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Density and Total Density of the Torsion Part of a Compact Topological Group (**) (***)

SUMMARY. — We study the class $\mathfrak{J}$ of compact groups having dense torsion part. We also consider the subclass $\mathfrak{J}'$ of $\mathfrak{J}$ consisting of those elements G of $\mathfrak{J}$ such that every closed subgroup of G belongs to $\mathfrak{J}$. We give various characterizations of these classes, and we show that $\mathfrak{J}'$ can be considered as a natural extension of the class of all compact Lie groups. As an application of our results, we give a new direct proof of Prodanov's conjecture asserting that every element of $\mathfrak{J}'$ is finite-dimensional.

Densità e densità totale della parte di torsione di un gruppo topologico compatto

RIASSUNTO. — Nel presente lavoro si studiano la classe $\mathfrak{J}$ dei gruppi compatti aventi parte di torsione densa e la sottoclasse $\mathfrak{J}'$ di $\mathfrak{J}$ formata dai gruppi compatti in cui ogni sottogruppo chiuso appartiene alla classe $\mathfrak{J}$. Come applicazione si ottiene una dimostrazione diretta della congettura di Prodanov secondo la quale ogni gruppo della classe $\mathfrak{J}'$ è di dimensione finita. Si dimostra che la classe $\mathfrak{J}'$ ha ulteriori proprietà che permettono di considerarla come una generalizzazione naturale della classe dei gruppi compatti di Lie.

0. - INTRODUCTION

A Hausdorff topological group G is said to be *minimal* (*totally minimal*) if every continuous isomorphism (epimorphism) $G \to H$ onto a Hausdorff topological group H is open. Clearly every compact group is totally minimal. Examples of non-compact totally minimal groups can be obtained by means of the following stronger notion of density. A subset H of a topological group G is said to be *totally dense* (*essential*) if for every closed non-trivial normal subgroup N of G the intersection $N \cap H$ is dense in N ($N \cap H \neq$

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$\neq \{1\}$). A dense subgroup H of a compact group G is totally minimal (minimal) iff H is totally dense (essential) in G ([DP1]). In this way all minimal and totally minimal abelian groups are described since minimal abelian groups are precompact by Prodanov-Stoyanov's theorem ([PS]).

Describing the minimal torsion abelian groups Prodanov and the author [PD2] proved that for every compact abelian group G the following conditions are equivalent: (*a*) the torsion subgroup $T(G)$ of G is totally dense; (*b*) $T(G)$ is essential; (*c*) G contains copies of the group $\mathbb{Z}_p$ of p-adic integers for no prime p. These groups, named *exotic tori*, were thoroughly studied in [DP2]. Further results in this direction were obtained in [D] where the class $\mathcal{C}$ of compact abelian groups containing copies of $\mathbb{Z}_p^{\mathbb{N}}$ for no prime p was characterized as completions of the minimal abelian groups of countable free-rank. It was proved earlier in [P] that the subclass of $\mathcal{C}$ consisting of the groups containing copies of $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}}$ for no prime p coincides with the completions of the countable minimal abelian groups.

Khan [K] characterized the locally compact abelian groups satisfying (*c*) in terms of the dual groups. Stoyanov and the author [DS] showed that the conditions (*a*)-(*c*) are not equivalent for arbitrary compact groups. On the other hand, they showed that the equivalence holds even for arbitrary locally compact groups if in (*a*) and (*b*) one takes all closed subgroups in checking total density or essentiality respectively (see Theorem 3.1 below).

Here we study the class $\mathfrak{I}$ of compact groups having dense torsion part. Since $\mathfrak{I}$ is not closed with respect to closed subgroups, we consider also the subclass $\mathfrak{I}'$ of $\mathfrak{I}$ consisting of those groups $G \in \mathfrak{I}$ such that $H \in \mathfrak{I}$ for every closed subgroup H of G. It was proved in [DS] that a compact group G satisfies (*c*) iff $G \in \mathfrak{I}'$. Clearly $\mathfrak{I}'$ contains all compact Lie groups, so following [DS] we call a group $G \in \mathfrak{I}'$ an *exotic Lie group*. In 1985 Prodanov conjectured that exotic Lie groups are finite-dimensional. Proving that a compact connected group is an exotic Lie group iff it is finite-dimensional and its center is an exotic torus Stoyanov and the author [DS] answered positively. As an application of our results on $\mathfrak{I}$ we give a new proof of Prodanov's conjecture.

In § 1 we give some preliminary facts about properties of compact connected groups and exotic tori. In § 2 we give various characterizations of $\mathfrak{I}$, in particular we give a chain of other six classes between $\mathfrak{I}$ and $\mathfrak{I}'$. Exotic Lie groups are studied in § 3.

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1. - Preliminaries

Let G be a group, we denote by $Z(G)$ the *center* of G and by $T(G)$ *the set of torsion elements* of G (the latter is a subgroup if G is abelian). If G is a topological group, we denote by G_0 the *connected component* of 1 and by G'—the *closure of the commutator* (derived group) of G. $\mathfrak{Z}$ will denote the class of all (compact) topological groups G such that $G = \overline{T(G)}$ (that is $T(G)$ is dense in G). We will use also the following notations: $\mathbf{Z}$—the additive group of the integers, $\mathbf{Z}(n) = \mathbf{Z}/n\mathbf{Z}$, $\mathbf{T}^n$—the n-dimensional torus, $\mathbf{P}$—the set of the prime numbers, $\mathbf{N}$—the naturals (that is the non-negative integers), $\cong$—isomorphism between topological groups, G^*—the group of the continuous characters of G, $H^\perp$—the annihilator of H ($H^\perp \subset G^*$ for $H \subset G$). Throughout the paper $p_1, p_2, \ldots, p_n, \ldots$ will be the usual ordering of the primes with $p_n < p_{n+1}$ for all n. For $n \in \mathbf{N}$ we denote by $SU(n)$ the group of $n \times n$ unitary matrices A with $\det A = 1$.

Compact connected groups: We often use the fact that for every locally compact group G, $\dim G = \operatorname{ind} G = \operatorname{Ind} G$ ([Pa]), and for every closed subgroup H of G, $\dim G = \dim H + \dim G/H$ ([Y]).

The following properties of the compact connected groups will be used very often too.

1.1. Proposition: Let G be a compact connected group. Then:

(*a*) ([W], ch. V) $G \cong (A \times L)/N$, where A is a compact connected abelian group, L is a product of compact connected and simply connected simple Lie groups, and N is a closed totally disconnected central subgroup of $A \times L$. If in addition $\dim G < \infty$, then $\dim A < \infty$ and L is a Lie group, in particular G is metrizable.

(*b*) ([CR], Theorem 4.2) $G/Z(G) \cong \prod_{i \in I} H_i$, where every H_i is a compact connected algebraically simple Lie group;

(*c*) ([M]) The maximal connected abelian subgroups of G are maximal abelian subgroups, they are pairwise conjugate and cover G.

Remark: Note that in spite of (*c*) the maximal abelian subgroups even in compact connected simple Lie groups are not necessarily connected. One can take $SO(3)$ for example. This corrects an erroneous result in [M].

Proposition 1.1 implies also that for every continuous epimorphism $g\colon G \to H$ of compact connected groups G and H we have $g(Z(G)) = Z(H)$, $g(Z(G)_0) = Z(H)_0$ and $g(G') = H'$.

In the representation given in (*a*) without loss of generality we may assume that $(A \times \{e\}) \cap N = \{(0, e)\}$, where e denotes the neutral element of L. In this case consider the projections π_1 and π_2 of $A \times L$ on A and L, respectively. Let $K = \pi_2(N)$. For $y \in K$ take the only $x \in \pi_1(N)$ with $(x, y) \in N$ and set $\varphi(y) = x$. In this way we get a continuous homomorphism $\varphi\colon K \to A$. Following Baum [B] we denote N by (K, φ). Thus, $G = (A \times L)/ /(K, \varphi)$, where K is a closed totally disconnected central subgroup of L, $\varphi\colon K \to A$ is a continuous homomorphism and $N = \{(\varphi(y), y) : y \in K\}$.

Denote by f the epimorphism $A \times L \to G$ with $\operatorname{Ker} f = N$, then

$$A \simeq f(A) = Z(G)_0 \quad \text{and} \quad f(L) = G'$$

since $L = L'$ and also $Z(G)_0 \cap G' = f(K) = f(\varphi(K))$, in particular $Z(G)_0 \cap \cap G' \simeq \varphi(K)$. Now the equality $G = Z(G)_0 \cdot G'$ gives the following isomorphisms

$$\text{(1)} \qquad G/Z(G)_0 \simeq G'/G' \cap Z(G)_0 , \qquad G/Z(G) \simeq G'/G' \cap Z(G) ,$$
$$G/G' \simeq Z(G)_0/G' \cap Z(G)_0 \simeq Z(G)/G' \cap Z(G)$$
$$\text{and } Z(G)/Z(G)_0 \simeq G' \cap Z(G)/G' \cap Z(G)_0 .$$

The last isomorphism follows from $Z(G) = Z(G)_0 \cdot (G' \cap Z(G))$. The second isomorphism implies $Z(G') = Z(G) \cap G'$.

Note that in the representation $G \simeq A \times L/(K, \varphi)$ the groups A and L are uniquely determined. In fact, $A \simeq Z(G)_0$, on the other hand by Proposition 1.1 (*b*) $L \simeq \prod_{i \in I} L_i$, where for every $i \in I$, L_i is the simply connected covering group of H_i.

Exotic tori: We collect in the next three propositions some of the main results in [DP2].

1.2. Proposition: For a compact abelian group G the conditions (*a*), (*b*) and (*c*) from the introduction are equivalent, moreover they are equivalent to each of the following conditions:

(i) $n = \dim G < \infty$ and for every continuous epimorphism $f \colon G \to \mathbf{T}^n$ we have $\operatorname{Ker} f \simeq \prod_{p \in P} G_p$, where G_p is a compact abelian p-group;

(ii) $n = \dim G < \infty$ and there exists a continuous epimorphism f as in (i).

This yields in particular that every totally disconnected exotic torus is a product of compact abelian p-groups. Let us mention here that the implication (*b*) $\Rightarrow$ (*a*) was proved separately in [DP2].

1.3. Proposition: Let G be a compact abelian group and $n = \dim G < \infty$. Then G is an exotic torus iff there exists a sequence $H_0 \supset H_1 \supset \ldots \supset H_m \supset \ldots$ of closed subgroups of G such that

$$\text{(2)} \qquad \bigcap_{m=0}^{\infty} H_m = (0) , \quad G/H_0 \simeq \mathbf{T}^n \text{ and } H_{m-1}/H_m \text{ is a } p_m\text{-group } (m > 1) .$$

If these conditions are fulfilled, then $G/H_m \simeq \mathbf{T}^n \times B_m$ for any $m > 1$, where B_m is a compact group such that $p_1^{k_1} \ldots p_m^{k_m} B_m = (0)$ for some $k_1, \ldots, k_m \in \mathbf{N}$.

If $\{\hat{H}_m\}_m$ is another sequence of closed subgroups of G satisfying (2), then $\hat{H}_m = H_m$ for all sufficiently large m. If G is connected, then $B_m = (0)$ and H_{m-1}/H_m is finite for every $m > 1$.

Remark that Proposition 1.3 may be formulated also in terms of projective limits as in [DP2] in fact. Namely, the exotic tori are projective limits of projective systems

$$G_0 \xleftarrow{\sigma_1} G_1 \xleftarrow{\sigma_2} \dots \longleftarrow G_{n-1} \xleftarrow{\sigma_n} G_n \longleftarrow \dots$$

where $G_0 \simeq \mathrm{T}^n$, $\operatorname{Ker}\sigma_n$ is a compact p_n-group and $G_m \simeq \mathrm{T}^n \times B_m$ with B_m as above ($m = 1, 2, \dots$). In § 3 we obtain a similar representation for connected exotic Lie groups.

1.4. Proposition: Let $0 \to G_1 \to G \to G_2 \to 0$ be an exact sequence of compact abelian groups and continuous homomorphisms. Then G is an exotic torus iff both G_1 and G_2 are exotic tori.

2. - Compact groups with dense torsion part

In this section we study the class $\mathfrak{J}$ of the compact groups with dense torsion parts. Clearly, $\mathfrak{J}$ is closed under products and quotients and every compact Lie group is in $\mathfrak{J}$. Therefore by Proposition 1.1 for every compact connected group G we have $G' \in \mathfrak{J}$, $Z \cap G' \in \mathfrak{J}$. The isomorphisms (1) imply also that $G/Z(G)_0$, $G/Z(G)$ and $Z(G)/Z(G)_0$ are in $\mathfrak{J}$.

The abelian groups in $\mathfrak{J}$ are characterized by Pontryagin duality. The next lemma is folklore.

2.1. Lemma: If G is a compact abelian group, then $G \in \mathfrak{J}$ iff $\bigcap_{n=1}^{\infty} nG^* = (0)$.
As an immediate consequence we get the following.

2.2. Corollary: Let G be a compact abelian connected group. Then:

1) ([St]) $\overline{T(G)}$ splits and $G/\overline{T(G)} \simeq \mathrm{Q}^{*\alpha}$ for some cardinal α;
2) For G the following conditions are equivalent:
   i) $G \in \mathfrak{J}$,
   ii) there exists a closed subgroup H of G such that H and G/H belong to $\mathfrak{J}$,
   iii) there is no continuous epimorphism $G \to \mathrm{Q}^*$,
   iv) G does not contain a direct summand isomorphic to Q^*.

Examples: *a*) T^{N} contains a subgroup isomorphic to Q^*, nevertheless $T^{\mathrm{N}} \in \mathfrak{J}$, so in iv) it is important to exclude only the direct summands isomorphic to Q^*.

b) The following example shows that the connectedness of G is important for ii). Let $X = \bigoplus_{n=1}^{\infty} \mathbf{Z}(p^n)$ and for every $n \in \mathrm{N}$ choose a generator e_n of $\mathbf{Z}(p^n)$ in such a way that e_n is identified with pe_{n+1} by the natural identification of $\mathbf{Z}(p^n)$ with $p\mathbf{Z}(p^{n+1})$. Set $Y = \langle p^{n-1}e_n - p^n e_{n+1} : n \in \mathrm{N}\rangle$ and $M = X/Y$. Then $e_1 + Y \in nM$ for every $n \in \mathrm{N}$, therefore $G = M^* \notin \mathfrak{J}$ by Lemma 1.1. On the other hand, $S = \langle e_1 + Y\rangle \simeq \mathbf{Z}(p)$ and $M/S \simeq X$, so $H = S^\perp \in \mathfrak{J}$ and $G/H \simeq \mathbf{Z}(p) \in \mathfrak{J}$.

The last example shows that $\mathfrak{J}$ is not closed under extensions. In fact, it can be shown that the smallest class of compact abelian groups containing $\mathfrak{J}$ and closed with respect to extensions is the class $\mathfrak{J}'$ of all compact abelian groups G such that G^* is reduced. Equivalently, $G \in \mathfrak{J}'$ iff $T(G/H) \neq (0)$ for every proper closed subgroup H of G. On the other hand, *a*) shows that $\mathfrak{J}$ is not closed with respect to closed subgroups. By Proposition 1.2 the largest subclass of $\mathfrak{J}$ which is closed with respect to closed subgroups is the class of the exotic tori.

In spite of the counterexample given above we have the following positive result.

2.3. Lemma: Let G be a compact abelian group and let H be a closed connected subgroup of G. Then $H, G/H \in \mathfrak{J}$ imply $G \in \mathfrak{J}$.

Proof: Let $f: G \to G/H$ be the canonical homomorphism. To show that $f(T(G)) = T(G/H)$ take $t \in T(G/H)$. Then $t = f(x)$ for some $x \in G$ and $x^n \in H$ for some $n \in \mathrm{N}$. Since every compact connected group is divisible, there exists $h \in H$ with $x^n = h^n$. Then $t' = xh^{-1} \in T(G)$ and $f(t') = f(x) = t$.

Now the compactness of G implies $f(\overline{T(G)}) = \overline{T(G/H)}$, therefore by $G/H \in \mathfrak{J}$ we get $G = \overline{T(G)} + H$. If in addition $H \in \mathfrak{J}$, then clearly $G \in \mathfrak{J}$.

Q.E.D.

To finish the first part of this section related to the abelian case, consider the following situation. Let G be a compact abelian group and let H and H_1 be closed subgroups of G such that $G = H + H_1$ and $H_1 \in \mathfrak{J}$. Now $H \in \mathfrak{J}$ yields $G \in \mathfrak{J}$, while $G \in \mathfrak{J}$ implies $H/H \cap H_1 \in \mathfrak{J}$ and none of these implications is reversible even when $H = G_0$ (for the first implication see the part $(f) \not\Rightarrow (e)$ in the proof of Theorem 2.8, the group $Z = Z_0(Z \cap G')$; for the second one see the part $(g) \not\Rightarrow (f)$ of the same proof, the same group). Some positive results in this direction for particular choices of H and H_1 in the non-abelian case will be given below.

2.4. Lemma: Let G be a compact group and let H be a closed central subgroup of G with $H \in \mathfrak{J}$. If H/H_0 is an exotic torus, then $G \in \mathfrak{J}$ iff $G/H \in \mathfrak{J}$.

Proof: By virtue of Proposition 1.2 we may assume that $H/H_0 = \prod_{p \in P} F_p$, where each F_p is a compact abelian p-group. Let $\psi: H \to H/H_0$ and $\varphi: G \to G/H$

be the canonical homomorphisms and suppose that $G/H \in \mathfrak{Z}$. To show that $G \in \mathfrak{Z}$ take an arbitrary element x of G and an open neighbourhood U of x in G. Then $\varphi(U)$ is an open neighbourhood of $\varphi(x)$ in G/H, so by $G/H \in \mathfrak{Z}$ there exists $u' \in T(G/H) \cap \varphi(U)$. Choose any $u \in U$ with $\varphi(u) = u'$, then there exists an $n \in \mathbf{N}$ with $u^n \in H$. Let $S = \{p \in \mathbf{P} : p | n\}$, then $\psi(u^n) = yz$ with $y \in \prod_{p \in S} F_p$ and $z \in \prod_{p \in \mathbf{P} \setminus S} F_p$. Clearly, $y^m = 1$ for some $m \in \mathbf{N}$ and $z = z_1^n$ for some $z_1 \in H/H_0$. Take $v \in H$ with $\psi(v) = z_1$, then

$$\psi(u^{mn}) = (\psi(u^n))^m = (yz)^m = z^m = z_1^{mn} = \psi(v^{mn}),$$

so $(uv^{-1})^{mn} \in H_0$. The latter group is compact and connected, hence it is divisible. Therefore $(uv^{-1})^{mn} = c^{mn}$ for some $c \in H_0$, then $t = uv^{-1}c^{-1} \in T(G)$. By $H \in \mathfrak{Z}$ for $vc \in H$ there exists a net (t_α) in $T(H)$ converging to vc. Then clearly $tt_\alpha \to u$, which by $u \in U$ implies $tt_\alpha \in U$ for some α. Since t and t_α belong to $T(G)$ and t_α is central, this yields $tt_\alpha \in T(G)$. Q.E.D.

The next theorem characterizes the connected groups in $\mathfrak{Z}$.

2.5. Theorem: Let G be a compact connected group. Then the following conditions are equivalent:

(*a*) $G \in \mathfrak{Z}$;

(*b*) $G/G' \in \mathfrak{Z}$;

(*c*) $B \in \mathfrak{Z}$ for every maximal connected abelian subgroup B of G;

(*d*) $\bigcap_{n=1}^{\infty} nG^* = (0)$;

(*e*) there exist no continuous epimorphisms $G \to \mathbf{Q}^*$.

Proof: (*a*) ⇒ (*b*) is obvious, (*c*) ⇒ (*a*) follows from Proposition 1.1 (*c*). On the other hand, (*b*) ⇒ (*d*) by virtue of Lemma 2.1, since $G^* = (G/G')^*$. By Corollary 2.2 (*b*) means that there does not exist a continuous epimorphism $G \to \mathbf{Q}^*$, thus (*b*) is equivalent to (*e*). To conclude the proof it remains to verify the implication (*b*) ⇒ (*c*). The latter follows from the next lemma which is probably known. Q.E.D.

2.6. Lemma: Let G be a compact connected group. Then every maximal connected abelian subgroup of G is topologically isomorphic to $G/G' \times T$, where T is a maximal connected abelian subgroup of G', that is T is a finite-dimensional torus if G' is a Lie group, and $T = \mathbf{T}^{w(G')}$ otherwise. Every continuous epimorphism $\varphi : G \to H$, where H is a compact connected group, preserves the maximal connected abelian subgroups.

Proof: Every maximal connected abelian subgroup of G contains $Z(G)$. Consider a representation $G \cong A \times L/N$ as in Proposition 1.1 (*a*), then the canonical homomorpgism

$$g\colon A\times L \to L/Z(L) \tag{3}$$

preserves the maximal connected abelian subgroups since L is a product of compact connected Lie groups. Therefore the epimorphism $f\colon A\times L\to G$ also preserves the maximal connected abelian subgroups. Moreover, by Proposition 1.1 (c) every maximal connected abelian subgroup of G is an image of a subgroup of $A\times L$ of the same type.

Take an arbitrary maximal connected abelian subgroup T_1 of L, then T_1 is a torus if L is a Lie group, otherwise $T_1\simeq \mathbf{T}^{w(L)}$. Since $\operatorname{Ker} f$ is totally disconnected, $T=f(T_1)\simeq T_1$ and T (resp. $f(A\times T_1)$) is a maximal connected abelian subgroup of G' (resp. G) by the above argument. Since every torus splits, $f(A\times T_1)\cong H\times T$, where H is topologically isomorphic to $f(A\times T_1)/T$. So it remains to show that $f(A\times T_1)/T\simeq G/G'$. We have $f(A\times T_1)/T=(f(A)+f(T_1))/f(T_1)\simeq f(A)/f(A)\cap f(T_1)$. We will check that

$$f(A)\cap f(T_1)=f(K)=Z_0\cap G'. \tag{4}$$

Clearly, $f(K)=f(\varphi(K))\subset f(A)\cap f(T_1)$. To verify the other inclusion take $a\in A$ and $t\in T_1$ with $f(a)=f(t)$. Then $a-t$ belongs to N, therefore there exists $y\in K$ such that $a-t=\varphi(y)+y$. Then $a-\varphi(y)=y+t\in A\cap L=(0)$, so $a=\varphi(y)\in\varphi(K)$. This proves (4). Now we have $f(A)/f(A)\cap f(T_1)=$ $=Z_0/Z_0\cap G'\simeq G/G'$ which proves the first part of the lemma.

Now consider a continuous epimorphism $\varphi\colon G\to H$, where H is a compact connected group. First assume that $H=G/Z(G)$ and φ is the canonical epimorphism. Since every maximal connected abelian subgroup of G is an image of a maximal connected abelian subgroup of $A\times L$ and the maximal connected abelian subgroups of $A\times L$ are mapped on maximal connected subgroups of $L/Z(L)\cong G/Z(G)$ (see (3)), it is clear that $G\to G/Z(G)$ preserves the maximal connected abelian subgroups.

In the general case consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ G/Z(G) & \xrightarrow{\varphi_1} & H/Z(H) \end{array}$$

where t and s are the canonical epimorphisms and φ_1 is the epimorphism induced by φ $\big($recall that $\varphi(Z(G))=Z(H)\big)$. By Proposition 1.1 (b), $G/Z(G)$ and $H/Z(H)$ are products of compact connected algebraically simple Lie groups, so φ_1 preserves the maximal connected abelian subgroups. Now the preceding argument applied to t and s proves that for every maximal connected abelian subgroup T of G, $t(G)$ and $\varphi_1(t(T))$ are maximal connected abelian subgroups of $G/Z(G)$ and $H/Z(H)$, respectively. Thus $s(\varphi(T)) = \varphi_1(t(T))$ is a maximal connected abelian subgroup of $H/Z(H)$. Since every maximal connected abelian subgroup of H contains $Z(H)$ and $s(\varphi(T))$ is an image under s of a maximal connected abelian subgroup of H containing $\varphi(T)$, this proves that $\varphi(T)$ is a maximal connected abelian subgroup of H. Q.E.D.

The next corollary strengthens Lemma 2.4 for connected group.

2.7. Corollary: Let G be a compact connected group and let H be a closed normal subgroup of G with $H \in \mathfrak{J}$. Then $G \in \mathfrak{J}$ iff $G/H \in \mathfrak{J}$.

Proof: Suppose $G/H \in \mathfrak{J}$. According to the above theorem to show $G \in \mathfrak{J}$ it suffices to establish that $G/G' \in \mathfrak{J}$. Consider the subgroup HG'/G' of G/G', it is isomorphic to $H/H \cap G'$, so it belongs to $\mathfrak{J}$. On the other hand, $(G/G')/(HG'/G') \simeq G/HG'$ is isomorphic to a quotient of G/H, so it also belongs to $\mathfrak{J}$. Now by Corollary 2.2, $G/G' \in \mathfrak{J}$. Q.E.D.

The next corollary strengthens a result of Stewart [St] (see Corollary 2.2 (1) above).

2.8. Corollary: For every compact connected group G with $\dim G' < \infty$ $\overline{T(G)}$ is a closed connected normal subgroup of G contaihing G' and $G \cong \overline{T(G)} \times \mathbf{Q}^{*\alpha}$ for some cardinal α. In particular, $G \in \mathfrak{J}$ iff $Z(G)_0 \in \mathfrak{J}$.

Proof: Since $G' \in \mathfrak{J}$ we have $G' \subset \overline{T(G)}$. Next consider the equality $G = Z(G)_0 \cdot G'$. By Corollary 2.2 there exists a closed subgroup H of $Z(G)_0$, isomorphic to $\mathbf{Q}^{*\alpha}$ for some cardinal α, such that $Z(G)_0 = H \cdot \overline{T(Z(G)_0)}$ and $H \cap \overline{T(Z(G)_0)} = (1)$. Take $x \in H \cap \left(\overline{T(Z(G)_0)} \cdot G'\right)$, then $x = zg$, where $z \in \overline{T(Z(G)_0)}$ and $g \in G'$. Clearly, g belongs to the group $G' \cap Z(G)_0$ which is finite, so $g^n = 1$ for some $n \in \mathbf{N}$. Since $z \in Z(G)$, we have $x^n = z^n g^n = z^n$, so $x^n = z^n \in H \cap \overline{T(Z(G)_0)} = (1)$. Therefore, $x = 1$ since H is torsion-free. So $G = H \times H_1$, where $H_1 = \overline{T(Z(G)_0)} \cdot G'$. Clearly, H_1 is a closed connected normal subgroup of G and $H_1 \in \mathfrak{J}$, hence $H_1 \subset \overline{T(G)}$. On the other hand, $\overline{T(G)} \subset H_1$ since H is torsion-free. Q.E.D.

We have seen above that, in particular, $\overline{T(G)} = \overline{T(Z(G)_0)} \cdot G'$. This follows also from $T(G) = T(Z(G)_0) \cdot T(G')$ which results directly from

$$\operatorname{card}\,(Z(G)_0 \cap G') < \infty .$$

The next theorem shows that this may fail if $\dim G' = \infty$, it shows also that $G \in \mathfrak{J}$ does not imply $Z(G)_0 \in \mathfrak{J}$ in general.

2.9. Theorem: Let G be a compact connected group. Then each of the next conditions implies the following one and none of the implications is reversible:

(*a*) $Z(G)$ is an exotic torus;

(*b*) $Z(G_0)$ is an exotic torus;

(*c*) $Z(G)_0 \cap G'$ is an exotic torus and $Z(G)_0/Z(G)_0 \cap G'$ (which is isomorphic to G/G') belongs to $\mathfrak{Z}$;

(*d*) $Z(G)_0 \cap G'$ and $Z(G)_0/Z(G)_0 \cap G'$ belong to $\mathfrak{Z}$;

(*e*) $Z(G)_0 \in \mathfrak{Z}$;

(*f*) $Z(G) \in \mathfrak{Z}$;

(*g*) $G \in \mathfrak{Z}$.

Proof: The implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$ are obvious, $(d) \Rightarrow (e)$ follows from Corollary 2.2. Since $Z/Z_0 \in \mathfrak{Z}$, the implication $(e) \Rightarrow (f)$ follows from Lemma 2.3. Finally, Theorem 2.5 (*b*) and the isomorphism $G/G' \simeq Z(G)/Z(G) \cap G'$ yield $(f) \Rightarrow (g)$.

To provide counterexamples for the reverse implications we are going to use the representation from Proposition 1.1 (*a*) with $L = \prod_{n=1}^{\infty} L_n$ and $L_n = SU(k_n)$ for $n \geq 1$. Then $Z(L) \simeq \prod_{n=1}^{\infty} \mathbf{Z}(k_n)$. Choosing appropriate compact connected abelian groups A, closed subgroups K of $Z(L)$ and continuous homomorphisms $\varphi\colon K \to A$ we obtain the necessary groups $G = (A \times L)/(K, \varphi)$.

$(b) \nRightarrow (a)$. Take $A = (0)$, $k_n = n$ and $K = (0)$. Then $Z(G)_0 = (0)$ is an exotic torus and $Z(L) = \prod_{n=1}^{\infty} \mathbf{Z}(n)$ is not.

$(c) \nRightarrow (b)$. Let q be a prime and M be a subgroup of $\mathbf{Q}^*$ isomorphic to $\mathbf{Z}_q$. Then for $A = \mathbf{Q}^*/M$ we have $A \in \mathfrak{Z}$. Choose L, K and φ as above, then $Z(G)_0$ is not an exotic torus but $Z(G)_0/Z(G)_0 \cap G' = Z(G)_0 \in \mathfrak{Z}$ and $Z(G)_0 \cap G'$ is an exotic torus.

$(d) \nRightarrow (c)$. Choose a prime number p and set $A = \mathbf{T}^{\mathbf{N}}$, $k_n = p^n$ $(n = 1, 2, \ldots)$, $K = Z(L)$ and let $\varphi\colon K \to A$ be any continuous monomorphism. Now $Z(G)_0 \cap G' \simeq K \in \mathfrak{Z}$ and $Z(G)_0 \in \mathfrak{Z}$ but $Z(G)_0 \cap G'$ is not an exotic torus.

$(e) \nRightarrow (d)$. Let p and q be distinct prime numbers. Take A as in $(c) \nRightarrow (b)$ and set $k_n = p^n$ for any n. Denote by c_n a generator of $\mathbf{Z}(p^n)$ and consider the element $c = (c_n)_{n=1}^{\infty}$ of $Z(L)$. Let K be the closed subgroup of $Z(L)$ generated by c. Then $K \simeq \mathbf{Z}_p$ and by $p \neq q$ there exists a continuous monomorphism $\varphi\colon K \to A$. Now $Z(G)_0 \simeq A \in \mathfrak{Z}$ and $Z(G)_0 \cap G' \simeq K$ is torsion-free.

$(f) \not\Rightarrow (e)$. Set $A = \mathbf{Q}^*$ and $k_n = n$. Then $Z(L)$ contains a closed subgroup $B \simeq \prod_{p \in \mathbf{P}} \mathbf{Z}_p$. Let K be the closed subgroup of B corresponding to $\prod_{p \in \mathbf{P}} p\mathbf{Z}_p$ by this isomorphism. Fix a non-trivial continuous character $\chi: \mathbf{Q}^* \to \mathbf{T}^1$, then $\operatorname{Ker} \chi = K$. Let $\varphi: K \to \operatorname{Ker} \chi$ be a fixed isomorphism. Then $Z(G)_0 \simeq \mathbf{Q}^*$ is torsion-free and we have to show that $Z(G) \in \mathfrak{Z}$.

Let $H = \overline{T(Z(G))}$ and consider the canonical homomorphism $f: A \times L \to G$. Now $Z(L) \in \mathfrak{Z}$ implies $f(Z(L)) \in \mathfrak{Z}$, thus $f(Z(L)) \subset H$. Then $H_1 = f^{-1}(H)$ is a closed subgroup of $A \times Z(L)$ containing $\varphi(K)$. For every $p \in \mathbf{P}$ choose an element y_p of $B \setminus pB$, clearly $py_p \in K \setminus pK$. Now for every $p \in \mathbf{P}$ chose $x_p \in A$ with $px_p = \varphi(py)$. Then $t_p = (x_p, y_p) \in A \times Z(L)$ and $pt_p \in (K, \varphi)$. This means that $f(t_p) \in T(Z(G))$, so $f(t_p) \in H$ and consequently $x_p \in H_2$, where H_2 is a closed subgroup of A containing $\varphi(K)$ and such that $H_1 = H_2 \times Z(L)$. Moreover $x_p \notin \varphi(K)$, otherwise $x_p = \varphi(z)$ for some $z \in K$ would imply $\varphi(pz) = \varphi(py_p)$ and $pz = py_p$, which implies $y_p = z \in K \subset pB$—a contradiction (remind that B is torsion-free). It was established in this way that for every $p \in \mathbf{P}$ there exists an element $x_p \in H_2 \setminus \varphi(K)$ with $px_p \in \varphi(K) = \operatorname{Ker} \chi$. Therefore $\chi(H_2)$ contains the socle of $\mathbf{T}^1$, so it coincides with $\mathbf{T}^1$. Since $H_2 \supset \operatorname{Ker} \chi$, this gives $H_2 = A$. Thus $Z(G) \in \mathfrak{Z}$.

$(g) \not\Rightarrow (f)$. Take $A = \mathbf{Q}^*$ and L and K as in $(e) \not\Rightarrow (d)$. Now $K \simeq \mathbf{Z}_p$, so there exists a continuous monomorphism $\varphi: K \to A$. Then $Z(G)_0 / Z(G)_0 \cap G' \simeq A/\varphi(K) \in \mathfrak{Z}$. By the isomorphism $G/G' = Z(G)_0 / Z(G)_0 \cap G'$ and Theorem 2.5 we get $G \in \mathfrak{Z}$.

We show next that $T(Z(G)) = T(Z(G) \cap G')$. This will imply $\overline{T(Z(G))} = \overline{T(Z(G) \cap G')} = Z(G) \cap G'$ (note that the latter is totally disconnected) and therefore $\overline{T(Z(G))} \neq Z(G)$.

Let $f: A \times L \to G$ be the canonical homomorphism and consider its restriction $f_1: A \times Z(L) \to Z(G)$. If $f_1(x, y) \in T(Z(G))$ is non-trivial, then there exists $n > 1$ with $ny \in K$ and $nx = \varphi(ny)$. Since for every $q \in \mathbf{P} \setminus \{p\}$, $A \times Z(L)$ has no non-trivial q-torsion elements and both $A \times Z(L)$ and (K, φ) are q-divisible, we may assume without loss of generality that $n = p^s$. If $x = 0$, then $\varphi(p^s y) = 0$. Since φ is a monomorphism, this gives $p^s y = 0$. So $y \in T(Z(L))$ and $f(x, y) \in T(Z(G) \cap G')$. Assume now that $x \neq 0$, then y is not torsion. Since K is a $\mathbf{Z}_p$-module, there exist an invertible element ξ of the ring $\mathbf{Z}_p$ and $t > 0$ such that $p^s y = p^t \xi c$ in K. Then clearly $t > s$, so we can define $z = p^{t-s} \xi c - y$. Now $p^s z = 0$, so $z \in T(Z(L))$. On the other hand, $p^s x = \varphi(p^s y) = \varphi(p^s p^{t-s} \xi c) = p^s \varphi(p^{t-s} \xi c)$. Since A is torsion-free, this yields $x = \varphi(p^{t-s} \xi c)$, that is $f(x, p^{t-s} \xi c) = 0$ in G. Hence $f(x, y) = f(x, p^{t-s} \xi c - z) = f(x, p^{t-s} \xi c) - f(0, z) = -f(0, z)$, so $f(x, y) \in T(Z(G) \cap G')$. Q.E.D.

The last example shows, in particular, that there are compact connected groups G with $\overline{T(G)} \supsetneq \overline{T(Z) \cdot G'}$.

3. - Exotic Lie groups

Here we study the class $\mathfrak{I}'$ under various aspects—its locally compact version and its relation to (and other possible generalizations of) the exotic tori. In Theorem 3.4 we prove Prodanov's conjecture and in Theorem 3.5 we give two versions of Proposition 1.3 for exotic Lie groups.

The torsion part of an arbitrary connected Lie group need not be dense (take for example the reals). This fact and the next result (in particular (*e*)) show that compactness is not an essential restriction in the definition of exotic Lie groups.

3.1. Theorem ([DS]): For a locally compact group G the following conditions are equivalent:

(*a*) for every closed subgroup H of G $\overline{T(H)} = H$;

(*b*) for every closed non-trivial subgroup H of G $T(H) \neq \{1\}$;

(*c*) every closed non-trivial subgroup of G contains a minimal closed non-trivial subgroup of G;

(*d*) every element of G is contained in a compact subgroup of G and G contains copies of $\mathbf{Z}_p$ for no prime p;

(*e*) there exists a compact open subgroup N of G satisfying (*a*) such that every infinite cyclic sybgroup of G meets $N\setminus\{1\}$.

Locally compact groups satisfying (*c*) (i.e. with atomic lattice of closed subgroups) were described earlier by Muhin and Homenko. Their paper is inaccessible to the author.

We consider now another possibility to generalize exotic tori in the non-abelian case—in the spirit of (*a*) and (*b*) from the Introduction, i.e. the lattice of closed normal subgroups is atomic in terms of (*c*) of the above theorem.

3.2. Theorem: For a compact connected group G the following conditions are equivalent:

(*a*) $T(G)$ is totally dense in G;

(*b*) $T(G)$ is essential in G;

(*c*) every closed non-trivial normal subgroup of G contains a minimal closed non-trivial normal subgroup of G;

(*d*) G does not contain closed normal subgroups topologically isomorphic to $\mathbf{Z}_p$ for every $p \in \mathbf{P}$;

(*e*) $Z(G)$ is an exotic torus.

Proof: Obviously, $(a) \Rightarrow (b) \Rightarrow (d) \Rightarrow (e)$ and $(c) \Rightarrow (d)$ holds for every topological group G.

Suppose that G is compact and connected. To prove the theorem we have to establish the implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$.

Let (b) holds and let K be a closed non-trivial normal subgroup of G. If $K_1 = Z(G) \cap K \neq (1)$, then K_1 is a non-trivial closed normal subgroup of G, so it contains a torsion element $t \neq 1$. Now the subgroup $\langle t \rangle$ is finite, therefore it contains a minimal non-trivial subgroup S. Then S is closed and normal since $S \subset Z(G)$. Further, assume

$$Z(G) \cap K = (1). \tag{5}$$

Observe first that without loss of generality we may assume that K is connected. In fact, by (5), K can be embedded in $G/Z(G)$ and by Proposition 1.1 (b) K is not totally disconnected. Thus, $K_0 \neq (1)$ will be also a closed normal subgroup of G.

Now the connectedness of G and K and (5) yield $Z(K) = (1)$. By Proposition 1.1 (b), K contains minimal closed non-trivial normal subgroups. It remains to apply Theorem 3 from [1] which says that for a connected group G and a compact normal subgroup K of G every closed normal subgroup of K is a normal subgroup of G. This proves the implication $(b) \Rightarrow (c)$.

Now suppose that $Z(G)$ is an exotic torus. To show that $T(G)$ is totally dense consider a closed normal subgroup K of G. Then $K/K \cap Z(G)$ is a closed normal subgroup of $G/Z(G)$. By Proposition 1.1 (b), $T(G/Z(G))$ is totally dense in $G/Z(G)$, therefore $K/K \cap Z(G) \in \mathfrak{I}$. On the other hand, $K \cap Z(G)$ is an exotic torus being a closed subgroup of $Z(G)$. Now Lemma 2.4 can be applied to K and its central subgroup $K \cap Z(G)$. This gives $K \in \mathfrak{I}$ and proves the implication $(c) \Rightarrow (a)$. Q.E.D.

Remarks: 1) It is clear now that the non-equivalent conditions in Theorem 2.9 vary from total density to density of the torsion part of G.

2) In contrast with Theorem 2.5 the total density of the torsion part of a compact connected group G does not imply the total density of the torsion part of the maximal connected abelian subgroups of G. In fact, the latter are isomorphic to $G/G' \times \mathbf{T}^{\alpha}$ and the torsion part of $\mathbf{T}^{\alpha}$ is not totally dense whenever $\dim G' = \infty$ (in this case $\alpha = w(G')$). On the other hand, the torsion part of G/G' is totally dense.

3) The fact that a compact group with totally dense torsion part may have an infinite dimension shows that the exotic Lie groups are the « right » non-abelian version of the exotic tori (see also Theorem 3.4 below).

The next proposition is the non-abelian version of Proposition 1.2.

3.3. Proposition: Let

$$1 \to H \to G \xrightarrow{\varphi} K \to 1 \tag{6}$$

be an exact sequence of compact groups and continuous homomorphisms. Then G is an exotic Lie group iff H and K are exotic Lie groups.

PROOF: The necessity remains true if we replace « exotic Lie group » by « locally compact group satisfying the equivalent conditions in Theorem 3.1 ». In fact, if G satisfies (*d*) from Theorem 3.1, then clearly every element of K is contained in a compact subgroup of K. Assume that K contains a subgroup N isomorphic to $\mathbf{Z}_p$ for some $p \in \mathbf{P}$. Choose $x \in G$ such that $g(x)$ generates N as a closed subgroup. Then the closed subgroup N_1 of G generated by x is compact and $g(N_1) = N$. By the well known projective property of $\mathbf{Z}_p$ there exists a subgroup of N_1 isomorphic to $\mathbf{Z}_p$—a contradiction.

The sufficiency is obvious for compact groups. Q.E.D.

REMARKS: 1) We do not know if the sufficiency can be proved under the weaker assumption of local compactness.

2) By Propositions 1.1 (*b*) and 1.2 and Theorem 3.2 it follows easily that an analogous result is true for compact connected groups with totally dense torsion part.

Contrary to the abelian case we have no satisfactory description of the totally disconnected exotic Lie groups.

The next theorem generalizes Proposition 1.2 and shows that for every exotic Lie group G there exists an exact sequence of the type (6) with K—a compact Lie group and H—a totally disconnected exotic Lie group.

3.4. THEOREM: For every compact group G the following conditions are equivalent:

(*a*) G is an exotic Lie group;

(*b*) $n = \dim G < \infty$ and for every continuous epimorphism $g: G \to L$, where L is a compact Lie group with $\dim L = n$, $\operatorname{Ker} g$ is a totally disconnected exotic Lie group;

(*c*) $n = \dim G < \infty$ and there exists g as in (*b*);

(*d*) $n = \dim G < \infty$, $Z(G)_0$ is an exotic torus and G/G_0 is an exotic Lie group.

PROOF: Let G be an exotic Lie group, then $\dim G = \dim G_0$. Since $Z(G_0)$ is an exotic torus, we have $\dim Z(G_0) < \infty$. On the other hand, the maximal connected abelian subgroups of G_0 are exotic tori, hence they are finite-dimensional. By Lemma 2.6 this gives $\dim G_0' < \infty$, hence $\dim G_0 = \dim G_0' + \dim Z(G_0) < \infty$. The rest of (*b*) follows from Proposition 3.3. This proves the implication $(a_1 \Rightarrow (b)$.

For $(b) \Rightarrow (c)$ it suffices to note that if $n = \dim G < \infty$, then there exists a continuous epimorphism $g: G \to L$, where L is a compact Lie group with $\dim L = n$ (cf. for example [Po], ch. 8, § 47, Theorem 69).

The implications $(c) \Rightarrow (a)$ and $(a) \Rightarrow (d)$ follow from the above argument and Proposition 3.3.

To verify $(d) \Rightarrow (a)$ suppose (d) holds for G. It is enough to show G_0 is an exotic Lie group. By $\dim G_0 < \infty$ we have that $G_0/Z(G_0)$ is a Lie group, therefore by Proposition 3.3 G_0 is an exotic Lie group. Q.E.D.

Let G be an exotic Lie group and let H be a maximal connected Lie group in G. Then H is a normal subgroup of G and $K = G/H$ contains no infinite Lie groups. This decomposition is in some sense opposite to that one given above.

By Proposition 1.1 (a) every finite-dimensional compact connected group G is the projective limit of a projective system

$$G_0 \xleftarrow{\sigma_1} G_1 \xleftarrow{\sigma_2} G_2 \leftarrow \dots \leftarrow G_{n-1} \xleftarrow{\sigma_n} G_n \leftarrow \dots, \tag{7}$$

where G_n are compact connected Lie groups with $\dim G_n = \dim G$ and σ_n are continuous epimorphisms with finite kernels.

3.5. Theorem: Let G be a compact connected finite-dimensional group.

(a) The following conditions are equivalent:

$(a1)$ G is an exotic Lie group;

$(a2)$ for every projective system (7) with $G = \varprojlim (G_n, \sigma_n)$ and for every $p \in \mathbf{P}$ the set

$$F_p = \{n \in \mathbf{N}\colon p \text{ divides } |\operatorname{Ker} \sigma_n|\}$$

is finite;

$(a3)$ there exists a projective system (7) with $G = \varprojlim (G_n, \sigma_n)$ such that F_p is finite for every $p \in \mathbf{P}$;

$(a4)$ there exists a compact connected Lie group G_0 and a projective system (7) as in $(a3)$ with $G_n = G_0$ for each $n \in \mathbf{N}$.

(b) G is an exotic Lie group iff there exists a projective system (7) with $G = \varprojlim (G_n, \sigma_n)$ such that $\operatorname{Ker} \sigma_n$ is a finite p_n-group for each $n \in \mathbf{N}$.

Proof: (a) To prove $(a1) \Rightarrow (a2)$ assume that G is an exotic Lie and (7) is a projective system with $G = \lim (G_n, \sigma_n)$.

Construct by induction maximal connected abelian subgroups A_n of G_n such that $A_n \supset Z(G_n)$ and $\sigma_{n+1}(A_{n+1}) = A_n$ for every $n \in \mathbf{N}$ (see the proof of Lemma 2.6). Let us note that $\operatorname{Ker} \sigma_n$ is a totally disconnected subgroup of G_n, thus by [H] we have $\operatorname{Ker} \sigma_n \subset Z(G_n)$.

Suppose that there is $p \in \mathbf{P}$ such that F_p is infinite. Let $k = \dim A_n$. Then $A_n \simeq \mathbf{T}^k$ and k does not depend on n since the Lie groups G_n are pairwise locally isomorphic. For $n \in \mathbf{N}$ denote by T_n the p-torsion part of $\operatorname{Ker} \sigma_n$. Then for each $n \in \mathbf{N}$ the restriction $\bar{\sigma}_n : T_n \to T_{n-1}$ is surjective. Then (7) gives the projective system

$$T_0 \xleftarrow{\bar{\sigma}_1} T_1 \xleftarrow{\bar{\sigma}_2} T_2 \xleftarrow{\bar{\sigma}_3} \dots \xleftarrow{\bar{\sigma}_{n-1}} T_{n-1} \xleftarrow{\bar{\sigma}_n} \dots \,. \tag{8}$$

Clearly $T = \varprojlim (T_n, \sigma_n)$ is isomorphic to a subgroup of G, so T is an exotic Lie group by Proposition 3.3. Now observe that each T_n is a subgroup of $\mathbf{Z}(p^\infty)^k$. Since finite abelian groups are selfdual we get the injective system

$$T_0 \xrightarrow{\bar{\sigma}_1^*} T_1 \xrightarrow{\bar{\sigma}_2^*} T_2 \xrightarrow{\bar{\sigma}_3^*} \dots \xrightarrow{\bar{\sigma}_{n-1}^*} T_{n-1} \xrightarrow{\bar{\sigma}_n^*} \dots$$

by taking the dual of (8). Moreover, σ_n^* is not an isomorphism iff $n \in F_p$. Thus our assumption yields that $\hat{T} = \varinjlim (T_n, \bar{\sigma}_n^*)$ is an infinite subgroup of $\mathbf{Z}(p^\infty)^k$. Then $\hat{T}$ contains a copy of the group $\mathbf{Z}(p^\infty)$. Since the latter group is divisible, this means that there exists a surjective homomorphism $\hat{T} \to \mathbf{Z}(p^\infty)$. Taking the duals we get an embedding $\mathbf{Z}_p \to T$ which contradicts Theorem 3.1.

$(a3) \Rightarrow (a1)$. Assume that G is not an exotic Lie group. Then by Theorem 3.1 there exists $p \in \mathbf{P}$ and a subgroup L of G topologically isomorphic to $\mathbf{Z}_p$. Suppose that (7) is a projective system satisfying $(a3)$. Let $\psi_n : G \to G_n$ be the natural homomorphism ($n \in \mathbf{N}$). Since G_n is a Lie group, $\psi_n(L)$ is a finite p-group. Moreover, by the choice of L for every $n \in \mathbf{N}$ there exists $k_n \in \mathbf{N}$ such that $L \cap \operatorname{Ker} \psi_n = p^{k_n} L$, in particular $|\psi_n(L)| = p^{k_n}$. The sequence $\{k_n\}_{n=1}^\infty$ is increasing by

$$\sigma_n(\psi_{n+1}(L)) = \psi_n(L) \qquad (n \in \mathbf{N}) \,. \tag{9}$$

Moreover, $n \in F_p$ if $k_n < k_{n+1}$. Since $\bigcap\limits_{n=1}^{\infty} \operatorname{Ker} \psi_n = \{1\}$, (9) yields $k_{n+1} > k_n$ for infinitely many $n \in \mathbf{N}$. Thus F_p is infinite—a contradiction.

Since $(a2) \Rightarrow (a3)$ and $(a4) \Rightarrow (a3)$ are trivial, it remains to prove that $(a3) \Rightarrow (a4)$. In fact, suppose that (7) is a projective system satisfying $(a3)$. Since all groups G_n are locally isomorphic, there exists a sequence $k_1 < k_2 < \dots < k_n < \dots$ such that all groups G_{k_n} are pairwise topologically isomorphic. Now the projective system (G_{k_n}, ϱ_n), where ϱ_n is the composition of the homomorphisms $\sigma_{k_{n-1}+1}, \dots, \sigma_{k_n}$, satisfies $(a4)$ since

$$|\operatorname{Ker} \varrho_n| = \prod \{|\operatorname{Ker} \sigma_i| : k_{n-1} < i \le k_n\} \,.$$

This proves (a).

(b) The sufficiency follows directly from (a). To prove the necessity let $A = Z(G)_0$. Then $G \cong (A \times L)/(K, \varphi)$, where L is a semisimple compact connected Lie group, K is a finite central subgroup of L and $\varphi : K \to A$ is a homomorphism. We can assume that φ is a monomorphism, otherwise we can replace L by L/Ker. Suppose that G is an exotic Lie group. Then A is an exotic torus and by Proposition 1.3 there exist closed subgroups

$$N_1 \supset N_2 \supset \ldots \supset N_n \supset \ldots$$

of A such that $\bigcap_{n=1}^{\infty} N_n = \{1\}$, and for $d = \dim A$ we have $A/N_n \simeq \mathrm{T}^d$ and N_{n+1}/N_n is a finite p_n-group for every $n \in \mathrm{N}$. Since $\varphi(K)$ is a finite subgroup of A, we may assume that $\varphi(K) \cap N_1 = \{1\}$. Consider the subgroups

$$H_n = N_n \cdot (K, \varphi) \qquad (n = 1, 2, \ldots)$$

of $A \times L$. If $(x, y) \in \bigcap_{n=1}^{\infty} H_n$, then $y \in K$ and there exists $x_n \in N_n$ for each $n \in \mathrm{N}$ such that $(x, y) = (x_n, 1) \cdot (\varphi(y), y)$. Hence $x = x_n \cdot \varphi(y)$ which yields $x \cdot \varphi(y)^{-1} \in \bigcap_{n=1}^{\infty} N_n = \{1\}$. Thus $x = \varphi(y)$ and $(x, y) \in (K, \varphi)$. Therefore

$$(K, \varphi) = \bigcap_{n=1}^{\infty} H_n . \tag{10}$$

Let $f\colon A \times L \to G$ be the natural homomorphism. Since $\mathrm{Ker} f = (K, \varphi)$ (10) implies $\bigcap_{n=1}^{\infty} f(H_n) = \{1\}$. Moreover,

$$G/(H_n) \simeq (A \times L/(K, \varphi))/(H_n/(K, \varphi)) \simeq (A \times L)/(N_n(K, \varphi)) .$$

Denote by φ_n the composition of φ and the natural homomorphism $A \to A/N_n$. Then φ_n is a monomorphism, and

$$G_n = G/f(H_n) \simeq ((A/N_n) \times L)/(K, \varphi_n) \simeq (\mathrm{T}^d \times L)/(K, \varphi_n)$$

is a Lie group locally isomorphic to $\mathrm{T}^d \times L$. Moreover, the kernel of the natural homomorphism $\sigma_n\colon G_{n+1} \to G_n$ is a finite p_n-group since $\mathrm{Ker}\, \sigma_n = f(H_{n+1})/f(H_n) \simeq N_{n+1}/N_n$.

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