{sb}(x_0) + F_j' \psi(x_0) \quad (a.e. \ x_0 \in \Sigma)$$

for any $\psi_s = \psi_{sb} dx^b \in L_1^p(\Sigma)$.

On the other hand, if $\psi_j = du_j$ with u_j being in $C^{1+\lambda}(\Sigma)$, then it follows from Lemma 1 that

$$\begin{aligned} \vartheta_j(x) &= \mu[2\delta_{ij}\Theta_b(du_b) + \mathcal{C}_{ij}(du) + \mathcal{C}_{ji}(du)]dx^i = \\ &= \left[-\delta_{ij}q + \mu \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right) \right] dx^i, \quad x \notin \Sigma, \end{aligned}$$

where w_j, q are defined by (3.1). This implies that, in the present case with p replaced by q in the definition of the boundary operator T_j :

$$(3.5) \quad \lim_{x \rightarrow x_0^+} * \vartheta_j(x) = [T_j w]_{\pm}(x_0).$$

Therefore from (3.4), (3.5) it follows that

$$2a_{jsb}(x_0) \frac{\partial u_s}{\partial x_b}(x_0) = [T_j w]_{+}(x_0) - [T_j w]_{-}(x_0)$$

and, because of Liapunov-Tauberian theorem (see, e.g., [16]):

$$a_{jsb}(x_0) \frac{\partial u_s}{\partial x_b}(x_0) = 0, \quad \forall x_0 \in \Sigma.$$

Due to the arbitrariness of $u \in C^{1+\lambda}(\Sigma)$, we conclude that: $a_{jsb}(x_0) \equiv 0$. In view of (3.3), we then have the assertion.

We are now in a position to discuss the Dirichlet problem for the Stokes system

$$(3.6) \quad \begin{cases} \mu \Delta_3 u - \text{grad } p = 0 & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u|_{\Sigma} = f & \text{on } \Sigma, \end{cases}$$

where the given data f is assumed to be in $W^{1,p}(\Sigma)$ satisfying the compatibility condition:

$$\int_{\Sigma} f_j v_j d\sigma = 0.$$

We again seek a solution u of the equation (3.6) in the form of a simple layer potential:

$$(3.7) \quad u_b(x) = \int_{\Sigma} \varphi_j(y) \gamma_{bj}(x, y) d\sigma_y.$$

Then the boundary condition in (3.6) leads us to a system of integral equations of the first kind:

$$(3.8) \quad \int_{\Sigma} \varphi_j(y) \gamma_{ij}(x, y) d\sigma_y = f_i(x), \quad x \in \Sigma.$$

Taking the differential of both sides of (3.8), we obtain the following system of singular integral equations

$$(3.9) \quad \int_{\Sigma} \varphi_j(y) d_x[\gamma_{ij}(x, y)] d\sigma_y = df_i(x), \quad x \in \Sigma$$

in which the unknown is $(\varphi_1, \varphi_2, \varphi_3) \in [L^p(\Sigma)]^3$ and the given data is $(df_1, df_2, df_3) \in [L_1^p(\Sigma)]^3$. Let us denote by $F\varphi$ the left hand side of (3.9).

THEOREM 2: *The singular integral operator $F: [L^p(\Sigma)]^3 \rightarrow [L_1^p(\Sigma)]^3$ can be reduced on the left. Namely, we have:*

$$F'F\varphi = -\frac{1}{4}\varphi + K^2\varphi \quad \text{on } \Sigma,$$

where F' is the operator previously introduced in (3.3), and K is the compact operator defined by

$$K_j\varphi(x) = \int_{\Sigma} \varphi_b(y) T_{jx}[\gamma^b(x, y)] d\sigma_y, \quad x \in \Sigma.$$

PROOF: From the definition (3.3), we have

$$\begin{aligned} F_j'F\varphi &= \mu[2\delta_{ij}\Theta_b(F_b\varphi) + \mathcal{H}_{ij}(F\varphi) + \mathcal{H}_{ji}(F\varphi)]v_i = \\ &= \mu[2\delta_{ij}\Theta_b(du_b) + \mathcal{H}_{ij}(du) + \mathcal{H}_{ji}(du)]v_i, \end{aligned}$$

where u_b is the simple layer potential defined by (3.7). Moreover, by Lemma 2. $F_j'F\varphi$ is the restriction of $*\vartheta_i$ and

$$\vartheta_i(x) = \left[-\delta_{ij}q + \mu \left(\frac{\partial w_i}{\partial x_i} + \frac{\partial w_j}{\partial x_i} \right) \right] dx^i, \quad x \in \Omega,$$

where w is given by (3.1). Hence $F_j'F\varphi = T_j[w]$ on Σ .

On the other hand, we have from the Green's representation formula for $x \in \Omega$,

$$w_j(x) = \int_{\Sigma} u_b(y) T'_{j\bar{b}}[\gamma^b(x, y)] d\sigma_y = u_j(x) + \int_{\Sigma} \gamma_{j\bar{b}}(x, y) T_b[u(y)] d\sigma_y$$

(see, e.g., [14, p. 54]). Therefore, on Σ , we find that

$$\begin{aligned} T_j[w] &= T_j \left[u(x) + \int_{\Sigma} \gamma^b(x, y) T_b[u(y)] d\sigma_y \right] = \\ &= \left(1 - \frac{1}{2} \right) T_j u + \int_{\Sigma} T_{j\bar{x}}[\gamma^b(x, y)] T_b[u(y)] d\sigma_y = \\ &= -\frac{1}{4} \varphi_j(x) + \int_{\Sigma} T_{j\bar{x}}[\gamma^b(x, y)] d\sigma_y \int_{\Sigma} T_{b\bar{y}}[\gamma^s(y, z)] \varphi_s(z) d\sigma_z. \end{aligned}$$

Since for a Lyapunov surface Σ of index d , the kernel of the above integral operator has the singular behaviour, $T_{j\bar{x}}[\gamma^b(x, y)] = O(|x - y|^{-2+d})$. The latter then defines a compact operator, and thus the theorem is proved.

We now state our main result:

THEOREM 3: *Let $f \in W^{1,p}(\Sigma)$ satisfy $\int_{\Sigma} f_j v_i d\sigma = 0$. Then the solution of (3.6) is given by a single layer potential (3.7) with the density φ being a solution of (3.9).*

In order to prove this theorem, it is sufficient to show that (3.9) has a solution. We shall proceed as follows. First, we note that because of Theorem 2, the range of F is closed in $L_1^p(\Sigma)$; therefore there exists a solution of (3.9) if and only if the data df_i satisfy the compatibility conditions:

$$(3.10) \quad \int_{\Sigma} \psi_j \wedge df_i = 0, \quad \forall \psi \in [L_1^{p/(p-1)}(\Sigma)]^3 \text{ such that } F^* \psi = 0.$$

By a solution ψ of the adjoint equation, $F^* \psi = 0$, we mean that ψ satisfies the equation

$$(3.11) \quad \int_{\Sigma} \psi_j(y) \wedge d_y[\gamma_{j\bar{i}}(x, y)] = 0 \quad \text{a.e. } x \in \Sigma.$$

As will be shown, the condition (3.11) implies that the «weak» differential of ψ_j , $d\psi_j$, exists and $d\psi_j = c v_i$ for some constant c ; for ease of reading, we shall defer the proof of

that in the Appendix (see Theorem A.1). Therefore, for any $\psi \in \mathcal{N}(F^*)$, we have

$$(3.12) \quad \int_{\Sigma} \psi_j \wedge du_j = c \int_{\Sigma} u_j v_j d\sigma$$

for any smooth function u . By using density arguments, we have (3.12) for any $u \in W^{1,p}(\Sigma)$, and therefore (3.10) holds.

To conclude this section, we remark that a different approach to the Stokes system by using differential forms is given in [5]. Furthermore, by following [8], solutions to the Stokes systems in \mathbb{R}^2 and \mathbb{R}^3 are also obtained in the form of single-layer potentials (see [12] and [10]). For general results see [14].

4. - THE LAMÉ EQUATIONS

We now extend our approach to the Lamé equations in linear elasticity for the isotropic material. To be more precise, we shall study the first boundary-value problem for the displacement field $u(x) = (u_1(x), u_2(x), u_3(x))$:

$$(4.1) \quad \begin{cases} \Delta_3 u + k \operatorname{grad} \operatorname{div} u = 0 & \text{in } \Omega \ (k > 1/3), \\ u = f & \text{on } \Sigma, \end{cases}$$

by means of a simple layer potential. Here $k = (\lambda + \mu)/\mu$ is given in terms of the Lamé constants λ and μ . We assume that the given data f is in the space $[W^{1,p}(\Sigma)]^3$. The simple layer potential now reads:

$$(4.2) \quad u_j(x) = \int_{\Sigma} \varphi_b(y) E_{jb}(x, y) d\sigma_y,$$

where $((E_{jb}))$ is the Somigliana tensor:

$$E_{jb}(x, y) = -\frac{1}{4\pi} \left[\frac{\delta_{jb}}{|x-y|} - \frac{k}{2(1+k)} \frac{\partial^2}{\partial x_i \partial x_j} |x-y| \right].$$

First, we need some properties of the double-layer potential

$$w_j(x) = \int_{\Sigma} u_b(y) L_{jy} [E^b(x, y)] d\sigma_y$$

where $E^b(x, y)$ is again the column vector whose components are $E_{jb}(x, y)$ and L_j is the operator:

$$Lu = (k - \xi) \operatorname{div} uv + (1 + \xi) \frac{\partial u}{\partial \nu} + \xi(v \wedge \operatorname{rot} u),$$

where ξ is a fixed real constant and a particular choice of ξ will be made later.

Similar to Lemma 1, we now have the following result:

LEMMA 3: Let $u \in [W^{1,p}(\Sigma)]^3$. Then

$$(4.3) \quad \frac{\partial w_j}{\partial x_s} = \mathcal{K}_{js}(du), \quad x \in \Omega,$$

where

$$\mathcal{K}_{js}(\psi) = \Theta_s(\psi_j) - \delta_{ibp}^{123} \int_{\Sigma} \frac{\partial}{\partial x_s} [K_{ij}(x, y)] \wedge \psi_b(y) \wedge dy^p,$$

$$K_{ij}(x, y) = \frac{1}{4\pi} \left[\frac{k - \xi(2+k)}{2(1+k)} \delta_{ji} + \frac{(1+\xi)k}{2(1+k)} \frac{\partial|x-y|}{\partial y_j} \frac{\partial|x-y|}{\partial y_i} \right] \frac{1}{|x-y|},$$

and Θ_s are the operators introduced in Section 3.

PROOF: A simple manipulation shows that

$$\begin{aligned} L_{jy}[E^b(x, y)] = & -\frac{1}{4\pi} \left\{ \left[\frac{3(1+\xi)k}{2(1+k)} \frac{\partial|x-y|}{\partial y_j} \frac{\partial|x-y|}{\partial y_b} + \delta_{jb} \frac{2+(1-\xi)k}{2(1+k)} \right] \right. \\ & \left. + \frac{\partial}{\partial v_y} \frac{1}{|x-y|} + \frac{k - \xi(2+k)}{2(1+k)} \left[v_j(y) \frac{\partial}{\partial y_b} - v_b(y) \frac{\partial}{\partial y_j} \right] \frac{1}{|x-y|} \right\}. \end{aligned}$$

Then by making use of a substitution similar to the ones in Lemma 1, we may rewrite the double layer potential in the form,

$$w_j(x) = - \int_{\Sigma} u_j(y) \frac{\partial}{\partial v_y} S(x, y) d\sigma_y - \int_{\Sigma} K_{ji}(x, y) M^{ib}[u_b(y)] d\sigma_y.$$

Here $S(x, y)$ is the fundamental solution for the Laplacian Δ , and M^{ib} denotes the same operator introduced in the proof of Lemma 1. As a consequence, we have

$$\frac{\partial w_j}{\partial x_s}(x) = \Theta_s(du_j) - \delta_{ibp}^{123} \int_{\Sigma} \frac{\partial}{\partial x_s} [K_{ij}(x, y)] \wedge du_b(y) \wedge dy^p = \mathcal{K}_{js}(du),$$

i.e., (4.3).

In the same manner, let us now define the following singular integral operator:

$$R' : [L_1^p(\Sigma)]^3 \rightarrow [L^p(\Sigma)]^3,$$

$$R'_i \psi = (k - \xi) \mathcal{K}_{ji}(\psi) v_i + (1 + \xi) \mathcal{K}_{ij}(\psi) v_j + \xi \delta_{sp}^{ij} v_j \mathcal{K}_{ps}(\psi).$$

We remark that both operators L and R' depend on the choice of the parameter ξ . As

will be shown (Theorem 4), with a special choice of ξ , the corresponding R' may be served as a reducing operator for the Lamé equations.

LEMMA 4: Let $\psi = (\psi_1, \psi_2, \psi_3) \in [L_1^p(\Sigma)]^3$ and ω_i be the following form:

$$\omega_i(x) = (k - \xi) \mathcal{K}_{ij}(\psi) dx^i + (1 + \xi) \mathcal{K}_{ij} \psi dx^j + \xi \delta_{ip}^j \mathcal{K}_{ps} \psi dx^j, \quad x \notin \Sigma.$$

Then the restriction of $\ast \omega_i(x)$ on Σ is $R'_i \psi$.

We omit the proof of the lemma, since the proof is almost identical to that of Lemma 2. The Liapunov-Tauberian theorem we need in this case can be found in [13, p. 408].

We return now to the boundary-value problem (4.1) by using a simple layer potential (4.2). We again arrive at a system of integral equations of the first kind,

$$(4.4) \quad \int_{\Sigma} \varphi_j(y) E_{ij}(x, y) d\sigma_y = f_i(x), \quad x \in \Sigma.$$

Taking the differential of both sides, we obtain the following system of singular integral equations:

$$(4.5) \quad \int_{\Sigma} \varphi_j(y) d_x [F_{ij}(x, y)] d\sigma_y = df_i(x), \quad x \in \Sigma.$$

Let us denote by $R\varphi$ the left hand side of (4.5). In an analogue to Theorem 2, we have the result:

THEOREM 4: The singular integral operator $R: [L^p(\Sigma)]^3 \rightarrow [L_1^p(\Sigma)]^3$ can be reduced on the left. Namely, we have:

$$R'_0 R\varphi = -\frac{1}{4}\varphi + K^2\varphi$$

where R'_0 is the operator R' with $\xi = k/(2+k)$, and the compact operator K is defined by

$$K_j \varphi = \int_{\Sigma} \varphi_b(y) L_{jx}^0 [E^b(x, y)] d\sigma_y,$$

$$L^0 u = \frac{k(1+k)}{2+k} \operatorname{div} u \nu + \frac{2(1+k)}{2+k} \frac{\partial u}{\partial \nu} + \frac{k}{2+k} \nu \wedge \operatorname{rot} u.$$

We remark that the proof of this theorem can be established step by step (with some obvious modifications) by repeating the proof of Theorem 2. It is well-known that in elasticity, the double-layer boundary integral operator corresponding to L_j is generally a singular integral operator. However, in terms of L_j^0 , the so-called *pseudo-stress operator*, it is a compact operator (see, e.g., [13] and [11]). Hence all the analysis employed for the case of Laplacian can be easily carried out here without difficulty.

THEOREM 5: Let $f \in [W^{1,p}(\Sigma)]^3$. Then the solution of (4.1) can be represented by a simple layer potential (4.2) in terms of φ , which is a solution of (4.5).

Theorem 4 implies that the range of R is closed. Therefore a solution of (4.5) exists if and only if (df_1, df_2, df_3) is orthogonal to any eigensolution of the adjoint system:

$$\int_{\Sigma} \psi_j(y) \wedge d_y [F_{ij}(x, y)] = 0 \quad \text{a.e. } x \in \Sigma.$$

This system implies that ψ_j ($j = 1, 2, 3$) are weakly closed forms (in order to prove that, one can simply follow the same proof given in [1, pp. 189-90] for the Laplace equation). We note that in contrast to the Stokes system, the given data here is not required to satisfy any compatibility condition. Details are omitted here.

Finally we remark that also in this case $\mathcal{N}(R'_0 R) = \mathcal{N}(R)$ and hence equation (4.5) and the Fredholm equation: $R'_0 R \varphi = R'_0(df)$ are equivalent.

APPENDIX: THE WEAK DIFFERENTIAL

We begin with the following auxiliary result.

LEMMA A.1: Let $u \in [C_0^\infty(\mathbb{R}^3)]^3$ be any infinitely differentiable function with compact support. Then the representation formula:

$$u_i(x) = \mu \int_{\mathbb{R}^3} \Delta u_j(y) \gamma_{ij}(x, y) dy + \int_{\mathbb{R}^3} \frac{\partial^2 u_j(y)}{\partial y_i \partial y_j} S(x, y) dy$$

holds for $x \in \mathbb{R}^3$, where $S(x, y)$ is the fundamental solution for the Laplacian Δ in \mathbb{R}^3 .

PROOF: It is well known that for $u \in [C_0^\infty(\mathbb{R}^3)]^3$, we have the representation formula:

$$u_i(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \Delta u_i(y) \frac{1}{|x-y|} dy$$

and therefore we can write

$$u_i(x) = \mu \int_{\mathbb{R}^3} \Delta u_j(y) \gamma_{ij}(x, y) dy - \frac{1}{8\pi} \int_{\mathbb{R}^3} \Delta u_j(y) \frac{\partial^2 |x-y|}{\partial y_i \partial y_j} dy.$$

Since the functions u_j has a compact support, the last integral is equal to

$$-\frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{\partial^2 u_j(y)}{\partial y_i \partial y_j} \Delta_y |x-y| dy = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\partial^2 u_j(y)}{\partial y_i \partial y_j} \frac{1}{|x-y|} dy.$$

The latter can be rewritten in terms of $S(x, y)$.

THEOREM A.1: *If $\psi \in [L_1^q(\Sigma)]^3$, $q = p / (p - 1)$ satisfies (3.11), then the weak differential of ψ_j exists and is equal to cv_j for same constant c , i.e., there exists a constant c such that*

$$\int_{\Sigma} \psi_j \wedge du_j = c \int_{\Sigma} u_j v_j d\sigma \quad \forall u \in C_0^\infty(\mathbb{R}^3).$$

PROOF: By using arguments similar to the one employed in [1, pp. 189-190], one can show that (3.11) implies that

$$(A.1) \quad \int_{\Sigma} \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] = 0, \quad \forall x \notin \Sigma.$$

Since $\Delta_x [\gamma_{ij}(x, y)] = -\frac{1}{\mu} \frac{\partial^2 S(x, y)}{\partial x_i \partial x_j}$ ($x \neq y$), we deduce that

$$\frac{\partial^2}{\partial x_i \partial x_j} \int_{\Sigma} \psi_j(y) \wedge d_y [S(x, y)] = 0, \quad \forall x \notin \Sigma.$$

Therefore, there exists a constant c such that

$$(A.2) \quad \frac{\partial}{\partial x_j} \Psi_j(x) = \begin{cases} -c, & x \in \Omega, \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega}, \end{cases}$$

where

$$\Psi_j(x) = \int_{\Sigma} \psi_j(y) \wedge d_y [S(x, y)].$$

Now Lemma A.1 yields that

$$\begin{aligned} \int_{\Sigma} \psi_j \wedge du_j &= \mu \int_{\mathbb{R}^3} \Delta u_j(x) dx \int_{\Sigma} \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] + \\ &+ \int_{\mathbb{R}^3} \frac{\partial^2 u_i}{\partial x_i \partial x_j}(x) dx \int_{\Sigma} \psi_j(y) \wedge d_y [S(x, y)], \quad \forall u \in [C_0^\infty(\mathbb{R}^3)]^3. \end{aligned}$$

The first term on the right hand side vanishes because of (A1), while the second one

can be written as

$$\int_{\Omega} \frac{\partial^2 u_i}{\partial x_i \partial x_j} \Psi_j dx + \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \frac{\partial^2 u_i}{\partial x_i \partial x_j} \Psi_j dx =$$

$$= \int_{\Sigma} \frac{\partial u_i}{\partial x_i} \Psi_j v_j d\sigma - \int_{\Omega} \frac{\partial u_i}{\partial x_i} \frac{\partial \Psi_j}{\partial x_j} dx - \int_{\Sigma} \frac{\partial u_i}{\partial x_i} \Psi_j v_j d\sigma - \int_{\mathbb{R}^3 \setminus \bar{\Omega}} \frac{\partial u_i}{\partial x_i} \frac{\partial \Psi_j}{\partial x_j} dx,$$

which follows from integration by parts. To justify these integrations, we see that the first of these integration by parts can be achieved by taking domains $\Omega_\varrho \subset \Omega$ ($0 < \varrho < \varrho_0$) with Lyapunov boundary $\Sigma_\varrho = \partial\Omega_\varrho$ such that $\Omega_\varrho \supset \Omega_{\varrho'}$ if $\varrho < \varrho'$ (with $\Omega_0 = \Omega$) and that $\int_{\Sigma_\varrho} \psi d\sigma \rightarrow \int_{\Sigma} \psi d\sigma$ (for a construction of such domains, see, e.g., [2]).

After integration by parts in Ω_ϱ , the resulting formula in Ω then follows by taking the limit as $\varrho \rightarrow 0^+$. The other integration by parts can be justified in the same manner.

Finally, collecting terms and by taking into account of (A.2), we then obtain the desired result:

$$\int_{\Sigma} \psi_j \wedge du_j = c \int_{\Omega} \frac{\partial u_i}{\partial x_i} dx = c \int_{\Sigma} u_i v_i d\sigma.$$

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