Rendiconti
Accademia Nazionale delle Scienze detta dei XL
*Memorie di Matematica*
108° (1990), Vol. XIV, fasc. 17, pagg. 327-339

D. E. EDMUNDS - A. KUFNER - JIONG SUN (*)

# Extension of Functions in Weighted Sobolev Spaces (**)

SUMMARY. — In this paper, weighted Sobolev spaces of functions defined on an $N$-dimensional domain $\Omega$ are investigated with weights which are functions of the distance from the piecewise smooth boundary $\partial\Omega$ of $\Omega$. Conditions on the weight functions are given which allow the extension of functions from $\Omega$ to the whole of $R^N$ preserving the Sobolev space.

# Prolungamento di funzioni in spazi di Sobolev con peso

SUNTO. — Questo lavoro è dedicato allo studio degli spazi di Sobolev con peso di funzioni definite su aperti $N$-dimensionali $\Omega$, con funzioni peso che dipendono dalla distanza dalla frontiera $\partial\Omega$ di $\Omega$. Si danno delle condizioni sulle funzioni peso che garantiscono la possibilità di estendere le funzioni di $\Omega$ su tutto $R^N$ conservando lo spazio di Sobolev.

## 0. - INTRODUCTION

0.1. The problem of the extension of a function in the Sobolev space $W^{k,p}(\Omega)$ « beyond the boundary $\partial\Omega$ » of the open subset $\Omega$ of $\mathbf{R}^N$ which « preserves the space », i.e. the problem of the construction of a (linear, bounded) operator

$$E\colon W^{k,p}(\Omega) \to W^{k,p}(\mathbf{R}^N)$$

such that

$$(Eu)(x) = u(x) \qquad \text{for } x \in \Omega\,, \tag{0.1}$$

is extensively described in the literature (see, e.g. [7], [2]). It is well-known that an important role is played by the geometrical properties of the bound-

(*) Indirizzi degli autori: DAVID E. EDMUNDS: Mathematics Division, University of Sussex, Brighton BN1 9QH, Sussex, England; ALOIS KUFNER: Mathematical Institute, Czech. Acad. Sci., Žitná 25, 115 67 Praha 1, Czechoslovakia; JIONG SUN: Department of Mathematics, Inner Mongolia University, Huhhot, People's Republic of China.

(**) Memoria presentata il 18 ottobre 1990 da Mario Troisi, socio dell'Accademia.

ISSN-0392-4106

ary $\partial\Omega$ of $\Omega$ and that satisfactory results can be obtained assuming (at least) that

$$\partial\Omega \text{ is (locally) Lipschitzian.}$$

0.2. The extension of functions in *weighted* Sobolev spaces $W^{k,p}(\Omega; S)$ defined—for a family

$$S = \{w_\alpha : |\alpha| < k\} \tag{0.2}$$

of *given* weight functions $w_\alpha$ (i.e. functions which are measurable and positive a.e. in $\Omega$)—as the linear space of functions $u$ satisfying

$$\|u\|_{k,p,S,\Omega} = \|u\|_{k,p,S} = \Big( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^p w_\alpha(x)\, dx \Big)^{1/p} < \infty \tag{0.3}$$

has not yet been investigated systematically. Weight functions (of a specific type) have appeared only in connection with the extension of functions from $W^{k,p}(\Omega)$: thus in [1], it is shown that under certain conditions on $\Omega$, a function $u \in W^{k,p}(\Omega)$ can be extended out of $\Omega$ to a function $Eu$ so that not only is $Eu$ in $W^{k,p}(R^N)$, but also it is infinitely differentiable in $R^N \setminus \bar{\Omega}$ and for any $\alpha$ with $|\alpha| > k$, we have

$$D^\alpha(Eu) \cdot w_\alpha \in L^p(R^N \setminus \bar{\Omega})$$

with the special family of weight functions $w_\alpha$:

$$w_\alpha(x) = [\operatorname{dist}(x, \partial\Omega)]^{\lambda(\alpha)}.$$

The number $\lambda(\alpha) = |\alpha| - k$ cannot be improved.

0.3. An extension of a function $u \in W^{k,p}(\Omega; S)$ requires also an appropriate *extension of the weight functions* $w_\alpha$. In this note, we shall deal with special weight functions of the form

$$w_\alpha(x) = \varrho_\alpha(\operatorname{dist}(x, \partial\Omega)), \qquad |\alpha| < k, \tag{0.4}$$

where $\varrho_\alpha = \varrho_\alpha(t)$ are functions defined (and positive) for $t > 0$. Consequently, $w_\alpha(x)$ is defined also for $x \in R^N \setminus \bar{\Omega}$, and we can ask whether there exists a (linear, bounded) extension operator $E$ such that

$$Eu \in W^{k,p}(R^N \setminus \bar{\Omega}; S) \tag{0.5}$$

for $u \in W^{k,p}(\Omega; S)$ with $S = \{w_\alpha : |\alpha| < k\}$ and $w_\alpha$ of the form (0.4).

0.4. Remark (more special weight functions): Weight functions of the form (0.4) are important and appear in many applications. They are used mainly in order to describe the behaviour of the function $u$ in the corresponding weighted Sobolev space *near the boundary* $\partial\Omega$. Therefore we shall deal with two types of functions $\varrho_\alpha = \varrho_\alpha(t)$, assuming that either

$$\varrho_\alpha(t)\downarrow 0 \text{ as } t\downarrow 0+, \qquad t\in(0,\delta) \tag{0.6}$$

or

$$\varrho_\alpha(t)\uparrow\infty \text{ as } t\downarrow 0+, \qquad t\in(0,\delta) \tag{0.7}$$

with some suitable (small) $\delta > 0$; in addition, we suppose that there exist two positive constants $c, C$ such that

$$c \leq \varrho_\alpha(t) \leq C \quad \text{for } t > \delta\,. \tag{0.8}$$

Consequently, the weight function $w_\alpha$ given by (0.4) does not play any role « far from $\partial\Omega$ », i.e. for $x\in\Omega\setminus\Omega_\delta$ where

$$\Omega_\delta = \{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega) < \delta\} \tag{0.9}$$

is a « boundary layer of width $\delta$ ».

0.5. Remark: The assumptions about the weight functions $\varrho_\alpha$, mentioned in Remark 0.4, guarantee that the weight spaces $W^{k,p}(\Omega; S)$ are well-defined since the weights influence behaviour in a neighbourhood of the boundary $\partial\Omega$ only, while « far from $\partial\Omega$ », $u$ behaves as being from the non-weighted space $W^{k,p}(\Omega)$.

On the other hand, the extension $Eu$ belongs to the space $W^{k,p}(\mathbb{R}^N; S)$ which has a somewhat different character. Even if we preserve the assumptions from Remark 0.4, which indicate that we need to work only with the space

$$W^{k,p}(\bar\Omega\cup\Omega^\delta; S) \tag{0.10}$$

where $\Omega^\delta$ is the « outer boundary layer of width $\delta$ »,

$$\Omega^\delta = \{x\in \mathbb{R}^N\setminus\bar\Omega;\ \operatorname{dist}(x,\partial\Omega) < \delta\}\,, \tag{0.11}$$

the following situation occurs: the set $\partial\Omega$ on which the influence of the weight functions is concentrated and realised, now lies *inside* the set $\bar\Omega\cup\Omega^\delta$.

Therefore, we have to use the result from [5] where it is shown that the space (0.10) is reasonably defined if, and only if,

$$w_\alpha^{1/(1-p)} \in L^1_{\mathrm{loc}}(\bar\Omega\cup\Omega^\delta)\,. \tag{0.12}$$

For example, if

$$\Omega = \mathbf{R}^N_+ = \{x = (x', x_N) : x' = (x_1, x_2, \dots, x_{N-1}) \in \mathbf{R}^{N-1},\ x_N > 0\}$$

and $\varrho_\alpha(t) = t^\lambda$ with $\lambda \in \mathbf{R}$, $t \in (0, \delta)$, the condition (0.12) indicates that

$$\int_M \Big( \int_0^\delta x_N^{\lambda/(1-p)} \, dx_N \Big) dx' < \infty \tag{0.13}$$

for any bounded domain $M \subset \mathbf{R}^{N-1}$. Indeed, we have

$$\operatorname{dist}(x, \partial\Omega) = |x_N| \qquad \text{and } w_\lambda(x) = |x_N|^\lambda .$$

The condition (0.13) implies that $\lambda/(1-p) + 1 > 0$, i.e.

$$\lambda < p - 1 ; \tag{0.14}$$

this last condition is *natural* since it guarantees the *existence of the trace* of the function $u \in W^{k,p}(\Omega; S)$ on $\partial\Omega$ (see [4], Example 9.16).

## 1. - A special case

1.1. In this section we shall describe the construction of the extension operator $E$ for weighted Sobolev spaces defined on very special subsets of $\mathbf{R}^N$; the passage to general spaces can be achieved—just as in [3], [6]—by a local description of the boundary $\partial\Omega$ and by the corresponding partition of unity, provided that the weight functions satisfy certain additional homogeneity conditions.

1.2. *The space* $V^{k,p}(\mathbf{R}^N_+; S)$. Let $p \in (1, \infty)$, $k \in \mathbf{N}$ and let $S = \{\varrho_\alpha : |\alpha| < k\}$ be a family of weight functions $\varrho_\alpha = \varrho_\alpha(t)$ defined for $t > 0$. The function

$$u = u(x) = u(x', x_N) , \qquad x' = (x_1, x_2, \dots, x_{N-1})$$

is said to belong to the space

$$V^{k,p}(\mathbf{R}^N_+; S) \tag{1.1}$$

if (i) there exists a number $R > 0$ such that

$$\operatorname{supp} u \subset \{x = (x', x_N) : |x'| < R,\ 0 < x_N < R\} ,$$

where $|x'| = (x_1^2 + \dots + x_{N-1}^2)^{1/2}$;

(ii) the norm

$$\|u\|_{k,p,S} = \left( \sum_{|\alpha| \le k} \int_{\mathbf{R}^{N-1}} \left( \int_0^\infty |D^\alpha u(x)|^p \varrho_\alpha(x_N)\, dx_N \right) dx' \right)^{1/p} \tag{1.2}$$

is finite.

1.3. REMARK: In (1.2), it obviously suffices to integrate over the set $B(R)\times(0, R)$ where $B(R)$ is the ball $\{x' : |x'| < R\}$ in $\mathbf{R}^{N-1}$.

1.4. *The space* $V^{k,p}(\mathbf{R}^N_+; \varrho)$. Suppose, for simplicity, that all the weight functions $\varrho_\alpha$ appearing in $S$ are the same:

$$\varrho_\alpha(t) = \varrho(t) \qquad \text{for all } \alpha,\ |\alpha| < k. \tag{1.3}$$

The corresponding space $V^{k,p}(\mathbf{R}^N_+; S)$ will be denoted by

$$V^{k,p}(\mathbf{R}^N_+; \varrho), \tag{1.4}$$

and we shall similarly denote $W^{k,p}(\Omega; S)$ by $W^{k,p}(\Omega; \varrho)$ with a corresponding change of notation for norms.

1.5. EXAMPLE: Let $\lambda \in \mathbf{R}$ and put

$$\varrho_\alpha(t) = t^\lambda \qquad \text{for } t > 0 \text{ and } |\alpha| < k. \tag{1.5}$$

The corresponding space from Subsection 1.4,

$$V^{k,p}(\mathbf{R}^N_+; x_N^\lambda), \tag{1.6}$$

will also be denoted simply by

$$V^{k,p}(\mathbf{R}^N_+; \lambda), \qquad \lambda \in \mathbf{R}. \tag{1.7}$$

1.6. *The weight function* $\varrho$. We shall consider *four* types of weight functions:

(i) Let $\varrho = \varrho(t)$ be *non-decreasing* in some small right neighbourhood of 0 and let

$$\lim_{t \to 0+} \varrho(t) = 0 \tag{1.8}$$

(for such functions, we shall also use the notation from (0.6): $\varrho(t)\downarrow 0$ as $t \downarrow 0+$). We shall say that such a function $\varrho$ is of *type I* if

$$\int_0 \varrho^{-1/(p-1)}(t)\, dt < \infty, \tag{1.9}$$

and of *type II* if

$$\int_0 \varrho^{-1/(p-1)}(t)\,dt = \infty . \tag{1.10}$$

[$\int_0$ means $\int_0^\delta$ for some sufficiently small positive $\delta$.]

(ii) Let $\varrho = \varrho(t)$ be *non-increasing* in some small right neighbourhood of 0 and let

$$\lim_{t\to 0+} \varrho(t) = \infty \tag{1.11}$$

(for such functions, we shall also use the notation from (0.7): $\varrho(t)\uparrow\infty$ as $t\downarrow 0+$). We shall say that such a function $\varrho$ is of *type III* if

$$\int_0 \varrho(t)\,dt < \infty , \tag{1.12}$$

and of *type IV* if

$$\int_0 \varrho(t)\,dt = \infty . \tag{1.13}$$

1.7. Example: The weight function $\varrho(t) = t^\lambda$ is

of type I for $0 < \lambda < p-1$,
of type II for $\lambda > p-1$,
of type III for $-1 < \lambda < 0$,
of type IV for $\lambda \leq -1$.

1.8. *The trace.* (i) Let $f = f(t)$ be defined on $(0, \delta)$ and suppose that the derivative $f'(t)$ exists and satisfies

$$\int_0^\delta |f'(t)|^p \varrho(t)\,dt < \infty \tag{1.14}$$

for a given weight function $\varrho$. Using the Hölder inequality, we obtain for $t, t+h \in (0, \delta)$, $h > 0$, the estimate

$$|f(t+h)-f(t)| = \left|\int_t^{t+h} f'(s)\,ds\right| = \left|\int_t^{t+h} f'(s)\varrho^{1/p}(s)\varrho^{-1/p}(s)\,ds\right| \leq$$

$$\leq \left(\int_t^{t+h} |f'(s)|^p \varrho(s)\,ds\right)^{1/p} \left(\int_t^{t+h} \varrho^{-1/(p-1)}(s)\,ds\right)^{(p-1)/p} .$$

Thus in view of (1.14) *the function $f$ is absolutely continuous and the limit*

$$\lim_{t\to 0+} f(t) = a_0 < \infty \tag{1.15}$$

*exists provided that $\varrho$ is of type I, III or IV.*

(ii) Counter-examples show that the last assertion does not hold if $\varrho$ is of type II (see [4], Example 9.16).

(iii) If we suppose that, in addition to (1.14), also

$$\int_0^\delta |f(t)|^p \varrho(t)\,dt < \infty \tag{1.16}$$

then for $a_0$ from (1.15) we necessarily have

$$a_0 = 0 \tag{1.17}$$

*provided that $\varrho$ is of type IV.*

(iv) Let $u \in V^{1,p}(R_+^N;\varrho)$ and apply the foregoing results to

$$f(t) = u(x', t)$$

for an arbitrary but fixed $x' \in R^{N-1}$. Then we can take

$$a_0(x') = \lim_{t\to 0+} u(x', t)$$

as the trace of $u$ on $\partial\Omega = R^{N-1}\times\{0\}$ and we obtain the following assertion:

*If $u \in V^{1,p}(R_+^N;\varrho)$ where $\varrho$ is of type I, III or IV, then the trace of $u$ on $\partial R_+^N$ exists. Moreover, if $\varrho$ is of type IV, then the trace is zero.*

(v) Similarly we obtain that for

$$u \in V^{k,p}(R_+^N;\varrho),$$

the trace of $D^\alpha u$—the function $(D^\alpha u)(x', 0)$—exists for $|\alpha| < k-1$ provided that $\varrho$ is of type I, III or IV and, moreover,

$$(D^\alpha u)(x', 0) = 0$$

if $\varrho$ is of type IV.

1.9. Remark: Let us consider the more general space

$$V^{k,p}(R_+^N; S) \quad \text{with } S = \{\varrho_\alpha : |\alpha| < k\}$$

and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1}, \alpha_N)$ be a fixed multi-index with $\alpha_N > 1$. If we put

$$\tilde{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{N-1}, \alpha_N - 1)$$

then we immediately see from Subsection 1.8(i)-(iii) that the weight function $\varrho_\alpha$ determines the *existence* of the trace $(D^{\tilde{\alpha}}(u)(x', 0))$ of $D^{\tilde{\alpha}}u$, while the weight function $\varrho_{\tilde{\alpha}}$ determines the *properties* of this trace.

1.10 *The extension operator E.* For $u \in V^{k,p}(\mathrm{R}^N_+; \varrho)$ define

$$(1.18)\qquad (Eu)(x) = (Eu)(x', x_N) = \left| \begin{array}{ll} u(x', x_N) & \text{if } x_N > 0\,, \\ \sum\limits_{j=1}^{k+1} \lambda_j u(x', -jx_N) & \text{if } x_N < 0\,, \end{array} \right.$$

where $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$ are real numbers uniquely determined by the system of lineqr equations

$$(1.19)\qquad \sum_{j=1}^{k+1} (-j)^m \lambda_j = 1\,, \qquad m = 0, 1, \dots, k\,.$$

Consequently, $Eu$ is defined for almost all $x = (x', x_N) \in \mathrm{R}^N$ with $x_N \neq 0$. Moreover, if the limit

$$\lim_{x_N \to 0+} (D^\alpha u)(x', x_N) \qquad (|\alpha| < k - 1)$$

exists and has the (finite) value $(D^\alpha u)(x', 0)$, then on account of (1.19) the limit

$$\lim_{x_N \to 0-} (D^\alpha Eu)(x', x_N)$$

also exists and has the same value $(D^\alpha u)(x', 0)$.

Thus, if $u \in V^{k,p}(\mathrm{R}^N_+; \varrho)$ with $\varrho$ of type I, III or IV, then $Eu$ has (distributional) derivatives of order $|\alpha| < k$ in the whole of $\mathrm{R}^N$ and we can easily check that for all $\alpha = (\alpha_1, \dots, \alpha_{N-1}, \alpha_N)$ with $|\alpha| < k$, we have

$$(1.20)\qquad D^\alpha(Eu) = \tilde{E}_\alpha(D^\alpha u)$$

where $\tilde{E}_\alpha$ is defined by

$$(1.21)\qquad (\tilde{E}_\alpha v)(x', x_N) = \left| \begin{array}{ll} v(x', x_N) & \text{if } x_N > 0\,, \\ \sum\limits_{j=1}^{k+1} (-j)^{\alpha_N} \lambda_j v(x', -jx_N) & \text{if } x_N < 0\,. \end{array} \right.$$

Moreover, if $u \in C^k(\overline{\mathrm{R}^N_+})$, then $Eu \in C^k(\mathrm{R}^N)$.

1.11 *The property* $(H)$. We say that a weight function $\varrho$ has *property* $(H)$ if, for every pair of given positive constants $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 < \varepsilon_2$, there are positive constants $C_1, C_2$ such that

$$\varepsilon_1 < t/s < \varepsilon_2 \quad \text{implies} \quad C_1 < \varrho(t)/\varrho(s) < C_2 . \tag{1.22}$$

1.12. THEOREM: Let $\varrho$ be a weight function of type I, III or IV, and let $\varrho$ have property $(H)$. Then the extension operator $E$ defined by formula (1,18) maps the space $V^{k,p}(\mathrm{R}_+^N; \varrho)$ linearly and continuously into the space $W^{k,p}(\mathrm{R}^N; \varrho)$.

PROOF: Obviously, $E$ is a linear operator.

Now, let $u \in V^{k,p}(\mathrm{R}_+^N; \varrho)$. Then in view of (1.20), (1.21), we have that

$$\begin{aligned}
(1.23)\qquad & \int_{\mathrm{R}^N} |D^\alpha(Eu)(x)|^p \varrho(|x_N|)\, dx = \\
& = \int_{\mathrm{R}_+^N} |D^\alpha u(x)|^p \varrho(x_N)\, dx + \int_{\mathrm{R}_-^N} \left|\sum_{j=1}^{k+1} (-j)^{\alpha_N} \lambda_j D^\alpha u(x', -jx_N)\right|^p \varrho(-x_N)\, dx < \\
& < \|D^\alpha u\|^p_{p,\varrho,\mathrm{R}_+^N} + \left(\sum_{j=1}^{k+1} |j^{\alpha_N} \lambda_j|^{p'}\right)^{p/p'} \sum_{j=1}^{k+1} \int_{\mathrm{R}_-^N} |D^\alpha u(x', -jx_N)|^p \varrho(-x_N)\, dx ,
\end{aligned}$$

where $\|v\|_{p,\varrho,\omega}$ stands for $\left(\int_\omega |v(x', x_N)|^p \varrho(x_N)\, dx'\, dx_N\right)^{1/p}$.

The substitution $y' = x'$, $y_N = -jx_N$ yields

$$I_{j,\alpha} := \int_{\mathrm{R}_-^N} |D^\alpha u(x', -jx_N)|^p \varrho(-x_N)\, dx = \int_{\mathrm{R}_+^N} |D^\alpha u(y', y_N)|^p \varrho(y_N/j) j^{-1}\, dy ;$$

since $\varrho$ has property $(H)$ and $1 < y_N/(y_N/j) < k+1$, there exist positive numbers $C_1, C_2$ such that

$$C_1 < \varrho(y_N)/\varrho(y_N/j) < C_2 ,$$

and we obtain

$$I_{j,\alpha} < (C_1 j)^{-1} \int_{\mathrm{R}_+^N} |D^\alpha u(y)|^p \varrho(y_N)\, dy = (C_1 j)^{-1} \|D^\alpha u\|^p_{p,\varrho,\mathrm{R}_+^N} . \tag{1.24}$$

Thus we have from (1.23), (1.24) that

$$\|D^\alpha(Eu)\|_{p,\varrho,\mathrm{R}^N} < \bar{K} \|D^\alpha u\|_{p,\varrho,\mathrm{R}_+^N}$$

where the constant $\bar{K}$ depends only on $p$, $k$ and $\alpha$. Consequently, we see that

$$Eu \in W^{k,p}(\mathrm{R}^N; \varrho) \tag{1.25}$$

and

$$\|u\|_{k,p,\varrho,\mathbb{R}^N} \leq K \|u\|_{k,p,\varrho,\mathbb{R}^N_+} \tag{1.26}$$

with $K$ independent of $u$. □

1.13. Remark: The extension operator $E$ given by formula (1.18) « preserves the space $C^k$ », i.e. $Eu \in C^k(\mathbb{R}^N)$ if $u \in C^k(\overline{\mathbb{R}^N_+})$. From the point of view of Sobolev spaces, this claim is superfluous, since we need only the *existence* of distributional derivatives and their *integrability* but not their *continuity*. For example, if we consider the space

$$V^{1,p}(\mathbb{R}^N_+;\varrho)$$

then the extension operator $E$ is given by

$$(Eu)(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N > 0 \\ 3u(x', -x_N) - 2u(x', -2x_N) & \text{if } x_N < 0 \end{cases}$$

but in fact we can use the simpler operator

$$(E_0 u)(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N > 0 \\ u(x', -x_N) & \text{if } x_N < 0 \end{cases} \tag{1.27}$$

provided that there exists a finite limit

$$\lim_{x_N \to 0+} u(x', x_N) = a_0(x') \qquad \text{for almost all } x' \in \mathbb{R}^{N-1}. \tag{1.28}$$

Indeed, it can be easily shown that the distributional derivative $\partial(E_0 u)/\partial x_N$ exists [the existence of the derivatives $\partial(E_0 u)/\partial x_i$ for $i = 1, \ldots, N-1$ is obvious due to (1.27)] and that

$$\int_{\mathbb{R}^N} \left| \frac{\partial(E_0 u)}{\partial x_N}(x) \right|^p \varrho(|x_N|)\, dx = 2 \int_{\mathbb{R}^N_+} \left| \frac{\partial u}{\partial x_N} \right|^p \varrho(x_N)\, dx. \tag{1.29}$$

Since the existence of the finite limit (1.28) is guaranteed if we suppose that $\varrho$ is of type I, III or IV, we immediately obtain the following analogue of Theorem 1.12:

*If $\varrho$ is a weight function of type I, III or IV, then the extension operator $E_0$ defined by formula* (1.27) *maps $V_+^{1,p}(\mathbb{R}^N;\varrho)$ linearly and continuously into $W^{1,p}(\mathbb{R}^N;\varrho)$.*

1.14. Remark: In fact, the last assertion is an *improvement* of Theorem 1.12 for the case $k = 1$: we do not need the very restrictive assumption that $\varrho$ has property $(H)$.

Unfortunately, this assumption is needed if we work with $V^{k,p}(\mathbf{R}^N_+;\varrho)$, $k>2$, and also if we consider general domains $\Omega$ instead of the special domain $\mathbf{R}^N_+$.

## 2. - The main result

2.1. In this section, we shall show how the extension operator can be constructed for weighted Sobolev spaces on more general (bounded) domains in $\mathbf{R}^N$. Let us start with some auxiliary remarks.

2.2. *Domains of class $C^k$.* (i) Let $k$ be a positive integer. For the exact definition of a domain of class $C^k$, we refer to [3], [6]. Here, we only recall that for such a domain $\Omega$, there exists a finite family of open sets $U_0, U_1, \dots, U_m$ such that

$$\bar{U}_0 \subset \Omega\,, \qquad \Omega \subset \bigcup_{i=0}^{m} U_i\,, \qquad \partial\Omega \subset \bigcup_{i=1}^{m} U_i \tag{2.1}$$

and that the set

$$\Gamma_i = \partial\Omega \cap U_i \tag{2.2}$$

can be described by a function $a_i \in C^k(\bar{\Delta})$ where $\Delta$ is an (open) ball in $\mathbf{R}^{N-1}$: we have

$$\Gamma_i = \{y = (y', y_N) : y' \in \Delta,\ y_N = a_i(y')\} \tag{2.3}$$

where the local coordinate system $y = (y', y_N)$ depends on $i \in \{1, 2, \dots, m\}$.

(ii) Using the partition of unity $\{\varphi_i\}_{i=0}^m$, subordinate to the covering $\{U_i\}_{i=0}^m$, so that

$$\varphi_i \in C_0^\infty(U_i)\,, \qquad 0 \le \varphi_i(x) \le 1 \quad \text{for } x \in \mathbf{R}^N\,, \tag{2.4}$$

the investigation of a function $u$ defined on $\Omega$ can be reduced to that of the function

$$v_i = u\varphi_i\,, \qquad v_i = v_i(y) = v_i(y', y_N) \tag{2.5}$$

defined on

$$V_i = \Omega \cap U_i\,. \tag{2.6}$$

Moreover, there exists a number $\delta_0 > 0$ such that for all $i \in \{0, 1, \dots, m\}$,

$$V_i = \{y = (y', y_N) : y' \in \Delta,\ a_i(y') - \delta_0 < y_N < a_i(y')\} \tag{2.7}$$

while

$$\tilde{V}_i = (\mathbf{R}^N \setminus \bar{\Omega}) \cap U_i = \{y = (y', y_N) : y' \in \Delta,\ a_i(y') < y_N < a_i(y') + \delta\}\,. \tag{2.8}$$

(iii) Finally, we can « straighten the boundary » $\partial\Omega$ using a regular mapping $\Phi_i$,

$$(2.9) \qquad z = \Phi_i(y), \qquad z = (z', z_N), \qquad y = (y', y_N)$$

with

$$(2.10) \qquad z_N = a_i(y') - y_N$$

which maps $V_i$ onto the half-space $R^N_+$; the part $\Gamma_i$ of the boundary $\partial\Omega$ is mapped onto a part of the hyperplane $z_N = 0$.

2.3. *The weight function.* Let $\Omega$ be a domain of class $C^{\kappa}$ and put

$$(2.11) \qquad d(x) = \operatorname{dist}(x, \partial\Omega).$$

We shall consider weight functions of the type

$$(2.12) \qquad \varrho(d(x))$$

where $\varrho = \varrho(t)$ is defined for $t > 0$. Moreover, we shall suppose that $\varrho$ is of the type described in Remark 0.4, so that only the behaviour of $\varrho(t)$ for small $t$ $[0 < t < \delta$, where $\delta = \delta_0/k$ with $\delta_0$ from Subsection 2.2(ii)] will be important.

Using these facts and the notation from Subsection 2.2, it can be shown that in $U_i$, the distance $d(x)$ and the « distance to $\partial\Omega$ in the direction of the $y_N$-axis »

$$(2.13) \qquad d_0(y) = |a(y') - y_N| \qquad \text{for } y = (y', y_N) \in U_i$$

are equivalent. More precisely: there exist constants $c_1, c_2 > 0$ such that for $x = y = (y', y_N) \in U_i \setminus \Gamma_i$,

$$(2.14) \qquad c_1 < \frac{d(x)}{d_0(y)} < c_2$$

(see [4], Lemma 4.6).

2.4. *The weighted Sobolev space.* Let us consider

$$(2.15) \qquad u \in W^{k,p}(\Omega; \varrho)$$

where $\Omega$ is of class $C^{\kappa}$, $\varrho$ is the function from Subsection 2.3 and, moreover, $\varrho$ has property $(H)$ (see Subsection 1.11). Then we can replace the investigation of

$$\int_{\Omega} |D^{\alpha} u(x)|^p \varrho(d(x))\, dx$$

by that of

$$\int_{V_i} |D^{\alpha} v_i(y)|^p \varrho(d_0(y))\, dx$$

[for $V_i, v_i$ see (2.5), (2.7); we use (2.14) due to property $(H)$]. Using now the mapping $\Phi_i$ from (2.9), we can restrict our considerations to the investigation of

$$\int_{\mathbf{R}_+^N} |D^\alpha v_i(\Phi_i^{-1}(\xi))|^p \varrho(\xi_N)\, d\xi'\, d\xi_N$$

since of view of (2.13) and (2.10), $d_0(y) = |\xi_N|$. Thus we arrive at the space

$$V^{k,p}(\mathbf{R}_+^N; \varrho) \tag{2.16}$$

from Subsection 1.4:

$$u \in W^{k,p}(\Omega; \varrho) \Leftrightarrow \tilde{v}_i = v_i \circ \Phi_i^{-1} \in V^{k,p}(\mathbf{R}_+^N; \varrho). \tag{2.17}$$

[Note that $\tilde{v}_i$ has the property (i) from Subsection 1.2 since $v_i = u\varphi_i$ with $\varphi_i \in C_0^\infty(U_i)$.]

Now we can construct the extension of $\tilde{v}_i$ from $\mathbf{R}_+^N$ to the whole of $\mathbf{R}^N$ according to Theorem 1.12; then, returning to the initial coordinates $x$ and to the initial function $u$, we immediately obtain

2.5. Theorem: Let $\Omega$ be a bounded domain in $\mathbf{R}^N$ of class $C^k$, for some $k \in \mathbf{N}$, let $\varrho = \varrho(t)$ be a (weight) function of type I, III or IV, let $\varrho$ have property $(H)$, and suppose that $1 < p < \infty$. Then there exists a bounded linear map

$$E: W^{k,p}(\Omega; \varrho) \to W^{k,p}(\mathbf{R}^N; \varrho)$$

such that for every $u \in W^{k,p}(\Omega; \varrho)$, the restriction of $Eu$ to $\Omega$ coincides with $u$.

## REFERENCES

[1] V. I. Burenkov, *On the density of infinitely differentiable functions in Sobolev spaces for an arbitrary open set*, Trudy Mat. Inst. Steklov, 131 (1974), pp. 39-50; English translation in Proc. Steklov Inst. Math., 13 (1974).

[2] V. I. Burenkov, *On a method of extending differentiable functions*, Trudy Mat. Inst. Steklov, 140 (1976), pp. 27-67; English translation in Proc. Steklov Inst. Math., 000 (1979).

[3] A. Kufner - O. John - S. Fučík, *Function Spaces*, Academia, Prague (1977).

[4] A. Kufner, *Weighted Sobolev spaces*, John Wiley and Sons Ltd., Chichester (1985).

[5] A. Kufner - B. Opic, *How to define reasonably weighted Sobolev spaces*, Comment. Math. Univ. Carolin, 25 (1984), pp. 537-554.

[6] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague and Masson et Cie, Paris (1967).

[7] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton (1970).