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MAURO CERASOLI (*)

Integer Sequences and Umbral Calculus (**)

SUMMARY. — The Note mainly concerns some identities for Bell, Fibonacci, Lucas and Montmort numbers. The proof is given via the umbral calculus revisited in the last years by G. C. Rota.

Successioni di interi e calcolo umbrale

SUNTO. — Vengono dimostrate alcune identità riguardanti numeri di Fibonacci, di Bell, di Lucas e di Montmort mediante il calcolo umbrale recentemente rielaborato da G. C. Rota.

1. - INTRODUCTION

We present some applications of the umbral calculus exposed by G. C. Rota and D. Taylor in [8]. From this paper we recall few definitions and lemmas.

The *umbral calculus* consists of a system $(A, D, \text{val}, \varepsilon)$, where A is a set, whose elements are called *umbrae*; D is a commutative integral domain whose quotient field is of characteristic zero; if $D[A]$ is the polynomial ring of A with coefficients in D , then $\text{val}: D[A] \rightarrow D$ is a linear functional, called *evaluation*, such that

- a) $\text{val}(1) = 1$ (1 is the identity of D)
- b) if $\alpha, \beta, \dots, \tau$ are distinct umbrae and i, j, \dots, k are natural numbers or elements of $N = \{0, 1, 2, \dots\}$, then

$$\text{val}(\alpha^i \beta^j \dots \tau^k) = \text{val}(\alpha^i) \text{val}(\beta^j) \dots \text{val}(\tau^k).$$

(*) Indirizzo dell'Autore: Mauro Cerasoli, Dipartimento di Matematica, via Vetoio, Università, I-67010 L'Aquila.

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At last, ε is a distinguished element, called *augmentation*, such that

$$\text{val}(x^n) = \delta_{n,0} \quad (\text{Kronecker delta}).$$

Umbral polynomials are elements of the polynomial ring $D[A]$. The support $\text{Sup}(p)$ of an umbral polynomial $p = p(a, \beta, \dots, \tau)$ is the set of umbrae that actually appear in the monomials of p .

Two polynomials p and q of $D[A]$ are called *umbrally equivalent* (in symbols $p \approx q$) when $\text{val}(p) = \text{val}(q)$; they are *exchangeable* (in symbols $p \equiv q$) when $\text{val}(p^n) = \text{val}(q^n)$ for every $n \in N$.

LEMMA 1.1: Let be f, g, r, s umbral polynomials; if

$$\text{Sup}(f) \cap \text{Sup}(g) = \text{Sup}(r) \cap \text{Sup}(s) = \emptyset$$

and $f \approx g$, $r \approx s$, then $f + r \approx g + s$ and $fg \approx rs$.

A sequence (a_n) , $n \in N$, of elements of D , is said umbrally represented by an umbra α when

$$\text{val}(\alpha^n) = a_n \quad \text{for every } n \in N.$$

Note that if the umbra α represents the sequence a_n , then $\alpha^n = a_n$ for every $n \in N$.

If (p_n) and (q_n) are two sequences of umbral polynomials, the formal power series

$$P(t) = \sum_{n \geq 0} p_n t^n \quad \text{and} \quad Q(t) = \sum_{n \geq 0} q_n t^n$$

in the indeterminate t , are umbrally equivalent, $P(t) \approx Q(t)$, if and only if, $p_n \approx q_n$ for every $n \in N$. If $\alpha^n = a_n$, then the exponential generating function

$$A(t) = \sum_{n \geq 0} a_n t^n / n!$$

of (a_n) is umbrally represented by

$$e^{\alpha t} = \sum_{n \geq 0} (\alpha t)^n / n!$$

that is $A(t) = e^{\alpha t}$.

2. - SATURATED UMBRAL CALCULUS

Given an umbra α , let $\alpha_1, \alpha_2, \dots, \alpha_k$ be k distinct umbrae such that $\alpha_i \approx \alpha$, for every $i = 1, 2, \dots, k$. The umbra product $k \cdot \alpha$, often called an auxiliary umbra,

is defined by the identity

$$k \cdot \alpha = \alpha_1 + \alpha_2 + \dots + \alpha_k.$$

Similary $(-k) \cdot \alpha$ is defined as the umbra exchangeable with the sum $\beta_1 + \beta_2 + \dots + \beta_k$, where $\beta_1, \beta_2, \dots, \beta_k$ are k distinct umbras each exchangeable with β , and $\alpha + \beta = \varepsilon$. At last, $0 \cdot \alpha$ is an umbra exchangeable with the augmentation ε . The product $k \cdot p$ is introduced similary for any polynomial p of $D[A]$, not containing auxiliary umbras.

LEMMA 2.1: If α' and α'' are distinct umbras exchangeable with α , then

$$1) (n+m) \cdot \alpha = n \cdot \alpha' + m \cdot \alpha'' \text{ for } n, m \in \mathbb{Z};$$

$$2) \text{ if } n \cdot \alpha = n \cdot \beta \text{ for some } n \neq 0 \text{ then } \alpha = \beta;$$

$$3) \text{ if } c \in D, \text{ then}$$

$$a) n \cdot (c\alpha) = c(n \cdot \alpha)$$

and

$$b) n \cdot (c) = nc \text{ for every } n \in \mathbb{Z};$$

$$4) \text{ if } \alpha + \beta = \varepsilon \text{ and } p(x) \text{ is a polynomial with coefficients in } D, \text{ then}$$

$$a) -\beta p(\alpha + \beta) = ap(\alpha + \beta)$$

$$b) (n \cdot \beta)p(n \cdot \alpha + n \cdot \beta) = n(\beta p(\alpha + \beta)).$$

For every $n \in N$ and $k \in \mathbb{Z}$ we define the polynomial $A_{n,k}(x)$ associated to the umbra α as follows

$$A_{n,k}(x) = (x + k \cdot \alpha)^n = \sum_{i \geq 0} a_{n,k,i} x^{n-i},$$

where

$$a_{n,k,i} = \binom{n}{i} \text{val}((k \cdot \alpha)^i).$$

LEMMA 2.2: The sequence of polynomials $A_{n,k}(x)$, $n \in N$, is an Appel sequence and if k_1, k_2, \dots, k_r are integers, such that $k_1 + k_2 + \dots + k_r = k$, then

$$A_{n,k}(x_1 + x_2 + \dots + x_r) = \sum \frac{n!}{i_1! \dots i_r!} A_{k_1, i_1}(x_1) \dots A_{k_r, i_r}(x_r)$$

and, if $a_s = \text{val}(\alpha^s)$,

$$a_{n,k,s} = \sum \frac{n!}{i_1! \dots i_r!} a_{i_1} \dots a_{i_r},$$

the sums being extended to $i_1, \dots, i_r \geq 0$ such that

$$i_1 + \dots + i_r = n.$$

PROOF: We have an Appel sequence because

$$D_x A_{n,k}(x) = n(x+k\cdot\alpha)^{n-1} = nA_{n-1,k}(x).$$

Moreover in the umbral notation the above identity is

$$(x_1 + \dots + x_r + k \cdot \alpha)^n = ((x_1 + k_1 \cdot \alpha_1) + \dots + (x_r + k_r \cdot \alpha_r))^n,$$

where $\alpha_1, \dots, \alpha_r$ are distinct umbrae, each exchangeable with α .

Now set $x_i = 0$ and $k_i = 1$ to obtain the second identity.

3. - THE GENERALIZED BERNOULLI UMBRA

For fixed natural $r > 0$ let be (β_r) the umbrae sequence such that

$$(3.1) \quad A' p(\beta_r) = D' p(e)$$

for every $p \in D[A]$. A' is the backward difference operator such that $A'f(x) = f(x+1) - f(x)$. If $r = 1$, we have the Bernoulli umbra β of [6]. Setting $p(x) = x^n$ it follows

$$Ax^n = (E - I)x^n = \sum_{j \geq 0} \binom{r}{j} (-1)^{r-j} E^j x^n = \sum_{j \geq 0} \binom{r}{j} (-1)^{r-j} \sum_{k \geq 0} \binom{n}{k} x^k j^{n-k}.$$

Likewise it results:

$$D' x^n |_{x=e} = n(n-1)(n-2) \dots (n-r+1) e^{n-r} = 1 \quad \text{if } r \leq n,$$

zero otherwise.

Therefore, if $b_{k,r} = \text{val}((\beta_r)_k)$, we obtain the identity

$$(4.1) \quad \sum_{j,k \geq 0} (-1)^{r-j} j^{n-k} \binom{r}{j} \binom{n}{k} b_{k,r} = \begin{cases} 1 & \text{if } r = n, \\ 0 & \text{if } r > n. \end{cases}$$

4. - THE BELL UMBRA

The Bell umbra θ is defined by

$$(4.1) \quad \binom{\theta}{n} = \frac{1}{n!} \quad \text{for every } n \in N.$$

The evaluations of θ , that is the numbers $B_n = \theta^n$, $B_0 = 1$, are called *Bell numbers*.

THEOREM 4.1: For every $n \in N$ the following identities hold:

$$(4.1a) \quad B_n = \sum_{k=0}^n S(n, k) \quad n > 0,$$

where $S(n, k)$ is the Stirling number of the second kind;

$$(4.1b) \quad \sum_{n=0}^{\infty} B_n t^n / n! = \exp(e^t - 1);$$

$$(4.1c) \quad (1 + \theta)^x = \theta^{x+1};$$

$$(4.1d) \quad \sum_{k=0}^n \binom{n}{k} B_k = B_{n+1};$$

$$(4.1e) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_{k+1} = B_n \quad (\text{Aitken formula});$$

$$(4.1f) \quad \sum_{k=0}^n s(n, k) B_k = 1,$$

where $s(n, k)$ is the Stirling number of the first kind;

$$(4.1g) \quad p(x + \theta + 1) = \theta p(x + \theta);$$

$$(4.1h) \quad \Delta^x p(x + \theta) = (\theta - 1)^x p(x + \theta)$$

for every polynomial $p \in D[A]$;

$$(4.1i) \quad eB_n = \sum_{k=0}^n \frac{k^n}{k!};$$

PROOF: The Bell umbra θ satisfies the identities

$$(\theta)_n = \theta(\theta - 1)(\theta - 2) \dots (\theta - n + 1) = 1 \quad \text{and} \quad \theta^n = \sum_{k=0}^n S(n, k)(\theta)_k.$$

Evaluating we obtain a); from this identity we can see that B_n is the number of partitions of a n -set. To prove b) note that if $x = e^t - 1$, then

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \approx e^{\theta t} = (1 + x)^{\theta} = \sum_{n=0}^{\infty} \binom{\theta}{n} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x = \exp(e^t - 1).$$

Likewise, c) is obtained in the following way:

$$e^{(\theta+1)} \approx \exp(e^t - 1 + t) = D \exp(e^t - 1) = D e^{\theta t} = \theta e^{\theta t}.$$

Now equate the corresponding coefficients in these generating functions; d) comes by evaluation of c). Again e) is a consequence of the obvious umbral identity

$$\theta^n \approx \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1 + \theta)^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \theta^{k+1}.$$

For a different derivation of the Aitken formula *e*) see Comtet p. 211. From

$$1 \approx (\theta)_n = \sum_{k=0}^n x(n, k) \theta^k \quad (5.1)$$

we obtain *f*). At last, *g*) comes from *c*), and *b*) is proved by induction. To prove *i*) see [6].

5. - THE FIBONACCI UMBRA

The Fibonacci umbra α satisfies the following identity for every natural n :

$$(5.1) \quad \alpha^{n+2} \approx \alpha^{n+1} + \alpha^n.$$

We put $\text{val}(\alpha^n) = F_n$: the n -th Fibonacci number, $F_0 = F_1 = 1$. With umbral reasonings we can prove easily some formulas involving these numbers.

THEOREM 5.1: The Fibonacci numbers F_n satisfies the following identities for every $n \in N$:

$$(5.1a) \quad F_{n+2} = F_{n+1} + F_n,$$

$$(5.1b) \quad F_0 + F_1 + F_2 + \dots + F_n = F_{n+2} - 1,$$

$$(5.1c) \quad F_0 + F_2 + \dots + F_{2n} = F_{2n+1},$$

$$(5.1d) \quad F_1 + F_3 + \dots + F_{2n-1} = F_{2n} - 1,$$

$$(5.1e) \quad F_0 - F_1 + F_2 - \dots + (-1)^{n-1} F_{n+1} = 1 + (-1)^{n-1} F_n,$$

$$(5.1f) \quad \sum_{k=0}^n \binom{n}{k} F_{k+n} = F_{2n+n},$$

$$(5.1g) \quad \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} F_{2k+n} = F_{n+n},$$

$$(5.1h) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} F_{k+n} = (-1)^n F_n.$$

PROOF: *a*) is obvious from definition (5.1) by evaluation; from (5.1), for $n = 0$, we have

$$\alpha^2 \approx \alpha + 1, \quad \alpha^2 - \alpha \approx 1 \quad \text{and} \quad \frac{1}{\alpha} \approx \alpha - 1;$$

therefore, analogously, *b*) comes from the

$$\text{The evaluation of } \sum_{k=0}^n \alpha^k = \frac{\alpha^{n+1} - 1}{\alpha - 1} = \frac{\alpha^{n+2} - \alpha}{\alpha^2 - \alpha} \approx \alpha^{n+2} - \alpha.$$

To derive c) consider the umbral identity

$$\sum_{k=0}^{n-1} \alpha^{2k} = \frac{\alpha^{2n} - 1}{\alpha^2 - 1} = \frac{\alpha^{2n} - 1}{\alpha} = (\alpha^{2n} - 1)(\alpha - 1) = \alpha^{2n+1} - \alpha^{2n} - \alpha + 1,$$

Moreover, from another point of view, we have

$$\sum_{k=0}^{n-1} \alpha^{2k} = \sum_{k=0}^{n-1} (1+\alpha)^k = \frac{(1+\alpha)^n - 1}{\alpha},$$

or

$$\sum_{k=0}^{n-1} \alpha^{2k+1} = (1+\alpha)^n - 1 = \alpha^{2n} - 1,$$

from which d) comes, always by evaluation. At last, the umbral identities

$$e) \sum_{k=0}^{n-1} (-1)^k \alpha^{k+2} = (-1)^{n-1} \alpha^n + 1,$$

$$f) \alpha^{2n+m} = \alpha^n (\alpha+1)^m = \sum_{k=0}^m \binom{m}{k} \alpha^{n+k},$$

$$g) \alpha^{n+m} = \alpha^n (\alpha^2 - 1)^m = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \alpha^{2k+m},$$

$$h) \alpha^n = \alpha^{n+m} (\alpha - 1)^m = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \alpha^{k+m+n}$$

give the other identities for the Fibonacci numbers.

6. - THE LUCAS UMBRAE

Given $a, b, c \in \mathbb{C}$, the Lucas umbrae $\lambda = \lambda(a, b, c)$ are defined by the identity

$$(6.1) \quad \lambda^{n+2} = a\lambda^{n+1} + b\lambda^n, \quad \lambda = c$$

for every $n \in \mathbb{N}$. The Lucas umbrae represent a unique sequence of complex numbers. Such a sequence, $L_n(a, b, c) = \text{val}(\lambda^n)$, $L_1(a, b, c) = c$ is a sequence of *Lucas numbers*. Note that $L_n(1, 1, 1) = F_n$ is a Fibonacci number.

THEOREM 6.1: The numbers $L_n(a, b, c)$ satisfies the following identities for all naturals m and n :

$$(6.1a) \quad L_{n+2} = aL_{n+1} + bL_n,$$

$$(6.1b) \quad \sum_{k=0}^n (-a)^k [L_{k+2} + (1-bc)L_k] = 1 + (-1)^n a^{n+1} L_{n+1},$$

$$(6.1c) \quad \sum_{k=0}^n a^k [L_{k+2} - (1+bc)L_k] = a^{n+1} L_{n+1} - 1,$$

$$(6.1d) \quad \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} L_{m+k} = L_{2n+m},$$

$$(6.1e) \quad \sum_{k=0}^n \binom{n}{k} (-b)^{n-k} L_{2k} = a^n L_n,$$

$$(6.1f) \quad \sum_{k=0}^n \binom{n}{k} (-a)^{n-k} L_{n+k} = b^n.$$

PROOF: All these numerical formulae are obtained by evaluating (6.1) and from the following identities involving the Lucas umbrae:

$$b) \quad \sum_{k=0}^n (-a\lambda)^k = \frac{1 + (-1)^n a^{n+1} \lambda^{n+1}}{1 + a\lambda} = \frac{1 + (-1)^n a^{n+1} \lambda^{n+1}}{\lambda^2 + 1 - bc},$$

$$c) \quad \sum_{k=0}^n (a\lambda)^k = \frac{(a\lambda)^{n+1} - 1}{\lambda^2 - 1 - bc},$$

$$d) \quad \lambda^n (a\lambda + b)^n = \lambda^{2n+m},$$

$$e) \quad (\lambda^2 - b)^n \lambda = a^n \lambda^n,$$

$$f) \quad (\lambda^2 - a\lambda)^n = b^n.$$

7. - THE MONTMORT UMBRA

The Montmort umbra δ is defined by the identity

$$(7.1) \quad \delta^n = n\delta^{n-1} + (-1)^n$$

for every natural $n \geq 1$. The Montmort numbers D_n (or *derangement numbers*) are the evaluations of δ^n , with $D_0 = 1$.

THEOREM 7.1: The Montmort numbers D_n satisfies the following identities:

$$(7.1a) \quad D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!} = nD_{n-1} + (-1)^n,$$

$$(7.1b) \quad \sum_{n \geq 0} \frac{D_n}{n!} t^n = \frac{e^{-t}}{1-t},$$

$$(7.1c) \quad D_{n+2} = (n+1)(D_{n+1} + D_n),$$

$$(7.1d) \quad D_n = \mathcal{A}^n x!|_{x=0},$$

$$(7.1e) \quad \sum_{k=0}^n \binom{n}{k} D_k = n!,$$

PROOF: From (7.1) we obtain by iteration

$$\delta^n = n\delta^{n-1} + (-1)^n = n[(n-1)\delta^{n-2} + (-1)^{n-2}] + (-1)^n = \sum_{k=0}^n (-1)^{n-k} (n)_k ,$$

that is (7.1a) after evaluation. Always from 7.1 we have

$$\frac{(\delta t)^n}{n!} = \frac{t(\delta t)^{n-1}}{(n-1)!} + \frac{(-t)^n}{n!} ,$$

that gives

$$e^{t\delta} - 1 = te^{\delta t} + e^{-t} - 1 ,$$

and (7.1b) is easily proved. Note that e^{-t} is the generating function of $(-1)^k/k!$ and $1/(1-t)$ is the generating function of the constant sequence equal to 1. Therefore, D_n is the convolution of this sequence multiplied by $n!$ and we obtain (7.1a) again. From (7.1) we have

$$\delta^{n+2} = (n+2)\delta^{n+1} + (-1)^{n+2} = (n+1)\delta^{n+1} + \delta^{n+1} + (-1)^{n+2} = (n+1)(\delta^{n+1} + \delta^n) ,$$

and (7.1c) is obtained by evaluation. The obvious proof of (7.1d) is omitted. To prove (7.1e) note that from (7.1) we have

$$\begin{aligned} (1+\delta)^n &= \sum_{k=0}^n \binom{n}{k} \delta^k = 1 + \sum_{k=1}^n \binom{n}{k} k\delta^{k-1} + \sum_{k=1}^n \binom{n}{k} (-1)^k = \\ &= n(1+\delta)^{n-1} = n(n-1)(1+\delta)^{n-2} = \dots = n! . \end{aligned}$$

Therefore, the umbral identity $(1+\delta)^n = n!$ is equivalent to (7.1e).

8. - A CHARACTERIZATION OF THE BINOMIAL POLYNOMIALS

We remember that a sequence of polynomials $p_n(x)$ of degree n , is called *binomial* if

a) $p_0(x) = 1$,

b) $p_n(x+y) = \sum_{k=0}^n \binom{n}{k} p_k(x) p_{n-k}(y) ,$

for every $x, y \in R$, $n \in N$. An example of binomial polynomials is given by the Abel polynomials

$$p_n(x) = x(x+nx)^{n-1} \quad x \in R .$$

We can characterize binomial polynomials by umbrae and Abel polynomials with the following

THEOREM 8.1: A sequence of polynomials $p_n(x)$ is binomial if, and only if, an umbra α exist such that

$$p_n(x) = x(x + n \cdot \alpha)^{n-1}.$$

PROOF: If $p_n(x)$ is binomial, then there exists an invertible shift invariant operator P , such that (Roman-Rota [5] p.132)

$$p_n(x) = xP^{-n}x^{n-1}$$

for all $n \geq 1$. Moreover, if P is shift invariant, then $P(1) = 1$ and $p(x) \approx P(x + \alpha)$ for every polynomial p and umbra α . From this we derive

$$P^{-1}p(x) \approx p(x + \alpha) \quad \text{and} \quad P^{-2}p(x) \approx p(x + \alpha + \alpha'),$$

with $\alpha = \alpha'$. Then, in general, we have by induction on n

$$P^{-n}p(x) \approx p(x + n \cdot \alpha).$$

Therefore $p_n(x) = xP^{-n}x^{n-1} \approx x(x + n \cdot \alpha)^{n-1}$.

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