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On Some Classes of Implicit Partial Differential Equations in Banach Spaces (**) (***)

Su alcune classi di equazioni differenziali alle derivate parziali in forma implicita negli spazi di Banach

RIASSUNTO. — In questo lavoro vien dato un contributo allo studio dell'esistenza, della dipendenza continua (dai dati) e dell'approssimazione delle soluzioni per un problema di Darboux, in forma implicita, del tipo $f(t, x, u, \partial u/\partial t, \partial u/\partial x, \partial^2 u/\partial t\,\partial x) = \theta_\Sigma$, $u(t_0, x) = \varphi(x)$, $u(t, x_0) = \theta_B$, essendo tanto la f quanto l'incognita a funzioni a valori in spazi di Banach.

INTRODUCTION

Let $I, J \subseteq \mathbf{R}$ be two compact intervals; B, Σ two real Banach spaces; Λ a non-empty set; f a function from $I \times J \times B^4 \times \Lambda$ into Σ. Further, let $t_0 \in I$, $x_0 \in J$ and let $\varphi\colon J \to B$ an absolutely continuous function, with $\varphi(x_0) = \theta_B$. For fixed $\lambda \in \Lambda$, consider the implicit Darboux problem

$$(\mathrm{P})_\lambda \qquad \begin{cases} f\left(t, x, u, \dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial t\,\partial x}, \lambda\right) = \theta_\Sigma\,, \\ u(t_0, x) = \varphi(x)\,, \\ u(t, x_0) = \theta_B\,. \end{cases}$$

A function $u\colon I \times J \to B$ is said to be a solution of $(\mathrm{P})_\lambda$ if $u(t_0, x) = \varphi(x)$, $u(t, x_0) = \theta_B$ for all $(t, x) \in I \times J$, the function $t \to u(t, \cdot)$ belongs to $AC(I, AC(J, B))$ and for almost every $t \in I$ there exists a set $J_t \subseteq J$ of measure zero

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such that

$$f\left(t, x, u(t, x), \frac{\partial u(t, x)}{\partial t}, \frac{\partial u(t, x)}{\partial x}, \frac{\partial^2 u(t, x)}{\partial t\, \partial x}, \lambda\right) = \theta_Z$$

for all $x \in J \setminus J_t$.

The main purpose of this paper is to establish some results about existence, continuous dependence (on φ and λ) and approximations of solutions of $(P)_\lambda$, provided that f can be expressed in a suitable form (see Theorems 2.4 and 3.4). Such results are obtained as consequences of more general theorems concerning differential inclusions (see Theorems 2.1, 2.3, 3.2, 3.3) which are, in turn, of independent interest.

To give an idea of the nature of our results, we state now, in a very simplified form, Theorem A, the simplest particular case of Theorem 2.4.

In Theorem A the space $(B, \|\cdot\|_B)$ is separable and the space $B \times B$ is endowed with the norm $\|(x, y)\|_{B\times B} = \|x\|_B + \|y\|_B$.

THEOREM A: *Let f_1 be a non-null continuous linear functional on B; f_2 a Lipschitzian functional on B; f_3 a Lipschitzian functional on $B \times B$; f_4 a Lipschitzian function from $B \times B$ into B. Let L and M be the Lipschitz constants of f_2 and f_3, respectively.*

Assume that $L < \|f_1\|_{B^*}$ and that $(M(1+l))/(\|f_1\|_{B^*} - L) < 1$, where l is the length of J.

Then, the problem

$$\begin{cases} f_1\left(\dfrac{\partial^2 u}{\partial t\, \partial x} + f_4\left(u, \dfrac{\partial u}{\partial x}\right)\right) + f_2\left(\dfrac{\partial^2 u}{\partial t\, \partial x} + f_4\left(u, \dfrac{\partial u}{\partial x}\right)\right) = f_3\left(\dfrac{\partial u}{\partial t}, \dfrac{\partial^2 u}{\partial t\, \partial x}\right) \\ u(t_0, x) = u(t, x_0) = \theta_B\,, \end{cases}$$

has at least one solution.

1. - NOTATION

Let X, Y be two non-empty sets. A multifunction Φ from X into Y (briefly, $\Phi\colon X \to 2^Y$) is a function from X into the family of all non-empty subsets of Y. The graph of Φ is the set $\mathrm{gr}(\Phi) = \{(x, y) \in X \times Y : y \in \Phi(x)\}$. If $A \subset X$, $\Omega \subset Y$, we put $\Phi(A) = \bigcup_{x\in A} \Phi(x)$ and $\Phi^-(\Omega) = \{x \in X : \Phi(x) \cap \Omega \neq \emptyset\}$. If $\Phi(X) = Y$, we say that Φ is onto (or, equivalently, surjective). If $X = Y$, a point $x \in X$ is said to be a fixed point of Φ if $x \in \Phi(x)$. We denote by $\mathrm{Fix}(\Phi)$ the set of all fixed points of Φ. When X, Y are two topological spaces, we say that Φ is lower (resp. upper) semicontinuous if, for every open (resp. closed) set $\Omega \subset Y$, the set $\Phi^-(\Omega)$ is open (resp. closed) in X. Φ is said to be continuous if it is simultaneously lower semicontinuous and upper semicon-

tinuous. When $(X, \mathscr{M})$ is a measurable space and Y is a topological space, we say that the multifunction Φ is measurable if $\Phi^{-}(\Omega) \in \mathscr{M}$ for every open subset $\Omega \subseteq Y$.

Let (Σ, δ) be a metric space. For every $x_0 \in \Sigma$ and $r > 0$, we put

$$B_\delta(x_0, r) = \{x \in \Sigma : d(x, x_0) < r\} .$$

For every $x \in \Sigma$ and non-empty $A, B \subseteq \Sigma$, we put:

$$\delta(x, A) = \inf_{z \in A} \delta(x, z) ;$$

$$\delta^*(A, B) = \sup_{z \in A} \delta(z, B) ;$$

$$\delta_H(A, B) = \max\left(\delta^*(A, B), \delta^*(B, A)\right) .$$

If $\{A_n\}_{n \in N}$ is a sequence of non-empty subsets of Σ, we put

$$\operatorname*{Li}_{n \to \infty} A_n = \left\{x \in \Sigma : \lim_{n \to \infty} \delta(x, A_n) = 0\right\} \quad \text{(see [2], p. 335)} .$$

If (X, d), (Y, ϱ) are two metric spaces, a multifunction $\Phi : X \to 2^Y$ is said to be Lipschitzian if there exists a positive real number L (Lipschitz constant) such that

$$\varrho_H(\Phi(x), \Phi(w)) < L d(x, w)$$

for every $x, w \in X$. If $L < 1$, Φ is a multi-valued contraction.

Now, let V be a vector space. We will denote by θ_V the null element of V. Let X, Y be two topological vector spaces. We denote by $\mathscr{L}(X, Y)$ the space of all continuous linear operators from X into Y. A convex process from X into Y is a multifunction whose graph is a convex cone containing $\theta_{X \times Y}$. If $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are two normed spaces and $\Phi : X \to 2^Y$ is a surjective convex process, we put

$$\alpha_\Phi = \sup\{d(\theta_X, \Phi^{-}(y)) : y \in Y, \|y\|_Y < 1\} ,$$

where d is the metric induced on X by the norm $\|\cdot\|_X$.

Now, let I be a compact real interval, $(B, \|\cdot\|_B)$ a Banach space, $p \in [1, \infty]$. Let us denote by $L^p(I, B)$ the space of all (equivalence classes of) strongly measurable functions $f : I \to B$ such that the real function $\|f(\cdot)\|_B$ belongs to $L^p(I)$. We will indicate by $\|\cdot\|_{L^p}$ the usual norm in this space,

$$\|f\|_{L^p} = \begin{cases} \left(\int_I \|f(t)\|_B^p \, dt\right)^{1/p} & \text{if } p < \infty , \\ \operatorname*{ess\,sup}_{t \in I} \|f(t)\|_B & \text{if } p = \infty . \end{cases}$$

Moreover, we will denote by $AC_p(I, B)$ the space of all strongly absolutely continuous functions $\varphi: I \to B$ such that the strong derivative φ' belongs to $L^p(I, B)$. In this space we will consider the norm

$$|\varphi|_{AC_p} = \max_{t \in I} |\varphi(t)|_B + |\varphi'|_{L^p}$$

and we will indicate by D the metric induced by this norm.

Finally, if Λ is a non-empty set, we will denote by $\mathscr{F}(\Lambda)$ the class of all symmetric functions $g: \Lambda \times \Lambda \to [0, +\infty[$ such that $g(\lambda, \lambda) = 0$ for every $\lambda \in \Lambda$.

2. - Results about existence and continuous dependence of solutions

We start with the following

THEOREM 2.1: *Let $(E, \|\cdot\|_E)$ be a separable Banach space, $(\Sigma, \|\cdot\|_\Sigma)$ another Banach space, $[a, b]$ a compact real interval, Λ a non-empty set, $p \in [1, \infty]$. Denote by d, d' the metrics induced on E, Σ respectively, by their norms. Moreover, let $\Phi: E \to 2^\Sigma$ be a surjective, closed-valued, upper semicontinuous convex process,*

$$\Psi: E \to 2^\Sigma, \quad G: [a, b] \times E \times \Lambda \to 2^\Sigma$$

two closed-valued multifunctions. Suppose that Ψ is Lipschitzian with Lipschitz constant M such that $\alpha_* M < 1$, and that either Ψ or G is compact-valued. Finally, let the following hypotheses be satisfied:

(α) *for every $(x, \lambda) \in E \times \Lambda$, the multifunction $G(\cdot, x, \lambda)$ is measurable;*

(β) *there exist a real function $\ell \in L^p([a, b])$ and a function $g \in \mathscr{F}(\Lambda)$, such that for almost every $t \in [a, b]$ and every $(x, \lambda), (y, \mu) \in E \times \Lambda$, one has*

$$d'_H(G(t, x, \lambda), G(t, y, \mu)) \leq \ell(t)(\|x - y\|_E + g(\lambda, \mu)) ;$$

(γ) *for every $\lambda \in \Lambda$, the real function $d'(0_\Sigma, G(\cdot, 0_E, \lambda))$ belongs to $L^p([a, b])$.*

Then, for every $t_0 \in [a, b]$, the following assertions hold:

i) *for every $(x_0, \lambda) \in E \times \Lambda$, the set*

$$\Gamma_{t_0}(x_0, \lambda) = \{\varphi \in AC_p([a, b], E) : (\Phi(\varphi'(t)) + \Psi(\varphi'(t))) \cap$$
$$\cap\, G(t, \varphi(t), \lambda) \neq \emptyset \text{ for almost every } t \in [a, b],\ \varphi(t_0) = x_0\}$$

is non-empty and closed;

(ii) *there exists a constant* $c>0$ *such that for every* $(x_0, \lambda), (y_0, \mu) \in E\times\Lambda$, *one has*

$$D_g(\Gamma_{p,t_0}(x_0, \lambda), \Gamma_{p,t_0}(y_0, \mu)) < c(\|x_0 - y_0\|_E + g(\lambda, \mu)).$$

PROOF: For every $(t, x, \lambda) \in [a, b]\times E\times\Lambda$, put

$$F(t, x, \lambda) = (\Phi + \Psi)^{-}(G(t, x, \lambda)).$$

Observe that for every $(t_0, x_0, \lambda) \in [a, b]\times E\times\Lambda$, we have

$$(1) \qquad \Gamma_{p,t_0}(x_0, \lambda) = \{\varphi \in AC_p([a, b], E) : \varphi'(t) \in F(t, \varphi(t), \lambda) \text{ for almost every } t \in [a, b],\ \varphi(t_0) = x_0\}.$$

Moreover, for every non-empty subset $A \subset \Sigma$ and $v \in E$, put

$$Q_A(v) = \Phi^{-}(A - \Psi(v)).$$

Plainly, we have $Q_A(v) \neq \emptyset$ and

$$(2) \qquad (\Phi + \Psi)^{-}(A) = \mathrm{Fix}\,(Q_A).$$

By [10] (see the Corollary on p. 131 and Theorem 6) it follows for every $v, z \in E$:

$$d_H(Q_A(v), Q_A(z)) < \alpha_\Phi d'_H(A - \Psi(v), A - \Psi(z)) < \alpha_\Phi d'_H(\Psi(v), \Psi(z)) < \alpha_\Phi M\|v - z\|_E.$$

Then, Q_A is a multi-valued contraction with Lipschitz constant $\alpha_\Phi M$. Now, let A be a closed subset of Σ; suppose that it is compact if Ψ is not compact-valued. Then, the set $A - \Psi(v)$ is closed, and hence so is the set $Q_A(v)$. By (2), $(\Phi + \Psi)^{-}(A)$ is non-empty and closed. Let B another subset of Σ with the same properties of A. By [10] and [3] (see, respectively, the Corollary on p. 131 and Lemma 1) we have

$$(3) \qquad d_H((\Phi + \Psi)^{-}(A), (\Phi + \Psi)^{-}(B)) < \frac{1}{1 - \alpha_\Phi M} \sup_{v \in E} d_H\big(\Phi^{-}(A - \Psi(v)), \Phi^{-}(B - \Psi(v))\big) < \frac{1}{1 - \alpha_\Phi M} \sup_{v \in E} \alpha_\Phi d'_H(A - \Psi(v), B - \Psi(v)) < \frac{\alpha_\Phi}{1 - \alpha_\Phi M} d'_H(A, B).$$

We can conclude that the multifunction F is closed-valued and that for every $(x, \lambda) \in E\times\Lambda$ the multifunction $F(\cdot, x, \lambda)$ is measurable, being the composition of the Lipschitzian multifunction $y \to (\Phi + \Psi)^{-}(y)$ with the measurable multifunction $G(\cdot, x, \lambda)$. Moreover, from (3) and (β) it follows that for

almost every $t \in [a, b]$ and every $(x, \lambda), (y, \mu) \in E \times \Lambda$, we have

$$d_H(F(t, x, \lambda), F(t, y, \mu)) \leq \frac{\alpha_\Phi M}{1 - \alpha_\Phi M} d_H'(G(t, x, \lambda), G(t, y, \mu)) \leq$$
$$\leq \frac{\alpha_\Phi d(t)}{1 - \alpha_\Phi M} (\|x - y\|_E + g(\lambda, \mu)) .$$

Finally, from (3) and (γ), it follows that, for every $\lambda \in \Lambda$, the real function $d(\theta_E, F(\cdot, \theta_E, \lambda))$ belongs to $L^p([a, b])$. Then, taking into account (1), our thesis follows by applying Theorem 2.1 of [5] to the multifunction F. ■

As a particular case, from Theorem 2.1, we get the following

THEOREM 2.2: *Let E, Σ, $[a, b]$, Λ as in Theorem 2.1. Let f be a continuous linear operator from E onto Σ, $h: E \to \Sigma$ a Lipschitzian operator, with Lipschitz constant M such that $M\alpha_f < 1$, $w: [a, b] \times E \times \Lambda \to \Sigma$ a function such that*

(α) *for every $(x, \lambda) \in E \times \Lambda$, the function $w(\cdot, x, \lambda)$ is measurable;*

(β) *there exist a real function $d \in L^p([a, b])$ and a function $g \in \mathscr{B}(\Lambda)$, such that for almost every $t \in [a, b]$ and every $(x, \lambda), (y, \mu) \in E \times \Lambda$, one has*

$$\|w(t, x, \lambda) - w(t, y, \mu)\|_\Sigma \leq d(t)(\|x - y\|_E + g(\lambda, \mu)) ;$$

(γ) *for every $\lambda \in \Lambda$, the real function $\|w(\cdot, \theta_E, \lambda)\|_\Sigma$ belongs to $L^p([a, b])$.*

Then, for every $t_0 \in [a, b]$, the following assertions hold:

(i) *for every $(x_0, \lambda) \in E \times \Lambda$, the set*

$$\Gamma_{w,\lambda}(x_0, \lambda) = \{\varphi \in AC_E([a, b], E) : f(\varphi'(t)) + h(\varphi'(t)) = w(t, \varphi(t), \lambda)$$
$$\text{for almost every } t \in [a, b],\ \varphi(t_0) = x_0\}$$

is non-empty and closed;

(ii) *there exists a constant $c > 0$ such that for every $(x_0, \lambda), (y_0, \mu) \in E \times \Lambda$, one has*

$$D_H(\Gamma_{w,\lambda}(x_0, \lambda), \Gamma_{w,\lambda}(y_0, \mu)) \leq c(\|x_0 - y_0\|_E + g(\lambda, \mu)) . \quad ■$$

Now, we want to present some applications of Theorem 2.1. In the following, we will denote by: $(B, \|\cdot\|_B)$, $(S, \|\cdot\|_S)$ two separable Banach spaces, $[a, b]$, $[r, s]$ two compact real intervals, Λ a non-empty set. We denote by d the metric induced on B by $\|\cdot\|_B$. Let $q \in [1, \infty[$. Let us introduce in the space $AC_q([r, s], B)$ the norm

$$\|\varphi\|_{AC_q} = (s - r)^{1/q} \max_{x \in [r, s]} \|\varphi(x)\|_B + \left(\int_r^s \|\varphi'(x)\|_B^q \, dx\right)^{1/q} .$$

We need the following

PROPOSITION 2.1: *Let $F\colon [r,s]\times B\times B\to 2^B$ a closed-valued multifunction. Suppose that:*

(α) *for every $u, v\in B$, the multifunction $F(\cdot, u, v)$ is measurable;*

(β) *there exists a positive real number ℓ such that for every $u_1, u_2, v_1, v_2\in B$ and for almost every $x\in[r,s]$ one has*

$$d_H(F(x,u_1,v_1), F(x,u_2,v_2))\leq \ell(\|u_1-u_2\|_B+\|v_1-v_2\|_B)\ ;$$

(γ) *the real function $x\to d(0_B, F(x,0_B,0_B))$ belongs to $L^s([r,s])$.*

Then, if we put, for every $\varphi\in AC_s([r,s],B)$,

$$\Psi(\varphi)=\{f\in L^s([r,s],B): f(x)\in F(x,\varphi(x),\varphi'(x)) \text{ for almost every } x\in[r,s]\},$$

$\Psi(\varphi)$ is non-empty and closed; moreover, the multifunction Ψ is Lipschitzian with Lipschitz constant ℓ.

PROOF: Let $\varphi\in AC_s([r,s],B)$. The multifunction $x\to F(x,\varphi(x),\varphi'(x))$ is closed-valued and, by (α), (β) and Proposition 1.2 of [4], it is measurable. Moreover, from (β) and (γ) it follows that the real function

$$x\to d(0_B, F(x,\varphi(x),\varphi'(x)) \quad \text{belongs to } L^s([r,s])\ .$$

Now, thanks to Proposition 2.1 of [6], there exists a measurable function $f\colon [r,s]\to B$ such that for almost every $x\in[r,s]$ we have $f(x)\in F(x,\varphi(x),\varphi'(x))$ and $\|f(x)\|_B\leq d(0_B, F(x,\varphi(x),\varphi'(x))+1)$. So, $f\in\Psi(\varphi)$. Of course, Ψ is also closed-valued. Finally, consider $\varphi, w\in AC_s([r,s],B)$, $f\in\Psi(\varphi)$, $\varepsilon>0$. For almost every $x\in[r,s]$, we have

$$d(f(x), F(x,w(x),w'(x)))\leq \ell(\|\varphi(x)-w(x)\|_B+\|\varphi'(x)-w'(x)\|_B)\ .$$

Then, again by Proposition 2.1 of [6], there exists a measurable function $g\colon [r,s]\to B$ such that, for almost every $x\in[r,s]$, $g(x)\in F(x,w(x),w'(x))$ and $\|f(x)-g(x)\|_B\leq \ell\|\varphi(x)-w(x)\|_B+\|\varphi'(x)-w'(x)\|_B+\varepsilon$. Hence,

$$\Big(\int_r^s \|f(x)-g(x)\|_B^s\,dx\Big)^{1/s}\leq \ell\Big((s-r)^{1/s}\max_{x\in[r,s]}\|\varphi(x)-w(x)\|_B+$$
$$+\Big(\int_r^s\|\varphi'(x)-w'(x)\|_B^s\,dx\Big)^{1/s}\Big)+\varepsilon(s-r)^{1/s}.$$

We can conclude that for every $f\in\Psi(\varphi)$ we have, if ϱ indicates the metric

induced by $\|\cdot\|_{AC_n}$:

$$\varrho(f, \Psi(w)) < \lambda \|\varphi - w\|_{AC_n}.$$

So,

$$\varrho^*(\Psi(\varphi), \Psi(w)) < \lambda \|\varphi - w\|_{AC_n}.$$

By this relation and the analogous one obtained interchanging the roles of ϱ and w, we get the following inequality:

$$\varrho_H(\Psi(\varphi), \Psi(w)) < \lambda \|\varphi - w\|_{AC_n},$$

that completes the proof. ■

The following result is a consequence of Theorem 2.1.

THEOREM 2.3: *Let $P: [r, s] \times B \times B \to 2^B$ be a closed-valued multifunction, $b: [a, b] \times [r, s] \times B \times B \times \Lambda \to B$ a (single-valued) function, η a non-null real number. Suppose that:*

(α_1) *for every $u, v \in B$, the multifunction $P(\cdot, u, v)$ is measurable;*

(α_2) *there exists $\lambda \in]0, |\eta|/(1+s-r)[$ such that for almost every $x \in [r, s]$ and for every $u_1, u_2, v_1, v_2 \in B$ one has*

$$d_H(P(x, u_1, v_1), P(x, u_2, v_2)) < \lambda(\|u_1 - u_2\|_B + \|v_1 - v_2\|_B);$$

(α_3) *the real function $x \to d(0_B, P(x, 0_B, 0_B))$ belongs to $L^p([r, s])$;*

(α_4) *for almost every $x \in [r, s]$ and for every $u, v \in B$, $\lambda \in \Lambda$, the function $b(\cdot, x, u, v, \lambda)$ is continuous;*

(α_5) *there exist a positive real number M and a function $\bar{g} \in \mathscr{F}(\Lambda)$ such that for every $t \in [a, b]$, for almost every $x \in [r, s]$ and for every $u_1, u_2, v_1, v_2 \in B$, $\lambda, \mu \in \Lambda$, one has*

$$\|b(t, x, u_1, v_1, \lambda) - b(t, x, u_2, v_2, \mu)\|_B \leq M(\|u_1 - u_2\|_B + \|v_1 - v_2\|_B + \bar{g}(\lambda, \mu));$$

(α_6) *for every $t \in [a, b]$, $u, v \in B$, $\lambda \in \Lambda$, the function $b(t, \cdot, u, v, \lambda)$ is measurable;*

(α_7) *for every $\lambda \in \Lambda$ there exists a real function $\chi_\lambda \in L^p([r, s])$ such that for every $t \in [a, b]$ and for almost every $x \in [r, s]$ one has*

$$\|b(t, x, 0_B, 0_B, \lambda)\| < \chi_\lambda(x).$$

Then, for every $p \in [1, \infty]$, $t_0 \in [a, b]$, $x_0 \in [r, s]$, if we put $E = \{\varphi \in AC_n([r, s], B): \varphi(x_0) = 0_B\}$, the following assertions hold:

(i) *for every* $(\varphi_0, \lambda) \in E \times \Lambda$, *the set*

$$\Gamma_{p,t_0,\varphi_0}(\varphi_0, \lambda) = \Big\{ u \in AC_p([a, b], E) : \textit{for almost every } t \in [a, b] \textit{ one has}$$

$$b\left(t, x, u(t, x), \frac{\partial u(t, x)}{\partial x}, \lambda\right) \in \eta \frac{\partial^2 u(t, x)}{\partial t\, \partial x} +$$

$$+ P\left(x, \frac{\partial u(t, x)}{\partial t}, \frac{\partial^2 u(t, x)}{\partial t\, \partial x}\right) \textit{ for almost every } x \in [r, s],\ u(t_0) = \varphi_0 \Big\}$$

is non-empty and closed in $AC_p([a, b], E)$;

(ii) *there exists a constant* $c > 0$ *such that for every* (φ_0, λ), $(\psi_0, \mu) \in E \times \Lambda$, *one has*

$$D_H(\Gamma_{p,t_0,\varphi_0}(\varphi_0, \lambda), \Gamma_{p,t_0,\psi_0}(\psi_0, \mu)) < c(\|\varphi_0 - \psi_0\|_{AC_p} + \bar{g}(\lambda, \mu)) .$$

Proof: We want to apply Theorem 2.1. To this end, choose as E the space $(E, \|\cdot\|_{AC_p})$ defined above and put $\Sigma = L^p([r, s], B)$. Let $w: E \to \Sigma$ be the operator defined by $w(\varphi) = \varphi'$ for every $\varphi \in E$, w is a linear homeomorphism from E onto Σ. So, since B is separable, E is separable too (see Remark 2.20, g-(r) of [2]). Now, let

$$\psi \in L^p([r, s], B), \qquad \text{we have } \|w^{-1}(\psi)\|_{AC_p} < (s - r)^{1/q} \max_{x \in [r, s]} \left\| \int_{x_0}^{x} \psi(\tau)\, d\tau \right\|_B + \|\psi\|_{L^p} .$$

But

$$\max_{x \in [r, s]} \left\| \int_{x_0}^{x} \psi(\tau)\, d\tau \right\|_B \leq (s - r)^{(q-1)/q} \|\psi\|_{L^p}; \qquad \text{so, } \alpha_w < s - r + 1 .$$

Put, for every $\varphi \in E$, $\Phi(\varphi) = \eta w(\varphi)$. Choose as $\Psi: E \to 2^E$ the multifunction defined by putting, for every $\varphi \in E$, $\Psi(\varphi) = \big\{ f \in \Sigma : f(x) \in P(x, \varphi(x), \varphi'(x))$ for almost every $x \in [r, s] \big\}$. Taking into account that $A(s - r + 1)/|\eta| < 1$, it is seen that Ψ satisfies the hypotheses of Theorem 2.1 thanks to Proposition 2.1. Finally, for every $(t, \varphi, \lambda) \in [a, b] \times E \times \Lambda$, put

$$G(t, \varphi, \lambda)(x) = b(t, x, \varphi(x), \varphi'(x), \lambda) .$$

G satisfies the hypotheses of Theorem 2.1 thanks to (α_b), (α_c), (α_τ). Then, for every $p \in [1, \infty]$, it is possible to apply Theorem 2.1. Now, to complete the proof, it suffices to observe that, for every $p \in [1, \infty]$, $t_0 \in [a, b]$, $(\varphi_0, \lambda) \in E \times \Lambda$, we have

$$\Gamma_{p,t_0,\varphi_0}(\varphi_0, \lambda) = \big\{ u \in AC_p([a, b], E) : G(t, u(t), \lambda) \in \Phi(u'(t)) + \Psi(u'(t))$$

$$\text{for almost every } t \in [a, b],\ u(t_0) = \varphi_0 \big\} . \quad \blacksquare$$

The last result of this section deals with partial differential equations.

THEOREM 2.4: *Let f be a continuous linear operator from B onto S, η a non-null real number, $\varphi: B \to S$, $\psi: [r, s] \times B \times B \to S$ and $b: [a, b] \times [r, s] \times B \times B \times \Lambda \to B$ three (single-valued) functions such that:*

(β_1) *for every $u, v \in B$, the function $\psi(\cdot, u, v)$ is measurable;*

(β_2) *the function φ is Lipschitzian with Lipschitz constant T such that $\alpha_f T < 1$;*

(β_3) *there exists a positive real number $\bar{\ell}$, with $\alpha_f \bar{\ell}/(1-\alpha_f T) < |\eta|/(1+s-r)$, such that for almost every $x \in [r, s]$ and for every $u_1, u_2, v_1, v_2 \in B$, one has*

$$\|\psi(x, u_1, v_1) - \psi(x, u_2, v_2)\|_B \leq \bar{\ell}\left(\|u_1 - u_2\|_B + \|v_1 - v_2\|_B\right);$$

(β_4) *the real function $x \to d\left(0_B, (f+\varphi)^{-1}(\psi(x, 0_B, 0_B))\right)$ belongs to $L^s([r, s])$;*

(β_5) *for almost every $x \in [r, s]$ and for every $u, v \in B$, $\lambda \in \Lambda$, the function $b(\cdot, x, u, v, \lambda)$ is continuous;*

(β_6) *there exist a positive real number M and a function $\bar{g} \in \mathscr{F}(\Lambda)$ such that for every $t \in [a, b]$, for almost every $x \in [r, s]$ and for every $u_1, u_2, v_1, v_2 \in B$, $\lambda, \mu \in \Lambda$, one has*

$$\|b(t, x, u_1, v_1, \lambda) - b(t, x, u_2, v_2, \mu)\|_B \leq M\left(\|u_1 - u_2\|_B + \|v_1 - v_2\|_B + \bar{g}(\lambda, \mu)\right);$$

(β_7) *for every $t \in [a, b]$, $u, v \in B$, $\lambda \in \Lambda$, the function $b(t, \cdot, u, v, \lambda)$ is measurable;*

(β_8) *for every $\lambda \in \Lambda$ there exists a real function $\chi_\lambda \in L^s([r, s])$ such that for every $t \in [a, b]$ and for almost every $x \in [r, s]$ one has $\|b(t, x, 0_B, 0_B, \lambda)\| < \chi_\lambda(x)$.*

Under such hypotheses, for every $p \in [1, \infty]$, $t_0 \in [a, b]$, $x_0 \in [r, s]$, if we put $E = \{u \in AC_p([r, s], B): u(x_0) = 0_B\}$, the following assertions hold:

(i) *for every $(\varphi_0, \lambda) \in E \times \Lambda$, the set*

$$\Gamma_{p, t_0, x_0}(\varphi_0, \lambda) = \Bigg\{u \in AC_p([a, b], E): \text{for almost every } t \in [a, b] \text{ one has}$$
$$f\left(b\left(t, x, u(t, x), \frac{\partial u(t, x)}{\partial x}, \lambda\right) + \eta \frac{\partial^2 u(t, x)}{\partial t\, \partial x}\right) +$$
$$+ \varphi\left(b\left(t, x, u(t, x), \frac{\partial u(t, x)}{\partial x}, \lambda\right) + \eta \frac{\partial^2 u(t, x)}{\partial t\, \partial x}\right) =$$
$$= \psi\left(x, \frac{\partial u(t, x)}{\partial t}, \frac{\partial^2 u(t, x)}{\partial t\, \partial x}\right) \text{ for almost every } x \in [r, s],\ u(t_0) = \varphi_0\Bigg\}$$

is non-empty and closed in $AC_p([a, b], E)$;

(ii) *there exists a constant $c > 0$ such that for every (φ_0, λ), $(\psi_0, \mu) \in E \times \Lambda$, one has*

$$D_H\left(\Gamma_{p, t_0, x_0}(\varphi_0, \lambda), \Gamma_{p, t_0, x_0}(\psi_0, \mu)\right) \leq c\left(\|\varphi_0 - \psi_0\|_{AC_p} + \bar{g}(\lambda, \mu)\right).$$

PROOF: The thesis follows by applying Theorem 2.3 to the function h and to the multifunction P defined by putting for every $(x, u, v) \in [r, s] \times B \times B$:

$$P(x, u, v) = (f + \varphi)^{-1}(\psi(x, u, v)) .$$

The hypotheses of Theorem 2.3 are immediately satisfied. In particular, (α_2) follows from (β_2) and (β_3), taking into account Theorem 2 of [9]: we will have $\delta = \alpha_f \bar{T}/(1 - \alpha_f T)$. ■

3. - SOME APPROXIMATION RESULTS

In this section we give some approximation results concerning the differential problems treated in the theorems of Section 2. Before establishing them, we need the following result.

THEOREM 3.1: *Let $(E, \|\cdot\|_E)$, $(\Sigma, \|\cdot\|_\Sigma)$ be two Banach spaces with E separable, D a dense subset of E. Denote by d and d' the metrics induced on E, Σ, respectively, by their norms. Let $\{\varphi_n\}_{n \in N}$ be a sequence of continuous linear operators from E onto Σ converging, in $\mathcal{L}(E, \Sigma)$, to a surjective operator φ, $\{\Psi_n\}_{n \in N}$ a sequence of closed-valued multifunctions from E into Σ, Ψ a closed-valued multifunction from E into Σ. Suppose that for every $n \in \mathbf{N}$, the multifunction Ψ_n is Lipschitzian with Lipschitz constant L_n, that the multifunction Ψ is Lipschitzian with Lipschitz constant L, and that the following conditions are satisfied:*

(α) *for every $x \in D$, $\Psi(x) \subseteq \operatorname{Li}_{n \to \infty} \Psi_n(x)$;*

(β) $\max\left(\sup_{n \in \mathbf{N}} \alpha_{\varphi_n} L_n, \alpha_\varphi L\right) < 1$.

Then, given a non-empty subset A of Σ and a sequence $\{A_n\}_{n \in N}$ of non-empty subsets of Σ, with $A \subseteq \operatorname{Li}_{n \to \infty} A_n$, one has

$$(\varphi + \Psi)^-(A) \subseteq \operatorname{Li}_{n \to \infty} (\varphi_n + \Psi_n)^-(A_n) .$$

PROOF: Let A, $\{A_n\}_{n \in N}$ be as in the statement. Let $\bar{x} \in (\varphi + \Psi)^-(A)$, $y \in A \cap (\varphi(\bar{x}) + \Psi(\bar{x}))$. Since $A \subseteq \operatorname{Li}_{n \to \infty} A_n$, there exists a sequence $\{y_n\}_{n \in N}$ with $y_n \in A_n$ for every $n \in \mathbf{N}$ and $\lim_{n \to \infty} y_n = y$. For every $n \in \mathbf{N}$, $x \in E$, we put

$$F_n(x) = \varphi_n^{-1}(y_n - \Psi_n(x)) ,$$
$$F(x) = \varphi^{-1}(y - \Psi(x)) .$$

Observe that, for every $n \in \mathbf{N}$, we have

$$(\varphi_n + \Psi_n)^-(y_n) = \operatorname{Fix}(F_n)$$

and

$$(\varphi + \Psi)^-(y) = \operatorname{Fix}(F) .$$

By Theorem 6 of [10] it follows that for every $n \in \mathbf{N}$, $x, w \in E$:

$$d_H(F_n(x), F_n(w)) < \alpha_{\varphi_n} L_n \|x - w\| ,$$
$$d_H(F(x), F(w)) < \alpha_\varphi L \|x - w\| ;$$

then, taking into account (β), the multifunctions F_n, F, are multi-valued contractions with the same Lipschitz constant, moreover, they are closed-valued. Now, we claim that for every $x \in D$ we have

$$F(x) \subseteq \operatorname*{Li}_{n \to \infty} F_n(x) . \tag{1}$$

Suppose the contrary, so, there exist $w \in F(x)$, $\gamma > 0$ and an infinite subset $\mathbf{N}'$ of $\mathbf{N}$ such that

$$F_n(x) \cap B_d(w, \gamma) = \emptyset \quad \text{for every } n \in \mathbf{N}' . \tag{2}$$

By the classical open mapping theorem, there exists $\delta > 0$ such that

$$B_E(\varphi(w), \delta) \subseteq \varphi(B_d(w, \gamma)) . \tag{3}$$

Let $m \in \mathbf{N}'$ and $u_m \in \Psi_m(w)$ be such that

$$\|\varphi - \varphi_m\|_{\mathscr{C}(E, E)} < \frac{\delta}{4(\|w\|_E + \gamma)} \tag{4}$$

and

$$\|y_m - \varphi(w) - u_m\|_E < \frac{\delta}{2} . \tag{5}$$

Let us show that

$$B_E\left(\varphi(w), \frac{\delta}{2}\right) \subseteq \varphi_m(B_d(w, \gamma)) . \tag{6}$$

In fact, suppose on the contrary that there exists $\bar{z} \in B_E(\varphi(w), \delta/2) \setminus \varphi_m(B_d(w, \gamma))$. Since $\varphi_m(B_d(w, \gamma))$ is a convex and open set, by a classical separation result there exists a non-null continuous linear functional f on Σ, such that $f(\bar{z}) < f(\zeta)$ for every $\zeta \in \varphi_m(B_d(w, \gamma))$. Let $r = f(\bar{z})$ and Σ^* the dual space of Σ, choose $\bar{g} \in \Sigma$ such that $f(\bar{g}) > \|f\|_{\Sigma^*}/2$ and $\|\bar{g}\|_\Sigma = 1$. Put $\zeta_0 = \bar{z} - (\delta/2)\bar{g}$. We have

$$\|\varphi(w) - \zeta_0\|_E < \|\varphi(w) - \bar{z}\|_E + \|\bar{z} - \zeta_0\|_E < \delta . \tag{7}$$

Now, taking into account a result of Ascoli (see [11], Lemma 1.2, p. 24) we have

$$d(\zeta_0, f^{-1}(r)) = \frac{|f(\zeta_0) - r|}{\|f\|_{\Sigma^*}} = \frac{\delta}{2} \frac{f(\bar{g})}{\|f\|_{\Sigma^*}} > \frac{\delta}{4} . \tag{8}$$

Moreover

$$d(z_0, f^{-1}(r)) = d(z_0, f^{-1}([r, \infty[)) . \tag{9}$$

In fact, suppose that there exists $z_1 \in f^{-1}([r, \infty[)$ such that $\|z_0 - z_1\| < d(z_0, f^{-1}(r))$. Since $d(z_0, f^{-1}(r)) = (r - f(z_0))/\|f\|_{E^*}$, we have:

$$f(z_1) - f(z_0) < \|f\|_{E^*} \|z_1 - z_0\|_E < r - f(z_0)$$

and so

$$f(z_1) < r ,$$

that is absurde. Then, taking into account that

$$\varphi_n(B_d(w, \gamma)) \subseteq f^{-1}([r, \infty[) ,$$

by (8) and (9) we obtain

$$d(z_0, \varphi_n(B(w, \gamma))) > \frac{\delta}{4} . \tag{10}$$

Now, fron (7) and (3) it follows that there exists $x_0 \in B_d(w, \gamma)$ such that $z_0 = \varphi(x_0)$. So, taking into account (4), we have

$$\|z_0 - \varphi_n(x_0)\|_E = \|\varphi(x_0) - \varphi_n(x_0)\|_E < \|\varphi - \varphi_n\|_{\mathscr{F}(E,E)} \|x_0\|_E <$$
$$< \frac{\delta}{4(\|x\|_E + \gamma)} (\|w\|_E + \gamma) = \frac{\delta}{4}$$

against (10). Then (6) is true. From (5) and (6) we obtain $y_n - u_n \in \varphi_n(B_d(w, \gamma))$, that contradicts (2). Then, (1) is proved. Hence, by Theorem 3.4 of [4], we get

$$(\varphi + \Psi)^-(y) \subseteq \operatorname*{Li}_{n \to \infty} (\varphi_n + \Psi_n)^-(y_n) ,$$

and so there exists a sequence $\{x_n\}_{n \in \mathbf{N}}$ in E such that $x_n \in (\varphi_n + \Psi_n)^-(y_n)$ for every $n \in \mathbf{N}$ and $\lim_{n \to \infty} x_n = \bar{x}$. So, our thesis is proved. ■

Now we can establish the following

THEOREM 3.2: *Let the hypotheses of Theorem 3.1 be satisfied. Moreover, let G_n, $G: E \to 2^E$ $(n \in \mathbf{N})$ be closed-valued multifunctions with the same Lipschitz constant δ. Suppose that either Ψ or G is compact-valued and, analogously, for every $n \in \mathbf{N}$, either Ψ_n or G_n is compact-valued. Finally, suppose that*

(γ) *for every $x \in E$, $G(x) \subseteq \operatorname*{Li}_{n \to \infty} G_n(x)$.*

Then, for every $p \in [1, \infty[$, $t_0 \in [a, b]$, $x_0 \in E$ and for every sequence $\{x_n\}_{n \in \mathbf{N}}$ in E converging to x_0, the following assertion holds: if we put

$$\Gamma = \{f \in AC_p([a, b], E) : (\varphi(f'(t)) + \Psi(f'(t))) \cap G(f(t)) \neq 0 \quad \text{for almost every } t \in [a, b],\ f(t_0) = x_0\}$$

and for every $n \in \mathbf{N}$

$$\Gamma_n = \{f \in AC_p([a, b], E) : (\varphi_n(f'(t)) + \Psi_n(f'(t))) \cap G_n(f(t)) \neq 0 \quad \text{for almost every } t \in [a, b],\ f(t_0) = x_n\},$$

one has $\Gamma \subset \operatorname{Li}_{n \to \infty} \Gamma_n$.

Proof: Let us denote by Λ the one-point compactification of $\mathbf{N}$ and for every $(x, \lambda) \in E \times \Lambda$ put

$$F(x, \lambda) = \begin{cases} (\varphi_n + \Psi_n)^-(G_n(x)) & \text{if } \lambda = n,\ n \in \mathbf{N}, \\ (\varphi + \Psi)^-(G(x)) & \text{if } \lambda = \infty. \end{cases}$$

It is easy to see that the multifunction $F(\cdot, \lambda)$ is Lipschitzian with Lipschitz constant $\alpha_{\varphi_n} A/(1 - \alpha_{\varphi_n} L_n)$ if $\lambda = n$, $n \in \mathbf{N}$, and with Lipschitz constant $\alpha_\varphi A/(1 - \alpha_\varphi L)$ if $\lambda = \infty$. On the other hand, the sequence $\{\alpha_{\varphi_n} A/(1 - \alpha_{\varphi_n} L_n)\}_{n \in \mathbf{N}}$ is bounded thanks to (β) and because the function $\sigma \to \alpha_\sigma$ is continuous on the subset of the surjective operators of $\mathscr{L}(E, \Sigma)$ (see [7], Lemma 3.2, (f)); then, there exists $\varrho > 0$ such that for every $x, y \in E$, $\lambda \in \Lambda$, we have

$$d_H(F(x, \lambda), F(y, \lambda)) < \varrho \|x - y\|.$$

Now observe that, thanks to Theorem 3.1, for every $x \in E$ we have

$$(\varphi + \Psi)^-(G(x)) \subset \operatorname{Li}_{n \to \infty} (\varphi_n + \Psi_n)^-(G_n(x)).$$

So, for every $x \in E$, the multifunction $F(x, \cdot)$ is lower semicontinuous. Then, if we put for every $(\xi, \lambda) \in E \times \Lambda$

$$\Gamma(\xi, \lambda) = \{f \in AC_p([a, b], E) : f'(t) \in F(f(t), \lambda) \quad \text{for almost every } t \in [a, b],\ f(t_0) = \xi\},$$

the multifunction $\Gamma(\xi, \cdot)$ is lower semicontinuous for every $\xi \in E$, by Theorem 3.2 of [6]; moreover, the multifunction $\Gamma(\cdot, \lambda)$ is Lipschitzian with a Lipschitz constant independent of λ, by Theorem 2.2 of [5]. So the multifunction Γ is lower semicontinuous in $E \times \Lambda$. In particular

$$\Gamma(x_0, \infty) \subset \operatorname{Li}_{n \to \infty} \Gamma(x_n, n)$$

that is our conclusion. ■

Observe that Theorem 3.2 deals with the same kind of differential problems that we have seen in Theorem 2.1. In Section 2 we have presented Theorem 2.3 which follows from Theorem 2.1. Analogously, we now present a result which follows from Theorem 3.2 and deals with the same kind of differential problems treated in Theorem 2.3. In the following, $[a, b]$, $[r, s]$ will denote two compact real intervals, $q \in [1, \infty[$, $\{\eta_n\}_{n\in\mathbf{N}}$ a sequence of non-null real numbers converging to a non-null real number η, $(B, |\cdot|_B)$, $(S, |\cdot|_S)$ two separable Banach spaces. We will denote by d and d' the metrics induced by $|\cdot|_B$ and $|\cdot|_S$ respectively. We will use the norm $\|\cdot\|_{AC_q}$ introduced in Section 2.

Let us prove the following

THEOREM 3.3: *Let P, P_n $(n \in \mathbf{N})$ be closed-valued multifunctions from $[r, s] \times B \times B$ into B, with $P(x, u, v) \subset \operatorname{Li}_{n\to\infty} P_n(x, u, v)$ for almost every $x \in [r, s]$ and for every $u, v \in B$; h, h_n $(n \in \mathbf{N})$ (single-valued) functions from $[r, s] \times B \times B$ into B, with $h(x, u, v) = \lim_{n\to\infty} h_n(x, u, v)$ for almost every $x \in [r, s]$ and every $u, v \in B$. Suppose that*

(α_1) *for every $n \in \mathbf{N}$, $u, v \in B$, the multifunctions $P(\cdot, u, v)$, $P_n(\cdot, u, v)$ are measurable;*

(α_2) *there exist λ, $\lambda_n > 0$ $(n \in \mathbf{N})$ such that for almost every $x \in [r, s]$ and for every $u_1, u_2, v_1, v_2 \in B$, $n \in \mathbf{N}$, one has*

$$d_H(P(x, u_1, v_1), P(x, u_2, v_2)) \leq \lambda(|u_1 - u_2|_B + |v_1 - v_2|_B);$$

$$d_H(P_n(x, u_1, v_1), P_n(x, u_2, v_2)) \leq \lambda_n(|u_1 - u_2|_B + |v_1 - v_2|_B)$$

with

$$\max\left(\sup_{n\in\mathbf{N}} \frac{1+s-r}{|\eta_n|}\lambda_n, \frac{1+s-r}{|\eta|}\lambda\right) < 1;$$

(α_3) *the real function $x \to \max\left(\sup_{n\in\mathbf{N}} d(\theta_B, P_n(x, \theta_B, \theta_B)), d(\theta_B, P(x, \theta_B, \theta_B))\right)$ belongs to $L^q([r, s])$;*

(α_4) *for every $n \in \mathbf{N}$, $u, v \in B$, the function $h_n(\cdot, u, v)$ is measurable;*

(α_5) *there exists $M > 0$ such that for almost every $x \in [r, s]$, for every $u, v \in B$, $n \in \mathbf{N}$, we have*

$$|h_n(x, u_1, v_1) - h_n(x, u_2, v_2)|_B \leq M(|u_1 - u_2|_B + |v_1 - v_2|_B);$$

(α_6) *the real function $x \to \sup_{n\in\mathbf{N}} |h_n(x, \theta_B, \theta_B)|_B$ belongs to $L^q([r, s])$.*

Under such hypotheses, for every $p \in [1, \infty[$, $t_0 \in [a, b]$, $x_0 \in [r, s]$, if we put $E = \{\chi \in AC_q([r, s], B) : \chi(x_0) = \theta_B\}$, the following assertion holds: for every $\chi \in E$

and every sequence $\{\chi_n\}_{n\in\mathbf{N}}$ *in* E *converging to* χ, *if we put*

$$\Gamma(\chi_n)=\Big\{u\in AC_p([a,b],E):\ \textit{for almost every } t\in[a,b] \textit{ one has}$$
$$b_n\Big(x,u(t,x),\frac{\partial u(t,x)}{\partial x}\Big)\in\eta_n\frac{\partial^2 u(t,x)}{\partial t\,\partial x}+$$
$$+P_n\Big(x,\frac{\partial u(t,x)}{\partial t},\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)\ \textit{for almost every } x\in[r,s],\ u(t_0)=\chi_n\Big\}\qquad(n\in\mathbf{N});$$

$$\Gamma(\chi)=\Big\{u\in AC_p([a,b],E):\ \textit{for almost every } t\in[a,b] \textit{ one has}$$
$$b\Big(x,u(t,x),\frac{\partial u(t,x)}{\partial x}\Big)\in\eta\frac{\partial^2 u(t,x)}{\partial t\,\partial x}+$$
$$+P\Big(x,\frac{\partial u(t,x)}{\partial t},\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)\ \textit{for almost every } x\in[r,s],\ u(t_0)=\chi\Big\},$$

one has $\Gamma(\chi)\subseteq\operatorname*{Li}_{n\to\infty}\Gamma(\chi_n)$.

Proof: We want to apply Theorem 3.2. To this end, choose as E the space $(E,\|\cdot\|_{E(\eta)})$ defined in the statement, and as Σ the space $L^s([r,s],B)$. Denote by $\pi_0: E\to\Sigma$ the operator that, to every element of E, associates its derivative, and put $\varphi=\eta\pi_0$, $\varphi_n=\eta_n\pi_0$ $(n\in\mathbf{N})$. Moreover, for every $f\in E$, $n\in\mathbf{N}$, put

$$\Psi_n(f)=\big\{\psi\in\Sigma:\ \psi(x)\in P_n(x,f(x),f'(x))\ \text{for almost every } x\in[r,s]\big\},$$
$$\Psi(f)=\big\{\psi\in\Sigma:\ \psi(x)\in P(x,f(x),f'(x))\ \text{for almost every } x\in[r,s]\big\}.$$

By (α_1), (α_2), (α_3) and thanks to Proposition 2.1, all the multifunctions Ψ_n, Ψ are closed-valued, Lipschitzian, with Lipschitz constant λ_n $(n\in\mathbf{N})$ and λ, respectively. Let us show, now, that for every $f\in E$ we have, in Σ: $\Psi(f)\subseteq\subseteq\operatorname*{Li}_{n\to\infty}\Psi_n(f)$. To this end, fix $\psi\in\Psi(f)$. Observe that

$$\text{(1)}\qquad \text{for almost every } x\in[r,s],\ \psi(x)\in P(x,f(x),f'(x))\subseteq\subseteq\operatorname*{Li}_{n\to\infty}P_n(x,f(x),f'(x)).$$

For every $n\in\mathbf{N}$, for almost every $x\in[r,s]$, put $\alpha_n(x)=d\big(\psi(x),P_n(x,f(x),f'(x))\big)$; α_n is measurable thanks to Lemma 2.1 of [8]. For almost every $x\in[r,s]$, one has $\lim_{n\to\infty}\alpha_n=0$, from (1). By (α_2), for every $n\in\mathbf{N}$ and for almost every $x\in[r,s]$, we have

$$\alpha_n(x)\le\|\psi(x)\|_B+d(\theta_B,P_n(x,\theta_B,\theta_B))+\lambda_n(\|f(x)\|_B+\|f'(x)\|_B)\le$$
$$\le\|\psi(x)\|_B+\sup_{n\in\mathbf{N}}d(\theta_B,P_n(x,\theta_B,\theta_B))+\frac{|\eta_n|}{1+s-r}(\|f(x)\|_B+\|f'(x)\|_B).$$

So, thanks to (α_3), we can conclude that $\lim_{n\to\infty} \alpha_n = 0$ in $L^1([r, s])$. Then, by Proposition 2.1 of [6], for every $n \in \mathbf{N}$, there exists a measurable function $\psi_n\colon [r, s] \to B$ such that, for almost every $x \in [r, s]$, we have

$$\psi_n(x) \in P_n(x, f(x), f'(x)) \quad \text{and} \quad d(\psi(x), \psi_n(x)) < \alpha_n(x) + \frac{1}{n}.$$

So, $\{\psi_n\}_{n\in\mathbf{N}}$ converges to ψ in Σ, that proves our claim. Now, observe that, as we saw in the proof of Theorem 2.3, we have

$$\alpha_\psi < \frac{s-r+1}{|\eta|}, \quad \alpha_{\psi_n} < \frac{s-r+1}{|\eta_n|} \quad (n \in \mathbf{N}).$$

From this relation and (α_2) it follows that condition (β) of Theorem 3.2 is satisfied. Now, for every $f \in E$, $n \in \mathbf{N}$ and almost every $x \in [r, s]$, put

$$g_n(f)(x) = h_n(x, f(x), f'(x)), \quad g(f)(x) = h(x, f(x), f'(x)).$$

By (α_4), (α_5), (α_6) and Proposition 4.1, all the functions g_n, g are Lipschitzian with Lipschitz constant M. Moreover, $\lim_{n\to\infty} g_n(f) = g(f)$ for every $f \in E$. Then, to conclude the proof, it suffices to apply Theorem 3.2 and observe that

$$\Gamma(\chi_n) = \Big\{u \in AC_p([a, b], E) : g_n(u(t)) \in \varphi_n(u'(t)) + \Psi_n(u'(t)) \\ \text{for almost every } t \in [a, b],\ u(t_0) = \chi_n\Big\} \quad (n \in \mathbf{N}),$$

$$\Gamma(\chi) = \Big\{u \in AC_p([a, b], E) : g(u(t)) \in \varphi(u'(t)) + \Psi(u'(t)) \\ \text{for almost every } t \in [a, b],\ u(t_0) = \chi\Big\}. \quad \blacksquare$$

We conclude with a particular case of Theorem 3.3.

THEOREM 3.4: *Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence of continuous linear operators from B onto Σ converging, in $\mathcal{L}(B, \Sigma)$, to a surjective operator f; $\varphi, \varphi_n\colon B \to S$ $(n \in \mathbf{N})$ (single-valued) functions such that, for every $u \in B$, one has $\lim_{n\to\infty} \varphi_n(u) = \varphi(u)$; $\Psi, \Psi_n\colon [r, s] \times B \times B \to S$ $(n \in \mathbf{N})$ functions such that for almost every $x \in [r, s]$ and every $u, v \in B$ one has $\lim_{n\to\infty} \psi_n(x, u, v) = \psi(x, u, v)$; $h, h_n\colon [r, s] \times B \times B \to B$ $(n \in \mathbf{N})$ functions such that for almost every $x \in [r, s]$ and every $u, v \in B$ we have $\lim_{n\to\infty} h_n(x, u, v) = h(x, u, v)$. Moreover, suppose that:*

(β_1) for every $n \in \mathbf{N}$, $u, v \in B$, the function $\psi_n(\cdot, u, v)$ is measurable;

(β_2) for every $n \in \mathbf{N}$, the function φ_n is Lipschitzian, with Lipschitz constant L_n; moreover, $\sup_{n\in\mathbf{N}} \alpha_{f_n} L_n < 1$;

(β_3) *there exists a sequence* $\{\bar{L}_n\}_{n\in\mathbf{N}}$ *of positive real numbers such that for almost every* $x\in[r,s]$, *for every* $u_1, u_2, v_1, v_2 \in B$, $n\in\mathbf{N}$, *one has*

$$\|\psi_n(x,u_1,v_1)-\psi_n(x,u_2,v_2)\|_B \leq \bar{L}_n(\|u_1-u_2\|_B+\|v_1-v_2\|_B),$$

moreover

$$\sup_{n\in\mathbf{N}} \frac{1+s-r}{|\eta_n|}\,\frac{\alpha_{t_n}}{1-\alpha_{t_n}L_n}\,\bar{L}_n<1;$$

(β_4) *the real function* $x\to\sup_{n\in\mathbf{N}} d\big(\theta_B,(f_n+\varphi_n)^{-1}(\psi_n(x,\theta_B,\theta_B))\big)$ *belongs to* $L^2([r,s])$;

(β_5) *for every* $n\in\mathbf{N}$, $u,v\in B$, *the function* $b_n(\cdot,u,v)$ *is measurable;*

(β_6) *there exists* $M>0$ *such that for almost every* $x\in[r,s]$, *for every* $u,v\in B$, $n\in\mathbf{N}$, *one has*

$$\|b_n(x,u_1,v_1)-b_n(x,u_2,v_2)\|_B \leq M(\|u_1-u_2\|_B+\|v_1-v_2\|_B);$$

(β_7) *the real function* $x\to\sup_{n\in\mathbf{N}}\|b_n(x,\theta_B,\theta_B)\|_B$ *belongs to* $L^2([r,s])$.

Under such hypotheses, for every $p\in[1,\infty[$, $t_0\in[a,b]$, $x_0\in[r,s]$, *if we put* $E=\{\varphi\in AC_p([r,s],B):\varphi(x_0)=\theta_B\}$, *the following assertion holds: for every* $\chi\in E$ *and every sequence* $\{\chi_n\}_{n\in\mathbf{N}}$ *converging to* χ, *if we put*

$$\Gamma(\chi_n)=\Big\{u\in AC_p([a,b],E):\text{ for almost every } t\in[a,b] \text{ one has}$$
$$f_n\Big(b_n\Big(x,u(t,x),\frac{\partial u(t,x)}{\partial x}\Big)+\eta_n\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)+$$
$$+\varphi_n\Big(b_n\Big(x,u(t,x),\frac{\partial u(t,x)}{\partial x}\Big)+\eta_n\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)=$$
$$=\psi_n\Big(x,\frac{\partial u(t,x)}{\partial t},\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)\text{ for almost every } x\in[r,s],\ u(t_0)=\chi_n\Big\}\quad(n\in\mathbf{N}),$$

$$\Gamma(\chi)=\Big\{u\in AC_p([a,b],E):\text{ for almost every } t\in[a,b] \text{ one has}$$
$$f\Big(b\Big(x,u(t,x),\frac{\partial u(t,x)}{\partial x}\Big)+\eta\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)+$$
$$+\varphi\Big(b\Big(x,u(t,x),\frac{\partial u(t,x)}{\partial x}\Big)+\eta\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)=$$
$$=\psi\Big(x,\frac{\partial u(t,x)}{\partial t},\frac{\partial^2 u(t,x)}{\partial t\,\partial x}\Big)\text{ for almost every } x\in[r,s],\ u(t_0)=\chi\Big\},$$

one has $\Gamma(\chi)\subset \operatorname*{Li}_{n\to\infty}\Gamma(\chi_n)$.

PROOF: Let us put, for every $(x, u, v) \in [r, s] \times B \times B$ and every $n \in \mathbf{N}$:

$$P(x, u, v) = (f + \varphi)^{-1}(\psi(x, u, v)), \quad P_n(x, u, v) = (f_n + \varphi_n)^{-1}(\psi_n(x, u, v)).$$

To prove our thesis, it suffices to apply Theorem 3.3 to the multifunctions P_n, P and to the functions h_n, h. In particular, let us observe that, if we put $L = \liminf_{n \to \infty} L_n$, taking into account ([7], Lemma 3.2, (f)), φ is Lipschitzian with Lipschitz constant L and $\alpha_f L < 1$. Analogously, iv we put $\bar{\ell} = \liminf_{n \to \infty} \bar{\ell}_n$, for almost every $x \in [r, s]$ and every $u_1, u_2, v_1, v_2 \in B$, one has

$$\|\psi(x, u_1, v_1) - \psi(x, u_2, v_2)\|_s < \bar{\ell}(|u_1 - u_2|_s + |v_1 - v_2|_s),$$

moreover, $((1 + s - r)/|\eta|)(\alpha_f/(1 - \alpha_f L))\bar{\ell} < 1$. Finally, thanks to Theorem 2 of [9], we have for almost every $x \in [r, s]$ and every $u, v \in B$:

$$P(x, u, v) \subseteq \operatorname{Li}_{n \to \infty} P_n(x, u, v);$$

moreover, hypothesis (α_2) of Theorem 3.3 is satisfied with

$$\ell = \frac{\alpha_f}{1 - \alpha_f L}\bar{\ell}, \quad \ell_n = \frac{\alpha_{f_n}}{1 - \alpha_{f_n} L_n}\bar{\ell}_n \quad (n \in \mathbf{N}). \quad \blacksquare$$

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