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Some Problems for Non-Homogeneous Fluids with Time Dependent Domains and Convex Sets (**) (***)

Qualche problema per i fluidi non omogenei in domini dipendenti dal tempo ed in insiemi convessi

RIASSUNTO. — Si dimostra l'esistenza di soluzioni deboli delle equazioni dei fluidi viscosi incomprimibili non omogenei in domini con frontiera dipendente dal tempo ed in insiemi convessi dipendenti dal tempo.

1. - INTRODUCTION

The purpose of this paper is to study some problems concerning the motion of a viscous incompressible non-homogeneous fluid. The equations which describe the motion are

$$
\begin{aligned}
&\varrho\partial_t u + \varrho u\cdot\nabla u - \mu\,\Delta u = \varrho f - \nabla p \\
&\partial_t\varrho + u\cdot\nabla\varrho = 0 \qquad\qquad \text{in } Q\,, \\
&\nabla\cdot u = 0
\end{aligned}
\tag{1.1}
$$

where $Q=(0,T)\times\Omega$, Ω is a bounded domain in R^3 with boundary Γ, $0<T<\infty$, and $\partial_t=\partial/\partial t$; moreover $\varrho=\varrho(t)=\varrho(x,t)$ is the density of the fluid, $u=u(t)=u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ the velocity, $f=f(t)=$ $=f(x,t)=(f_1(x,t),f_2(x,t),f_3(x,t))$ the external force, and $p=p(t)=p(x,t)$ the pressure; the constant μ is the viscosity coefficient.

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We complete the system (1.1) with the following initial-boundary conditions

$$
(1.2) \qquad \begin{aligned} u &= 0 \quad \text{on } \Gamma, \\ u(x,0) &= u_0 \quad \text{in } \Omega, \\ \varrho(x,0) &= \varrho_0 . \end{aligned}
$$

The existence of a weak solution for the system (1.1), (1.2) has been proved by S. N. Antonzev - V. A. Kajikhov in [1] with $0<\alpha<\varrho_0<\beta$ (α, β = constants). For this problem see also [9]. Later J. Simon, in [20], proved the existence of a weak solution with $0<\varrho_0<\beta$.

Moreover A. N. Kajikhov in [6], and O. A. Ladyzhenskaya - V. A. Solonnikov in [7] studied the system (1.1), (1.2) in a class of smoother functions.

In this paper we consider the following physically important problems:

PROBLEM 1: *Find a solution of the system* (1.1), (1.2) *when the space region* $\Omega(t)$ *filled by the fluid at time* t *depends on* t.

PROBLEM 2: *Find a solution for a variational inequality associated to system* (1.1), (1.2) *with a time dependent convex set.*

Some papers ([3], [5], [10], [11], [12]) appeared concerning Problem 1 for Navier-Stokes equations (ϱ is constant). In particular, H. Fujita - N. Sauer in [3] proved the existence in Hopf's class by a sort of a penalty method. Furthermore, as far as I know, the only existence result for Problem 2 has been obtained by M. Biroli in [2] for the evolution Navier-Stokes equations in the two dimensional case, and the convex set depends on time smoothly. The existence of a weak solution for Problem 1 was proved in [17] by using the Rothe method and an elliptic regularization assuming $0<\alpha<\varrho_0<\beta$. Furthermore Problem 1 was considered in [18] using the method of H. Fujita - N. Sauer for a diffusion model of an inhomogeneous fluid. Problem 2 was considered in [15] with particular time dependent convex sets, where it is also given the physical meaning of the variational inequality studied.

In this paper we prove the *existence of a weak solution of Hopf's class for Problem* 1 *with* $0<\varrho_0<\beta$ by using the method of H. Fujita - N. Sauer combined with the elliptic regularization. This fact permits to use directly compactness theorem, valid in cylindrical domains, proved by J. Simon in [20]. We notice the method of the elliptic regularization was used by the author in [16] and improved in [19] for the Navier-Stokes equations in non-cylindrical domains.

Problem 2 is investigated by using the penalty method.

The outline of the paper is as follows.

Section 2 is devoted to the notations, and to the statements of Problems 1, 2. In section 3 we prove the existence of a weak solution of Problem 1. Section 4 is devoted to the existence proof of a weak solution of Problem 2.

2. - Notations and preliminaries

We consider the flow of the fluid in a space domain $\Omega(t)$ at time $t \in [0, T]$. We assume that $\Omega(t)$ is a bounded domain in R^3. As t increases over $[0, T]$, $\Omega(t)$ generates a (t, x)-domain Ω_T and $\Gamma(t)$, the boundary of $\Omega(t)$, generates a (t, x)-hypersurface Γ_T. We assume Γ_T is a C^1-hypersurface.

Problem 1, in the classical form, is an initial-boundary value problem in which one requires to find u, ϱ, p satisfying

$$\text{(2.1)} \qquad \begin{aligned} &\varrho\partial_t u + \varrho u\cdot\nabla u - \mu\,\Delta u = \varrho f - \nabla p \\ &\partial_t\varrho + u\cdot\nabla\varrho = 0 \qquad\qquad \text{in } \Omega_T\,, \\ &\nabla\cdot u = 0 \end{aligned}$$

with the initial-boundary conditions

$$\text{(2.2)} \qquad \begin{aligned} &u = 0 && \text{on } \Gamma_T\,, \\ &u(x, 0) = u_0\,, \quad \varrho(x, 0) = \varrho_0 && \text{in } \Omega(0)\,, \end{aligned}$$

where

$$u\cdot\nabla u = \sum_{i=1}^{3} u_i\partial_{x_i} u\,, \qquad \nabla\cdot u = \sum_{i=1}^{3}\partial_{x_i} u_i\,.$$

Now we give the weak formulation of the Problem 1. To this end we need particular functional spaces. Let Ω be an arbitrary domain in R^3. We put

$$\begin{aligned} D(\Omega) &= \{\varphi : \varphi \in (C_0^\infty(\Omega))^3, \nabla\cdot\varphi = 0\}\,; \\ H(\Omega) &= \text{the completion of } D(\Omega) \text{ under the } (L^2(\Omega))^3\text{-norm}\,; \\ V(\Omega) &= \text{the completion of } D(\Omega) \text{ under the } (H^1(\Omega))^3\text{-norm}\,; \end{aligned}$$

($H^s(\Omega)$ is the usual Sobolev space of order s on $L^2(\Omega)$).

$$(u, v)_\Omega = \sum_{i=1}^{3}\int_\Omega u_i v_i\,dx\,; \qquad |v|_\Omega^2 = (v, v)_\Omega\,;$$

$$((u, v)) = \sum_{i,j=1}^{3}\int_\Omega \partial_{x_j} u_i\,\partial_{x_j} v_i\,dx\,; \qquad \|u\|_\Omega^2 = ((u, u))_\Omega\,.$$

and in general $\|\cdot\|_L$ denotes the norm in the space L.

Let G be an arbitrary domain in (x, t)-space closed in $t = T$, and $t = 0$. For functions u defined in G we define

$$v^2(u) = \sum_{i,j=1}^{3}\int_G \partial_{x_j} u_i\,\partial_{x_j} u_i\,dx\,dt$$

whenever the integral above makes sense. Then we introduce

$$D(G) = \{\varphi\colon \varphi \in (C^\infty(G))^3,\ \operatorname{supp}\varphi \subseteq G,\ \nabla\cdot\varphi = 0\},$$
$H(G) =$ the completion of $D(G)$ under the $(L^2(G))^3$ norm ;
$V(G) =$ the completion of $D(G)$ under the norm $\nu(\varphi)$.

We set the definition of weak solutions of Problem 1 (for semplicity we set $\mu = 1$).

(u, ϱ) is a *weak solution* of Problem 1 if

i) $u \in V(\Omega_T)$; $\varrho \in L^\infty(\Omega_T)$; $\sqrt{\varrho}\, u \in L^\infty(0, T; L^2(\Omega(t)))$.

ii) $\forall \varphi \in D(\Omega_T)$ with $\varphi(T) = 0$ the equality

$$\int_0^T \{(\varrho u, \partial_t \varphi)_{\Omega(t)} - ((u, \varphi))_{\Omega(t)} + (\varrho u, u\cdot\nabla\varphi)_{\Omega(t)} + (\varrho f, \varphi)_{\Omega(t)}\}\, dt = -(\varrho_0 u_0, \varphi(0))_{\Omega(0)} \tag{2.3}$$

holds:

iii) $\partial_t \varrho + u\cdot\nabla\varrho = 0$ in the distributions sense.

We remark there is no essential difference between Problem 1 with $u|_{\Gamma(t)} = 0$ and with $u|_{\Gamma(t)} = b$ so far as the existence of weak solutions is concerned.

Now we set Problem 2.

Let $C(t)$ be a closed convex set in $H(\Omega(t))$ at time t. As t increases over $[0, T]$, $C(t)$ generates a set $\bigcup_{0\le t\le T} C(t) \subseteq L^2(0, T; H(\Omega(t)))$.

We shall find functions u, ϱ such that

$$\begin{aligned} &(\varrho\partial_t u + \varrho u\cdot\nabla u, v - u) + ((u, v - u)) - (\varrho f, v - u) \geq 0,\\ &\partial_t \varrho + u\cdot\nabla\varrho = 0,\\ &u(0) = u_0, \quad \varrho(0) = \varrho_0, \end{aligned} \tag{2.4}$$

with u, v defined in $Q = \Omega\times[0, T]$ and $u, v \in C(t)$, $\forall t \in [0, T]$. Here

$$(u, v) = (u, v)_\Omega ; \quad ((u, v)) = ((u, v))_\Omega .$$

We give the definition of weak solutions of (1.2), (2.4).

(u, ϱ) is a *weak solution* of (1.2), (2.4) if (see [14])

$$\int_0^T \{(\varrho\partial_t v, v - u) + (\varrho u\cdot\nabla v, v - u) + ((u, v - u)) - (\varrho f, v - u)\}\, dt \geq$$
$$\geq -\tfrac{1}{2}|\sqrt{\varrho_0}(v(0) - u_0)|^2 ;$$

$$\begin{aligned} &\partial_t \varrho + u\cdot\nabla\varrho = 0 \quad \text{in the distributions sense} ;\\ &u \in L^2(0, T; V(\Omega)),\ u(t) \in C(t) \text{ a.e. in } (0, T); \quad \varrho \in L^\infty(Q);\\ &\sqrt{\varrho}\, u \in L^\infty(0, T; H(\Omega(t))); \quad v \in H^1(0, T; V(\Omega)),\ v(t) \in C(t)\\ &\qquad\qquad \text{a.e. in } (0, T) \text{ and } v(T) = 0 . \end{aligned} \tag{2.5}$$

Now we state the assumptions made throughout the present paper.

Assumption 1: *Γ_T of class C^1.*

Let $P(t)$ the projection operator from $H(\Omega)$ onto $C(t)$, and $m>0$ an integer. Now for every decomposition of $(0, T)$ in m intervals (t_i, t_{i+1}) we denote by C_i^m the hull convex of $\bigcup_{t_i<t\leq t_{i+1}} C(t)$.

We have

$$C^m = \bigcup_{i=1}^{m} C_i^m \supset \bigcup_{0<t<T} C(t)\,.$$

We assume

$$t_{i+1} - t_i = T/m \qquad (i \in (0, 1, 2, \ldots, m-1))\,,$$

$$\bigcap_{m=1}^{\infty} C^m = \bigcup_{0<t<T} C(t)\,, \quad 0 \in C(t)\,,\ \forall t>0\,.$$

We denote by P_i^m the projection operator from $H(G)$ onto C_i^m and by $\chi_i^m(t)$ the characteristic function of the interval (t_i, t_{i+1}). We state the following assumptions on the convex set.

Assumption 2:

$$\lim_{m\to\infty} \int_0^T \left| \sum_{i=1}^{m} \chi_i^m(t)(P_i^m - P(t))\, v(t) \right|^2 dt = 0 \qquad \forall v \in L^2(Q)\,.$$

Assumption 3:

$$\sup_{d>0} \frac{1}{d^3} \int_t^{t+d} |(P_{i+1}^m - P_i^m)\, u(t)|^2\, dt < c$$

uniformly with respect to u, t, i, and with $d = T/m$, and c a fixed constant (in the following c denotes different constants).

Now we state our results.

Theorem 1: *Let*

$$u_0 \in H(\Omega(0))\,; \quad f \in L^2(\Omega_T)\,; \quad \varrho_0 \in L^\infty(\Omega(0))\,; \quad 0 < \varrho_0 < \beta$$

(β is a positive constant). Furthermore the Assumption 1 *holds. Then there exists a weak solution of Problem* 1.

Theorem 2: *We assume*

$$u_0 \in H(\Omega) \cap C(0)\,; \quad f \in L^2(0, T; H(\Omega))\,;$$
$$\varrho_0 \in L^\infty(\Omega)\,; \qquad 0 < \varrho_0 < \beta\,;$$

0 is an interior point of $\bigcap_{0 \le t \le T} C(t)$. *Furthermore Assumption 2 holds. Then there exists a weak solution of Problem 2.*

THEOREM 3: *We assume*

$$u_0 \in H(\Omega) \cap C(0) ; \quad f \in L^2(0, T; H(\Omega)) ;$$
$$\varrho_0 \in L^\infty(\Omega) ; \quad 0 < \varrho_0 < \beta .$$

Furthermore the Assumptions 2 and 3 hold. Then there exists a weak solution of Problem 2.

3. - PROOF OF THEOREM 1

Let us begin by considering the following approximating problem.

3.1. - *Auxiliary problem.*

We introduce an auxiliary bounded domain B in R^3 such that the boundary ∂B is smooth, $\Omega(t) \subseteq B$, $\forall t \in [0, T]$, and $\operatorname{dist}(\partial B, \Gamma(t)) > \gamma > 0$, $\forall t \in [0, T]$ (dist = distance, and γ = constant). We put $B_T = [0, T] \times B$. Let $E = B_T - \Omega_T$, and χ_E the characteristic function of E. Furthermore $\hat{u}_0$ is the natural extension to B of u_0 that is $\hat{u}_0 = u_0$ in $\Omega(0)$ and $\hat{u}_0 = 0$ in $B - \Omega(0)$. We let $\varrho_0^\varepsilon \in C^1(B)$ with

$$0 < \sqrt{\varepsilon} < \varrho_0^\varepsilon < \alpha + \sqrt{\varepsilon} \tag{3.11}$$

and

$$\varrho_0^\varepsilon \to \varrho_0 \quad \text{strongly in } L^p(\Omega) \text{ for some } p > 1 .$$

We denote again by f the extension by zero to $B_T \setminus \Omega_T$ of f.

We consider the following auxiliary problem.

We look for u_m^ε, ϱ_m^ε defined in B_T such that

$$u_m^\varepsilon \in H^1(B_T) \cap L^2(0, T; V(B)) ; \tag{3.12}$$

$$\varrho_m^\varepsilon \in C^1(B_T) ; \tag{3.13}$$

$$\begin{aligned} \int_0^T \{ \varepsilon(\partial_t u_m^\varepsilon, \partial_t \varphi)_B + ((u_m^\varepsilon, \varphi))_B - (\varrho_m^\varepsilon u_m^\varepsilon, \bar{u}_m^\varepsilon \cdot \nabla \varphi)_B - \\ - (\varrho_m^\varepsilon u_m^\varepsilon, \partial_t \varphi)_B - (\varrho_m^\varepsilon f, \varphi)_B + m(\chi_E u_m^\varepsilon, \varphi)_B \} \, dt = \\ = (\varrho_0^\varepsilon \hat{u}_0, \varphi(0))_B - (\varrho_m^\varepsilon(T) u_m^\varepsilon, \varphi(T))_B ; \end{aligned} \tag{3.14}$$

$$\partial_t \varrho_m^\varepsilon + \bar{u}_m^\varepsilon \cdot \nabla \varrho_m^\varepsilon = 0 . \tag{3.15}$$

Here $\bar{u}_m^\varepsilon$ is obtained by a regularization of u_m^ε by a mollifier (in x and t) depending on ε, and by projection on $H(B)$.

Assuming u^ε_m is known, the existence of the solution ϱ^ε_m of the continuity equation satisfying (3.13), (3.15) follows from standard techniques, using the method of characteristics (see [7]).

Next we consider the existence of a solution of (3.12), (3.14). We set

$$a(\varrho^\varepsilon_m, u^\varepsilon_m, \varphi^\varepsilon_m) = \int_0^T \{\varepsilon(\partial_t u^\varepsilon_m, \partial_t \varphi)_B + ((u^\varepsilon_m, \varphi))_B -$$

$$- (\varrho^\varepsilon_m u^\varepsilon_m, \bar{u}^\varepsilon_m \cdot \nabla\varphi)_B - (\varrho^\varepsilon_m u^\varepsilon_m, \partial_t \varphi)_B + m(\chi_B u^\varepsilon_m, \varphi)_B\}\, dt + (\varrho^\varepsilon_m(T) u^\varepsilon_m(T), \varphi(T))_B ;$$

$$L(\varphi) = \int_0^T (\varrho^\varepsilon_m f, \varphi)_B\, dt + (\varrho^\varepsilon_0 u_0, \varphi(0))_B .$$

L is a continuous form in $H^1(B_T)$.

We note

$$a(\varrho^\varepsilon_m, u^\varepsilon_m, u^\varepsilon_m) > c_{\varepsilon m} \|u^\varepsilon_m\|_{H^1(B_T)} \tag{3.16}$$

and the form $u^\varepsilon_m \to a(\varrho^\varepsilon_m, u^\varepsilon_m, \varphi)$ is weakly continuous in

$$H^1(B_T) \cap L^2(0, T; V(B)) .$$

In fact, bearing in mind that if $v_n \to v$ weakly in $H^1(B_T)$ then $v_n \to v$ strongly in $L^2(B_T)$, the weak continuity of the form $a(\varrho^\varepsilon_m, u^\varepsilon_m, \varphi)$ is obvious.

Now from the continuity equation (3.15), we have

$$\frac{1}{2}\big((\varrho^\varepsilon_m(T) u^\varepsilon_m(T), u^\varepsilon_m(T))_B - (\varrho^\varepsilon_m(0) u^\varepsilon(0), u^\varepsilon(0))_B\big) =$$

$$= \int_0^T \{(\varrho^\varepsilon_m \bar{u}^\varepsilon_m \cdot \nabla u^\varepsilon_m, u^\varepsilon_m)_B + (\varrho^\varepsilon_m u^\varepsilon_m, \partial_t u^\varepsilon_m)_B\}\, dt .$$

Thus from (3.14) we have

$$a(\varrho^\varepsilon_m, u^\varepsilon_m, u^\varepsilon_m) = \int_0^T \{\varepsilon(\partial_t u^\varepsilon_m, \partial_t u^\varepsilon_m)_B + ((u^\varepsilon_m, u^\varepsilon_m))_B + m(\chi_B u^\varepsilon_m, u^\varepsilon_m)_B\}\, dt +$$

$$+ \frac{1}{2}\big((\varrho^\varepsilon_m(T) u^\varepsilon_m(T), u^\varepsilon_m(T))_B + (\varrho^\varepsilon_m(0) u^\varepsilon_m(0), u^\varepsilon_m(0))_B\big) > c_{\varepsilon m} \|u^\varepsilon_m\|_{H^1(B_T)} .$$

Then, for a well known theorem (see [4], p. 106), there exists a solution u^ε_m of (3.14).

To passing to the limit $m \to \infty$ and $\varepsilon \to 0$, we will need a priori estimates of the approximations u^ε_m, ϱ^ε_m.

3.2. - *Standard a priori estimates.*

From the method of the characteristics, (see [7]), by virtue of (3.11), one has

$$0<\sqrt{\varepsilon}<\varrho_m^\varepsilon<\alpha+\sqrt{\varepsilon}\,. \tag{3.21}$$

Now, we replace in (3.14) φ by u_m^ε, it gives

$$\int_0^T \{\varepsilon|\partial_t u_m^\varepsilon|_B^2+\|u_m^\varepsilon\|_B^2-(\varrho_m^\varepsilon u_m^\varepsilon,\partial_t u_m^\varepsilon)_B-(\varrho_m^\varepsilon \bar u_m^\varepsilon\cdot\nabla u_m^\varepsilon, u_m^\varepsilon)_B+$$
$$+m|\chi_E u_m^\varepsilon|_B^2-(\varrho_m^\varepsilon f, u_m^\varepsilon)_B\}\,dt=(\varrho_0^\varepsilon \bar u_0^\varepsilon, u_m^\varepsilon(0))_B-(\varrho_m^\varepsilon(T)u_m^\varepsilon(T), u_m^\varepsilon(T))_B\,.$$

Bearing in mind the continuity equation (3.15), after some calculations, one has

$$\begin{gathered}\int_0^T \varepsilon|\partial_t u_m^\varepsilon|_B^2\,dt<c\,;\quad \int_0^T \|u_m^\varepsilon\|_B^2\,dt<c\,;\quad |\sqrt{\varrho_m^\varepsilon(0)}\,u_m^\varepsilon(0)|_B<c\,;\\ \int_0^T m|\chi_E u_m^\varepsilon|_B^2\,dt<c\,;\quad |\sqrt{\varrho_m^\varepsilon(T)}\,u_m^\varepsilon(T)|_B<c\,.\end{gathered} \tag{3.22}$$

here c is a constant independent of m and ε.

(3.21), (3.22) and the regularity of ϱ_0^ε imply for $m\to\infty$

$$\begin{aligned} u_m^\varepsilon &\to u^\varepsilon && \text{weakly in } L^2(0,T;V(B))\,;\\ u_m^\varepsilon &\to 0 && \text{strongly in } L^2(E)\,;\\ \partial_t u_m^\varepsilon &\to \partial_t u^\varepsilon && \text{weakly in } L^2(B_T)\,;\\ \varrho_m^\varepsilon &\to \varrho^\varepsilon && \text{weakly* in } L^\infty(B)\,;\\ \varrho_m^\varepsilon &\to \varrho^\varepsilon && \text{weakly in } H^1(B_T)\,.\end{aligned} \tag{3.23}$$

From the compactness theorem in [8], p. 58, (3.23) implies

$$\begin{aligned} u_m^\varepsilon &\to u^\varepsilon && \text{strongly in } L^2(B_T)\\ \varrho_m^\varepsilon &\to \varrho^\varepsilon && \text{strongly in } L^2(B_T)\,.\end{aligned}$$

consequently

$$\begin{aligned} \varrho_m^\varepsilon \bar u_{im}^\varepsilon u_{jm}^\varepsilon &\to \varrho^\varepsilon \bar u_i^\varepsilon u_j^\varepsilon && \text{weakly in } L^2(B_T)\,;\\ \varrho_m^\varepsilon u_m^\varepsilon &\to \varrho^\varepsilon u^\varepsilon && \text{weakly in } L^2(B_T)\,.\end{aligned}$$

Furthermore, we note $u^\varepsilon=0$ on E whence

$$\|u^\varepsilon\|_{L^2(\Gamma(t))}<c\left(|u^\varepsilon|_E+|u^\varepsilon|_E^{1/2}\|u^\varepsilon\|_E^{1/2}\right)=0$$

consequently

$$u^\varepsilon = 0 \quad \text{on } \Gamma_T .$$

Passing to the limit $m \to \infty$, (3.23), (3.24) imply that u^ε satisfies

$$\text{(3.25)} \qquad \int_0^T \{\varepsilon(\partial_t u^\varepsilon, \partial_t \varphi)_B + ((u^\varepsilon, \varphi))_B - (\varrho^\varepsilon u^\varepsilon, \bar{u}^\varepsilon \cdot \nabla\varphi)_B - (\varrho^\varepsilon u^\varepsilon, \partial_t \varphi)_B - (\varrho^\varepsilon f, \varphi)_B\}\, dt = \\ = (\varrho_0^\varepsilon \hat{u}_0, \varphi(0))_B - (\varrho^\varepsilon(T) u^\varepsilon(T), \varphi(T))_B ;$$

$\forall \varphi \in H^1(B_T) \cap L^2(0, T; V(B))$ with support in Ω_T.

$$\text{(3.26)} \qquad \partial_t \varrho^\varepsilon + \bar{u}^\varepsilon \cdot \nabla \varrho_m^\varepsilon = 0 \quad \text{a.e. in } B_T .$$

We note $\varrho^\varepsilon \in C^1(B_T)$ again (see [7], p. 707).

Now by virtue of (3.22), there exists a subsequence of $(\varrho^\varepsilon, u^\varepsilon)$, still denoted by $(\varrho^\varepsilon, u^\varepsilon)$, such that

$$\begin{aligned} u^\varepsilon &\to u && \text{weakly in } V(B_T) ; \\ \varrho^\varepsilon &\to \varrho && \text{weak* in } L^\infty(B_T) ; \\ \partial_t \varrho^\varepsilon &\to \partial_t \varrho && \text{weakly in } L^2(0, T; H^{-1}(B)) . \end{aligned}$$

The compactness theorem in [8] p. 58 and (3.22) imply

$$\begin{aligned} \varrho^\varepsilon &\to \varrho && \text{strongly in } L^2(0, T; H^{-1}(B)) ; \\ \varrho^\varepsilon u^\varepsilon &\to \varrho u && \text{weakly in } L^2(B_T) . \end{aligned}$$

Now reexamining (3.25) and bearing in mind $u^\varepsilon = 0$ in E, we notice (3.25) continues to hold $\forall \varphi \in H^1(B_T) \cap L^2(0, T; V(B))$, with $\varphi = 0$ on Γ_T. Now a weak solution of Problem 1 could be obtined with u and ϱ provided that all terms in (3.25) converge to the corresponding terms in (2.3). We obtain this if $\varrho^\varepsilon u^\varepsilon$ converges to ϱu strongly in $L^2(0, T; H^{-1}(B))$, for example. To this we need to estimate a time difference quotient.

3.3. - *Time difference quotient estimates.*

We denote by $\bar{\bar{u}}^\varepsilon$ the extension by 0 of u^ε for $t < 0$. We let

$$u_h^\varepsilon = \frac{1}{h} \int_{t-h}^{t} \bar{\bar{u}}^\varepsilon(s)\, ds$$

where h is a positive number.

We consider furthermore the function ψ_h^ε solution of

$$
(3.31)\qquad
\begin{aligned}
-\Delta\psi_h^\varepsilon + \nabla p^\varepsilon &= 0 \quad \text{in } \Omega(t),\\
\nabla\cdot\psi_h^\varepsilon &= 0\\
\psi_h^\varepsilon &= u_h^\varepsilon \quad \text{on } \Gamma(t).
\end{aligned}
$$

To estimate $\partial_t\psi_h^\varepsilon$, we formally differentiate (3.31) with respect to t and we consider $\partial_t\psi_h^\varepsilon$ as a generalized solution of the problem

$$
(3.32)\qquad
\begin{aligned}
-\Delta\partial_t\psi_h^\varepsilon + \nabla\partial_t p^\varepsilon &= 0 && \text{in } \Omega(t),\\
\nabla\cdot\partial_t\psi_h^\varepsilon &= 0\\
\partial_t\psi_h^\varepsilon &= -(\tilde{u}^\varepsilon(x, t-b))/b && \text{on } \Gamma(t).
\end{aligned}
$$

Bearing in mind the smoothness of $\Gamma(t)$, from well known results on Stokes problem (see [21]), se have that

$$\psi_h^\varepsilon, \partial_t\psi_h^\varepsilon \in L^2\big(0, T, H^1(\Omega(t))\big)$$

and the following estimates hold

$$(3.33)\qquad \|\psi_h^\varepsilon\|_{L^2(0,T;H^1(\Omega(t)))} \leq c\|u_h^\varepsilon\|_{L^2(0,T,H^{1/2}(\Gamma(t)))} \leq c/\sqrt{b}\,\|u^\varepsilon\|_{L^2(0,T;V(B))};$$

$$
\begin{aligned}
(3.34)\qquad \|\partial_t\psi_h^\varepsilon\|_{L^2(\Omega(t))} &\leq c/b\,\|\tilde{u}^\varepsilon(t-b)\|_{H^{1/2}(\Gamma(t))} \leq\\
&\leq (c/b)\,\big(\text{measure}\,(\Omega(t)-\Omega(t-b))\big)^{1/2}\,\|\tilde{u}^\varepsilon(t-b)\|_{B}.
\end{aligned}
$$

Furthermore, we denote by θ_h^ε a function which satisfies the Stokes problem as (3.32) in $B-\Omega(t)$ with the following boundary conditions

$$
\begin{aligned}
\theta_h^\varepsilon &= 0 \quad \text{on } \partial B,\\
\theta_h^\varepsilon &= u_h^\varepsilon \quad \text{on } \Gamma(t).
\end{aligned}
$$

For θ_h^ε the estimates (3.33), (3.34) hold.

Now we denote by φ_h^ε the function equal to ψ_h^ε in $\Omega(t)$ and equal to θ_h^ε in $B-\Omega(t)$.

Now we can replace in (3.25) φ by $u_h^\varepsilon - \varphi_h^\varepsilon$ and we get

$$
\begin{aligned}
(3.35)\qquad &\int_0^T \Big\{\varepsilon/b(\partial_t u^\varepsilon, \tilde{u}^\varepsilon(t) - \tilde{u}^\varepsilon(t-b))_B - \varepsilon(\partial_t u^\varepsilon, \partial_t\varphi_h^\varepsilon)_B + ((u^\varepsilon, u_h^\varepsilon))_B -\\
&-((u^\varepsilon, \varphi_h^\varepsilon))_B + (\varrho^\varepsilon u^\varepsilon, \tilde{u}^\varepsilon\cdot\nabla(u_h^\varepsilon-\varphi)_h^\varepsilon)_B + (\varrho^\varepsilon u^\varepsilon, \partial_t\varphi_h^\varepsilon)_B -\\
&-1/b(\varrho^\varepsilon(t)u^\varepsilon(t), \tilde{u}^\varepsilon(t) - \tilde{u}^\varepsilon(t-b))_B + (\varrho^\varepsilon f, \varphi)_B\Big\}\,dt =\\
&= (\varrho_0^\varepsilon \hat{u}_0, \varphi_h^\varepsilon(0))_B - (\varrho^\varepsilon(T)u^\varepsilon(T), \varphi_h^\varepsilon(T))_B - (\varrho_0^\varepsilon\hat{u}_0, u_h^\varepsilon(0))_B + (\varrho^\varepsilon(T)u^\varepsilon(T), u_h^\varepsilon(T))_B.
\end{aligned}
$$

Now we denote by s^ε the extension of u^ε by $u^\varepsilon(0)$ for $t<0$.

By virtue of (3.22), Jensen inequality and the smoothness of Γ_T, one has

$$\varepsilon\left|\int_0^T \varrho^\varepsilon\big(\partial_t u^\varepsilon(t), (\bar{u}^\varepsilon(t)-\bar{u}^\varepsilon(t-h))/h\big)_H\, dt\right| <$$

$$<\int_0^T \varepsilon|\partial_t u^\varepsilon(t)|_H\Big(|(\bar{u}^\varepsilon(t)-v^\varepsilon(t-h))/h|_H+|(v^\varepsilon(t-h)-\bar{u}^\varepsilon(t-h))/h|_H\Big)dt <$$

$$<c\varepsilon\int_0^T |\partial_t u^\varepsilon|_H^2\, dt + c\varepsilon/(\sqrt{h})|u^\varepsilon(0)|_H^2 + c\varepsilon/\sqrt{h}\int_0^T |\partial_t u^\varepsilon|_H^2\, dt <$$

$$<c/\sqrt{h}+c/\sqrt{h}\,|\sqrt{\varrho^\varepsilon(0)}\,u^\varepsilon(0)|^2 < c/\sqrt{h}\,;$$

$$\left|\int_0^T ((u^\varepsilon, u_h^\varepsilon))_H\, dt\right| < \int_0^T \|u^\varepsilon\|_H \left\|1/h\int_{t-h}^t \bar{u}^\varepsilon(s)\, ds\right\|_H dt <$$

$$< 1/\sqrt{h}\int_0^T \|u^\varepsilon\|_H\left(\int_{t-h}^t \|\bar{u}^\varepsilon(s)\|_H^2\, ds\right)^{1/2} dt < c/\sqrt{h}\,;$$

(3.36)

$$\left|\int_0^T \left(\varrho^\varepsilon \bar{u}^\varepsilon\cdot\nabla\int_{t-h}^t \bar{u}^\varepsilon(s)\, ds, u^\varepsilon\right) dt\right| < c\int_0^T \|u^\varepsilon\|_H^2 \left|1/h\int_{t-h}^t \bar{u}^\varepsilon(s)\, ds\right|_H dt <$$

$$< c/\sqrt{h}\int_0^T \|u^\varepsilon\|_H^2\left(\int_{t-h}^t \|\bar{u}^\varepsilon(s)\|_H^2\, ds\right)^{1/2} dt < c/\sqrt{h}\,;$$

$$\left|\int_0^T (\varrho^\varepsilon f, u_h^\varepsilon)_H\, dt\right| < c\int_0^T |f|_H \left|1/h\int_{t-h}^t \bar{u}^\varepsilon(s)\, ds\right|_H dt < c/\sqrt{h}\,;$$

$$|(\varrho^\varepsilon(T)u^\varepsilon(T), u_h^\varepsilon(T))_H| < c|\sqrt{\varrho^\varepsilon(T)}\,u^\varepsilon(T)|_H \left|1/h\int_{T-h}^T \bar{u}^\varepsilon(s)\, ds\right|_H <$$

$$< c/\sqrt{h}\left(\int_0^T |u^\varepsilon|_H^2\, dt\right)^{1/2} < c/\sqrt{h}\,;$$

Thanks to (3.33), (3.34), similar estimates hold for the terms containing φ_h^ε in (3.35).

Finally we will estimate

$$-1/h\int_0^T (\varrho^\varepsilon(t)u^\varepsilon(t), \bar{u}^\varepsilon(t)-\bar{u}^\varepsilon(t-h))_H\, dt\,.$$

Thanks to (3.22) one has

$$
\begin{aligned}
(3.37)\qquad & -1/h\int_0^T\left(\varrho^\varepsilon(t)w^\varepsilon(t),\tilde{w}^\varepsilon(t)-\tilde{w}^\varepsilon(t-h)\right)_H dt=\\
&=-1/h\int_0^T\left|\sqrt{\varrho^\varepsilon(t)}\,\tilde{w}^\varepsilon(t)\right|_H^2 dt+1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\,\tilde{w}^\varepsilon(t)\right|_H^2+\\
&\quad+1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\,\tilde{w}^\varepsilon(t-h)\right|_H^2 dt-\\
&\quad-1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\left(\tilde{w}^\varepsilon(t)-\tilde{w}^\varepsilon(t-h)\right)\right|_H^2 dt=\\
&=-1/(2h)\int_0^T\left|\sqrt{\varrho^\varepsilon(t)}\,\tilde{w}^\varepsilon(t)\right|_H^2 dt+1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t-h)}\,\tilde{w}^\varepsilon(t-h)\right|_H^2 dt+\\
&\quad+1/(2h)\int_h^T\left(\left(\varrho^\varepsilon(t)-\varrho^\varepsilon(t-h)\right)\tilde{w}^\varepsilon(t-h),\tilde{w}^\varepsilon(t-h)\right)_H dt-\\
&\quad-1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\left(\tilde{w}^\varepsilon(t)-\tilde{w}^\varepsilon(t-h)\right)\right|_H^2 dt<c_1\sqrt{h}+\\
&\quad+\left|\int_h^T\left(\left(\nabla\cdot\int_{t-h}^t\tilde{w}^\varepsilon(s)\,\varrho^\varepsilon(s)\,ds\right)\tilde{w}^\varepsilon(t-h),\tilde{w}^\varepsilon(t-h)\right)_H dt\right|-\\
&\quad-1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\left(\tilde{w}^\varepsilon(t)-\tilde{w}^\varepsilon(t-h)\right)\right|_H^2 dt<c_1\sqrt{h}+\\
&\quad+2/h\left|\int_h^T\left(\left(\int_{t-h}^t\tilde{w}^\varepsilon(s)\,\varrho^\varepsilon(s)\,ds\right)\tilde{w}^\varepsilon(t-h),\nabla\tilde{w}^\varepsilon(t-h)\right)_H dt\right|-\\
&\quad-1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\left(\tilde{w}^\varepsilon(t)-\tilde{w}^\varepsilon(t-h)\right)\right|_H^2 dt<c_1\sqrt{h}+\\
&\quad+c_1\sqrt{h}\int_0^T\|\tilde{w}^\varepsilon\|_H^2\left(\int_{t-h}^t\|\tilde{w}^\varepsilon(s)\|_H^2\,ds\right)^{1/2}dt-\\
&\quad-1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\left(\tilde{w}^\varepsilon(t)-\tilde{w}^\varepsilon(t-h)\right)\right|_H^2 dt<c_1\sqrt{h}-\\
&\qquad\qquad-1/(2h)\int_h^T\left|\sqrt{\varrho^\varepsilon(t)}\left(\tilde{w}^\varepsilon(t)-\tilde{w}^\varepsilon(t-h)\right)\right|_H^2 dt\,.
\end{aligned}
$$

(3.36), (3.37) imply

$$\int_0^{T-h} |\sqrt{\varrho^\varepsilon(t+h)}(\bar{u}^\varepsilon(t+h)-\bar{u}^\varepsilon(t))|_B^2\,dt < c\sqrt{h}\,.$$

Now ϱ^ε belongs to a bounded set in $L^\infty(B_T)$ consequently

$$\int_0^{T-h} |\varrho^\varepsilon(t+h)(\bar{u}^\varepsilon(t+h)-\bar{u}^\varepsilon(t))|_B^2\,dt < c\sqrt{h}\,. \tag{3.38}$$

From (3.26), $\partial_t\varrho^\varepsilon$ belongs to a bounded set in $L^2(0, T; H^{-1}(B))$ then ϱ^ε satisfies the inequality

$$\|\varrho^\varepsilon(t+h)-\varrho^\varepsilon(t)\|_{L^2(0,T-h;H^{-1}(B))} < ch\,. \tag{3.39}$$

Now the mapping $(\varrho^\varepsilon, u^\varepsilon)\to\varrho^\varepsilon u^\varepsilon$ is continuous from $H^{-1}(B)\times H_0^1(B)$ to $W^{-1,r}(B)$ with $r<\frac{3}{2}$ ($W^{-1,r}$ is the dual of the Sobolev space $W_0^{1,r'}$) with $1/r+1/r'=1$. Hence adding (3.38), (3.39) we obtain

$$\int_0^{T-h} \|\varrho^\varepsilon(t+h)\bar{u}^\varepsilon(t+h)-\varrho^\varepsilon(t)\bar{u}^\varepsilon(t)\|_{W^{-1,r}(B)}^2\,dt < c\sqrt{h}\,.$$

Thanks to the compactness theorem due to J. Simon in [20], the immersion of

$$\Psi=\left\{\varphi:\varphi\in L^2(B_T);\ \sup_{h>0} 1/\sqrt{h}\int_0^{T-h}\|\varphi(t+h)-\varphi(t)\|_{W^{-1,r}(B)}\,dt<\infty\right\}$$

from $L^2(B_T)$ in $L^2(0, T; H^{-1}(B))$ is compact.

This theorem and the above estimates imply there exists a subsequence of $(\varrho^\varepsilon, u^\varepsilon)$, still denoted by $(\varrho^\varepsilon, u^\varepsilon)$, such that

$$\varrho^\varepsilon u^\varepsilon\to\varrho u \quad \text{strongly in } L^2(0, T; H^{-1}(B))\,. \tag{3.40}$$

By virtue of (3.32), (3.40), we have $\forall\varphi\in C_0^\infty(B_T)$

$$\lim_{\varepsilon\to 0}\int_0^T (\varrho^\varepsilon\bar{u}^\varepsilon, \bar{u}^\varepsilon\cdot\nabla\varphi)_B\,dt=\int_0^T(\varrho\bar{u}, u\cdot\nabla\varphi)_B\,dt\,.$$

Now it is a standard matter to pass to the limit $\varepsilon\to 0$ in (3.25), (3.26) and to prove (u, ϱ) satisfies the integral equation (2.3).

It remains to prove

$$\sup_t|\sqrt{\varrho}\,u|_{\Omega(t)}<c\,.$$

We let

$$u_{\bar t}^\varepsilon(t)=\begin{cases} u^\varepsilon(t) & \text{if } 0<t<\bar t\,,\\ 0 & \text{if } t>\bar t\,,\end{cases}$$

here $\bar t$ is an arbitrary point in $(0, T)$.

We replace in (3.25) φ by $u_{\bar t}^\varepsilon(t)$ and after some calculations we obtain

$$\begin{aligned}(3.41)\quad |\sqrt{\varrho^\varepsilon(\bar t)}\,u^\varepsilon(\bar t)|^2_{\Omega(\bar t)} &< \int_0^{\bar t}\varepsilon|\partial_t u^\varepsilon|^2_{\Omega(t)}\,dt+\varepsilon(\partial_t u^\varepsilon(\bar t), u^\varepsilon(\bar t))_{\Omega(\bar t)}+\left|\int_0^{\bar t}(\varrho^\varepsilon f, u^\varepsilon)_{\Omega(t)}\,dt\right|\\ &+|\sqrt{\varrho^\varepsilon(0)}\,u^\varepsilon(0)|^2_{\Omega(0)}+\int_0^{\bar t}\|u^\varepsilon\|^2_{\Omega(t)}\,dt<c+\varepsilon^{3/2}|\partial_t u^\varepsilon(\bar t)|^2_{\Omega(\bar t)}+1/2|\sqrt{\varrho^\varepsilon(\bar t)}\,u(\bar t)|^2_{\Omega(\bar t)}.\end{aligned}$$

Passing to limit in (3.41) one gets

$$\varliminf_{\varepsilon\to 0}|\sqrt{\varrho^\varepsilon(\bar t)}\,u^\varepsilon(\bar t)|^2_{\Omega(\bar t)}<c\,.$$

From (3.41) we have $\{|\sqrt{\varrho^\varepsilon(t)}\,u^\varepsilon(t)|_{\Omega(t)}\}$ is a bounded set in $L^\infty(0, T)$; thanks to (3.22), (3.40) we have

$$|\sqrt{\varrho(t)}\,u(t)|_{\Omega(t)}<c\,.$$

The proof of Theorem 1 is completed.

4. - Proofs of Theorems 2 and 3

Now we prove Theorem 2. We utilize the results in [14], and part of the procedure of the proof of Theorem 1.

As in Theorem 1 one obtains a solution (ϱ^m, u^m) of

$$(4.1)\quad\begin{aligned}&\int_0^T\Big\{(\varrho^m\partial_t v, u^m)+(\varrho^m u^m, \bar u^m\cdot\nabla v)-((u^m, v))-\frac{1}{m}(\partial_t u_m, \partial_t v)\\ &\qquad -m\sum_{i=1}^m(\chi^m(t)(I-P_i^m u^m, v)+(\varrho^m f, v))\Big\}\,dt=-(\varrho_0^m u_0, v(0))\,;\\ &\partial_t\varrho^m+\bar u^m\cdot\nabla\varrho^m=0\,,\end{aligned}$$

with $0<1/m<\varrho_0<\alpha+1/m$, $v\in H^1(0, T; V(\Omega))$ with $v(T)=0$, and $\bar u^m$ is an approximation of u^m by smooth functions of $V(\Omega)$.

Furthermore, the following estimates hold

$$\int_0^T \|u^m\|^2\,dt < c\,; \quad |\sqrt{\varrho^m}(t)u^m(t)| < c\,;$$

$$1/m < \varrho^m < \alpha + 1/m\,; \quad \int_0^T |\partial_t \varrho^m\|^2_{H^{-1}(\Omega)}\,dt < c\,; \tag{4.2}$$

$$m\int_0^T \sum_{i=1}^m |(\chi_i^m(t)(I - P_i^m)u^m)|^2\,dt < c\,,$$

with c a constant independent of m.

Consequently for $m \to \infty$

$$\begin{aligned} u^m &\to u \quad \text{weakly in } L^2(0, T; V(\Omega))\,; \\ \varrho^m &\to \varrho \quad \text{weak* in } L^\infty(Q)\,; \\ \partial_t \varrho^m &\to \partial_t \varrho \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega))\,. \end{aligned} \tag{4.3}$$

Whence, by the compactness theorem in [8], p. 58, we have

$$\varrho^m \to \varrho \quad \text{strongly in } L^2(0, T; H^{-1}(\Omega))\,;$$

consequently

$$\varrho^m u^m \to \varrho u \quad \text{weakly in } L^2(Q)\,.$$

Now we need the estimate

$$\int_0^{T-h} \varrho^m(t)(u^m(t+h)) - u^m(t)|^2\,dt < c\sqrt{h}\,. \tag{4.5}$$

We denote by $\tilde{u}^m$ the extension by zero for $t < 0$ and $t > T$. We let

$$u_h^m(t) = \int_t^{t+h} \tilde{u}^m(s)\,ds\,.$$

We replace in (4.1) v by u_h^m, and we obtain as in § 3

$$\left|\int_0^T ((\varrho^m u^m, \tilde{u}^m \cdot \nabla u_h^m) - ((u^m, u_h^m))(\varrho^m f, u_h^m)\}\,dt - (\varrho_0^m u_0, u_h^m(0))\right| < c\sqrt{h}\,. \tag{4.6}$$

Thanks to (4.2), (4.6), as in § 3, we have

$$(4.7) \qquad \int_0^{T-h} |\varrho^m(t)(u^m(t+h)) - u^m(t)|^2\,dt < \varepsilon\sqrt{h} + \\ + m\int_0^T \sum_{i=1}^m \chi_i^m(t)\left((I-P_i^m)u(t), \int_t^{t+h} \bar{u}^m(s)\,ds\right).$$

Now

$$\frac{1}{\sqrt[4]{h}} \int_t^{t+h} \bar{u}^m(t)\,dt \to 0 \quad \text{ad } h \to 0 \text{ strongly in } L^2(Q),$$

uniformly with respect to m. Hence for every $h < \bar{h}$ with a suitable fixed $\bar{h}$

$$\frac{1}{\sqrt[4]{h}} \int_t^{t+h} \bar{u}^m(s)\,ds \in \bigcap_{i=1}^m C_i.$$

This fact, by virtue of a classical property of P_i^m, implies the last term in (4.7) is $< \varepsilon\sqrt[4]{h}$.

We obtain so (4.5), with $\sqrt[4]{h}$.

By virtue of (4.2) and (4.5), Simon's compactness theorem gives

$$\varrho^m u_i^m u_j^m \to \varrho u_i u_j \quad \text{weakly in } L^2(0, T; L^{3/2}(\Omega)).$$

Now it remains to prove

$$u(t) \in C(t) \quad \forall t \in (0, T).$$

From (4.2) and the assumptions on P^m we have

$$\int_0^T |(I-P(t))\,u(t)|^2\,dt < \varliminf_{m\to\infty} \int_0^T |(I-P(t))\,u^m(t)|^2\,dt <$$

$$< \varliminf_{m\to\infty}\left(\sum_{i=1}^m \left(\int_0^T |\chi_i^m(t)(P_i^m - P(t))\,u^m(t)|^2\,dt\right) + \int_0^T |\chi_i^m(t)(I-P_i^m)u^m(t)|^2\,dt\right) = 0$$

whence

$$\int_0^T |(I-P(t)(u(t)|^2\,dt = 0.$$

Then

$$(I-P(t))\,u(t)=0 \quad \text{a.e. in } (0,T).$$

From this

$$u(t)\in C(t) \quad \text{a.e. in } (0,T).$$

Now, by choosing an arbitrary function $v\in H^1(Q)$, $v(t)\in C(t)$, and $v(T)=0$, after some calculations (see [14]) and passing to the limit $m\to\infty$ in (4.1), we obtain the result.

Now we prove Theorem 3.

To prove this theorem we utilize the proof of Theorem 2. We have to prove only estimates of the penalty term.

First if $b<1/m$

$$m\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left((I-P_i^m)u^m(t),\int_t^{t+h}\bar{u}^m(s)\,ds\right)dt<$$

$$<\sqrt{b}\,m\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left(|(I-P_i^m)u^m(t)|\left(\int_{t_i}^{t_{i+1}+h}|\bar{u}^m(s)|^2\,ds\right)^{1/2}\right)dt<c\sqrt{b}\,.$$

Now we consider the case $b>1/m$. Let $n(b)>0$ an integer, depending on b, such that $n(b)/m<b<(n(b)+1)/m$.

Bearing in mind (4.2), and $1/b\int_t^{t+h}P_i^m\bar{u}^m(s)\,ds\in C_i^m$, we have

$$\frac{m}{b}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left((I-P_i^m)u^m(t),\int_t^{t+h}\bar{u}^m(s)\,ds\right)dt<$$

$$<\frac{m}{b}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left((I-P_i^m)u^m(t),\int_t^{t+h}P_i^m\bar{u}^m(s)\,ds\right)dt+$$

$$+\frac{m}{b}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left((I-P_i^m)u^m(t),\int_t^{t+h}(I-P_i^m)\bar{u}^m(s)\,ds\right)dt<$$

$$<\frac{m}{b}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left((I-P_i^m)u^m(t),\int_t^{t+h}P_i^m\bar{u}^m(s)\,ds\right)dt+$$

$$+\,m/\sqrt{b}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\left(|(I-P_i^m)u^m(t)|\left(\int_t^{t+h}|I-P_i^m\bar{u}^m(s)|^2\,ds\right)^{1/2}\right)dt<c+$$

$$+\,m/\sqrt{b}\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}|(I-P_i^m)u^m(t)|\left(\int_{t_i}^{t_{i+1}+h}|(I-P_i^m)\bar{u}^m(s)|^2\,ds\right)^{1/2}dt<$$

$$<\varepsilon+c_1\sqrt{h}\left(\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}+h}|(I-P_i^m)\bar{u}^m(t)|^2\,dt\right)^{1/2}<\varepsilon+$$

$$+c_1\sqrt{h}\left(\sum_{i=1}^{m}\sum_{j=0}^{n(h)}\int_{t_i+jT/m}^{t_{i+1}+(j+1)T/m}|(I-P_{i+j}^m)\bar{u}^m(t)|^2\,dt\right)^{1/2}+$$

$$+c_1\sqrt{h}\left(\sum_{i=1}^{m}\sum_{j=0}^{n(h)}\sum_{k=0}^{j}\int_{t_i+jT/m}^{t_i+(j+1)T/m}|(P_{i+k}^m-P_{i+k+1}^m)\bar{u}^m(t)|^2\,dt\right)^{1/2}<$$

$$<\varepsilon+c_1\sqrt{h}\sum_{i=1}^{m}\sum_{j=0}^{n(h)}\sum_{k=0}^{j}\frac{1}{m^3}<\varepsilon+c\sqrt{h}\,.$$

Now, utilizing the last part of Theorem 2, it is a routine matter to prove the existence of a solution of (2.4).

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