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Butler Modules in Torsion Theories (***).

ABSTRACT. — The notion of Butler group is carried into the context of hereditary torsion theories τ over associative rings with identity R. Sufficient conditions are given for the class of τ-Butler R-modules to be included in the class of τ-torsion-free R-modules generated by finitely many τ-cocritical submodules, and for the converse inclusion. Two particular cases of coincidence of these two classes are investigated.

Moduli di Butler in teorie della torsione.

RIASSUNTO. — La nozione di gruppo di Butler viene estesa in questo lavoro al contesto delle teorie della torsione ereditarie τ su anelli associativi con identità R. Vengono date condizioni sufficienti affinchè la classe degli R-moduli di τ-Butler sia contenuta nella classe degli R-moduli privi di τ-torsione, che sono generati da un numero finito di sottomoduli τ-cocritici, e per l'inclusione opposta. Sono poi studiati due casi particolari, in cui queste due classi di moduli coincidono.

INTRODUCTION

Butler studied in [B1] the class of pure subgroups of finite direct sums of rank one torsion-free abelian groups; these groups are now generally referred to as « Butler groups ». In particular, he showed that every Butler group is generated by finitely many rank one (pure) subgroups or, equivalently, that it is a pure quotient of a finite direct sum of rank one torsion-free groups; conversely, he showed that every such group is a Butler group.

The class of Butler groups is one of the most investigated classes of torsion-free abelian groups of finite rank; see the papers by Arnold [A1], Arnold and Vinsonhaler [AV] and references there.

Recently, in the authors' paper [BS], Butler groups have been generalized to the infinite rank case; this subject received relevant contributions by many authors; see [BSS], [A2], [DR], [D], [AH] and [DHR].

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Our goal in this paper is to generalize Butler groups in a completely different direction. A very natural setting to generalize Butler groups is that of hereditary torsion theories τ over arbitrary associative rings with identity R, such that there exist τ-cocritical R-modules. Recall that an R-module M is τ-cocritical if it is τ-torsion-free, and every proper quotient of M is τ-torsion; so τ-cocritical modules are a good generalization of torsion-free abelian groups of rank one.

τ-cocritical modules were introduced by Goldman [Go1], [Go2], in order to generalize simple modules and modules with composition series; in fact, given the hereditary torsion theory τ in R-Mod, Goldman defined a τ-composition series for a module in the obvious way in terms of τ-cocritical modules; he also showed that for modules with finite τ-composition series the τ-length of the series is well defined. Hence the τ-torsion-free modules of finite τ-length are a good generalization of torsion-free groups of finite rank.

We will use also some properties of τ-semicocritical modules (i.e. submodules of finite direct sums of τ-cocritical modules), investigated by Lau [L] and Teply [T], as developed by Golan in [G].

In this context it is natural to define a τ-*Butler module* as a τ-pure submodule of a finite direct sum of τ-cocritical modules. We give sufficient conditions for a τ-Butler module to be generated by finitely many (τ-pure) τ-cocritical submodules, and for the converse.

As applications of these results, we show that in two particular and very different situations the two classes of τ-pure submodules and τ-pure quotients, respectively, of finite direct sums of τ-cocritical modules coincide, as in the case of abelian groups.

First we deal with the Dickson (semisimple) torsion theory over an associative ring R satisfying the following condition: every maximal left ideal P of R is left and right singly generalized by the same element: $P = pR = Rp$. In order that τ-cocritical modules do exist, it is also required that R is not τ-torsion. Some arithmetical properties of these rings are preliminarily investigated.

Then we deal with the usual torsion theory on a Prüfer domain (which coincides both with Lambek and Goldie torsion theories). In this case the localization technique, used by Butler for abelian groups, works nicely by virtue of some results on finite direct sums of uniserial modules over valuation domains obtained by Fuchs and the second author in [FS1]. Butler himself obtained in [B2] part of our results: in fact, an immediate consequence of [B2, Prop. 5] and of the fact that the lattice of ideals of a Prüfer domain R is distributive, is that the class of torsion-free R-modules generated by finitely many rank one submodules is closed under taking pure submodules. From the point of view of the present investigation, the most interesting result in [B2] is Theorem 3, which shows that the equality of the two classes of modules, holding in the two particular situations described above, does not hold in general.

1. - τ-Butler modules

Let R be an associative ring with identity; the category of unital left R-modules is denoted by R-Mod. We follow Golan [G] for terminology and notation on torsion theories in R-Mod. In particular, a torsion theory τ will always be hereditary; a submodule N of an R-module M is τ-dense (respectively τ-pure) if M/N is τ-torsion (respectively τ-torsion-free). The τ-torsion submodule of an R-module M is denoted by $T_\tau(M)$; the Gabriel filter of left τ-dense ideals of R is denoted by $\mathscr{L}_\tau(R)$ (or simply by $\mathscr{L}_\tau$). An R-module A is *τ-cocritical* if it is non-zero τ-torsion-free and every non-zero submodule of A is τ-dense. A τ-torsion-free module M has a *τ-composition series* if it has a finite chain of submodules: $0 = M_0 < M_1 < ... < M_n = M$, such that M_{i+1}/M_i is τ-cocritical for each $i < n$; evidently, each M_i is τ-pure in M. Any two τ-composition series of M have the same length, which is called the τ-length of M and is denoted by $\text{len}_\tau M$. Characterizations of modules of finite τ-length can be found in [G], [Be], [N] and [T].

A *τ-semicocritical* module is a submodule of a finite direct sum of τ-cocritical modules; the class of τ-semicocritical modules is contained (in general properly) in the class of modules of finite τ-length, and it is closed under taking τ-torsion-free quotients. For these notions and results see [L], [T] and [G].

Our goal is to extend the notion of Butler groups to the general context of a torsion theory τ over an arbitrary associative ring R. So we define a *τ-Butler* module M as a τ-pure submodule of a finite direct sum of τ-cocritical R-modules. The class of τ-Butler R-modules is denoted by $\mathscr{P}_\tau(R)$, or more simply by $\mathscr{P}_\tau$ if there is no danger of confusion.

Dually, we define a *τ-purely finitely generated* R-module as a τ-torsion-free quotient of a finite direct sum of τ-cocritical R-modules; the class of these modules is denoted by $\mathscr{F}_\tau(R)$, or more simply by $\mathscr{F}_\tau$.

Modules in $\mathscr{P}_\tau$ are obviously τ-semicocritical; the fact that also modules in $\mathscr{F}_\tau$ are τ-semicocritical follows by the quoted result that the class of τ-semicocritical modules is closed under τ-torsion-free quotients.

We are interested in the mutual inclusions of the two classes $\mathscr{P}_\tau$ and $\mathscr{F}_\tau$; if they coincide, then $\mathscr{P}_\tau = \mathscr{F}_\tau$ is the minimal class of τ-torsion-free R-modules containing the τ-cocritical modules and closed under τ-pure submodules and τ-torsion-free quotients.

In this section we give two sufficient conditions in order that a module in $\mathscr{P}_\tau$ (respectively in $\mathscr{F}_\tau$) belongs to $\mathscr{F}_\tau$ (respectively to $\mathscr{P}_\tau$).

We fix some notation. Let $A = \bigoplus_1^n A_i$ be a finite direct sum of R-modules, and B a submodule of A. For each $j < n$ we set:

$$A^j = \bigoplus_{i \neq j} A_i \quad \text{and} \quad B^j = B \cap A^j .$$

LEMMA 1.1: *If each A_i is τ-cocritical and $\mathrm{len}_\tau B > 2$, then for each $i < n$ the following statements hold:*

a) B^i is non-zero and τ-pure in B, and B/B^i is either zero or τ-cocritical;

b) there exists a j such that $B/(B^i + B^j)$ is τ-torsion.

PROOF: *a)* $B/B^i \simeq (B + A^i)/A^i < A/A^i \simeq A_i$ shows that B^i is τ-pure in B; since $\mathrm{len}_\tau A_i = 1 < \mathrm{len}_\tau B$, it follows that $B^i \neq 0$, and that either $B^i = B$, of B/B^i is τ-cocritical.

b) It is enough to show that there exists a j such that $B^j \nless B^i$. Assume that $B^j < B^i$ for all j; then $B^j = B^i$ for all j, since $B^j < B^i$ implies B/B^i τ-torsion, absurd by *a*). But $B^j = B^i$ for all j gives $0 \neq B^i = \bigcap_j B^j = 0$, absurd. □

The preceding lemma shows, in particular, that the quotient $B/\sum_i B^i$ is τ-torsion. It is of central importance to know whether this factor module is zero.

PROPOSITION 1.2: *Let B be a τ-pure submodule of $A = \overset{\bullet}{\bigoplus_i} A_i$, with A_i τ-cocritical for each i. If $B = \sum_i^{n} B^i$, then $B \in \mathscr{F}_\tau$.*

PROOF: We induct on n, which can be assumed to be larger than 2 (for $n = 2$, B is either equal to A or τ-cocritical, hence $B \in \mathscr{F}_\tau$ trivially). We can also assume $\mathrm{len}_\tau B > 2$, so Lemma 1.1 applies. B^i is τ-pure in A, hence in A^i; but $\mathrm{len}_\tau A^i < \mathrm{len}_\tau A$, hence $B^i \in \mathscr{F}_\tau$ by induction. Since $\mathscr{F}_\tau$ is closed under taking finite direct sums and τ-torsion-free quotients, $B \in \mathscr{F}_\tau$. □

Let now $C = C_1 + \dots + C_m$ be a τ-torsion-free sum of τ-cocritical modules C_i, that we assume to be different. We set, for each $i < m$: $\bar{C}_i = C/C_i$; let π: $C \to \bigoplus_i \bar{C}_i$ be the diagonal map of the canonical surjections, i.e. $\pi(c) = \sum_i (c + C_i)$ $(c \in C)$.

LEMMA 1.3: *If each C_i is τ-pure in C and $i \neq j$, then $C_i \cap C_j = 0$; hence, if $m > 2$, π is an injection.*

PROOF: $(C_i + C_j)/C_i \simeq C_j/(C_i \cap C_j)$ is a submodule of C/C_i, which is τ-torsion-free; so the fact that C_j is τ-cocritical shows that $C_i \cap C_j = 0$. The last claim is obvious, since $\ker \pi = \bigcap_i C_i$. □

PROPOSITION 1.4: *Let $C = C_1 + \dots + C_m$ be a τ-torsion-free sum of τ-pure τ-cocritical different submodules C_i. If $\pi(C)$ is τ-pure in $\bigoplus_i \bar{C}_i$, then $C \in \mathscr{F}_\tau$.*

PROOF: We induct on m, the case $m = 1$ being trivial. For each $i < m$ ($m > 2$) we have the natural epimorphism $\sum_{j \neq i} C_j \to \bar{C}_i$; let C_i^* be the τ-pure closure of the isomorphic copy of C_i in $\bar{C}_i$; then C_i^* is τ-pure in $\bar{C}_i$ and τ-cocritical, by [G, 14.3], so $\bar{C}_i \in \mathscr{P}_\tau$ by the inductive hypothesis. Then the claim follows, since $\mathscr{P}_\tau$ is closed under τ-pure submodules and finite direct sums. $\square$

The fact that the sufficient conditions of Propositions 1.2 and 1.4 come true seems to depend on arithmetical properties of the ring R with respect to the torsion theory τ. In the next sections we will show that these conditions are satisfied in two particular and very different situations.

We close this section by setting some problems. The first one has its motivation in a characterization of Butler groups by means of balanced exact sequences (see [B] and [BS]). We need the following definition: given the torsion theory τ in R-Mod and the exact sequence of R-modules.

$$0 \to A \to B \xrightarrow{\pi} C \to 0$$

with C τ-torsion-free, we say that the sequence is *τ-balanced* if every homomorphism $f\colon M \to C$, with M τ-cocritical, lifts through π to B.

PROBLEM 1: Investigate the class $\mathscr{B}_\tau$ of τ-torsion-free R-modules of finite τ-length C, such that every τ-balanced exact sequence $0 \to A \to B \to C \to 0$ with A τ-torsion splits; in particular, investigate connections between $\mathscr{B}_\tau$, $\mathscr{P}_\tau$ and $\mathscr{F}_\tau$.

PROBLEM 2: Find torsion theories τ over suitable rings R, such that $\mathscr{P}_\tau(R) \neq \mathscr{F}_\tau(R)$.

An example of a commutative integral domain R such that $\mathscr{F}_\tau(R)$ is not closed under taking τ-pure submodules (where τ denotes the usual torsion theory), hence $\mathscr{P}_\tau(R) \not\leqslant \mathscr{F}_\tau(R)$, is given in [B2].

PROBLEM 3: Are the sufficient conditions of Propositions 1.2 and 1.4, in order that $\mathscr{P}_\tau \leqslant \mathscr{F}_\tau$ and $\mathscr{F}_\tau \leqslant \mathscr{P}_\tau$, respectively, also necessary conditions?

2. - RINGS ALL WHOSE MAXIMAL LEFT IDEALS ARE TWO-SIDED PRINCIPAL IDEALS

The rings considered in this and in the next section are associative rings R with identity, which are not division rings, such that every maximal left ideal P is a two-sided ideal singly generated, on the left and on the right, by the same element p, i.e. $P = pR = Rp$.

Arithmetical properties of R.

We develope here some arithmetical properties of the rings described above.

LEMMA 2.1: *Let $P = Rp = pR$ be a maximal ideal of R. Then:*

(i) *if $a \notin P$, $b \notin P$, then $ab \notin P$;*

(ii) *$p^n \notin J$ for every $n \in \mathrm{N}$ and every maximal left ideal $J \neq P$;*

(iii) *if $a \notin P$, then $Ra + Rp^n = R$ for each $n \in \mathrm{N}$.*

PROOF: (i) There are elements $r, s \in R$ with $ra + sp = 1$ and so $rab + spb = b$ yields $b \in P$ whenever $ab \in P$.

(ii) Obviously, $p \notin J$. If $n > 2$ is the smallest integer for which $p^n \in J$, then $p^{n-1} \notin J$ and consequently $rp^{n-1} + j = 1$ for suitable elements $r \in R$ and $j \in J$; this yields a contradiction: $p = prp^{n-1} + pj = r'p^n + pj \in J$ $(r' \in R)$.

(iii) If, for some $n \in \mathrm{N}$, it is $Ra + Rp^n < R$, then there is a maximal left ideal J of R such that $Ra + Rp^n < J \neq P$, which contradicts (ii). □

LEMMA 2.2: *Let $P = Rp = pR$ be a maximal left ideal of R such that $P^{n+1} < P^n$ for each $n \in \mathrm{N}$. Then*

(i) *if $p^k x = x'p^k$, $x, x' \in R$, then $x \in Rp$ if and only if $x' \in Rp$;*

(ii) *if $a \in P^k \setminus P^{k+1}$ and $b \in P^h \setminus P^{h+1}$ for some $k, h \in \mathrm{N}$, then $ab \in P^{k+h} \setminus P^{k+h+1}$.*

PROOF: (i) Suppose that $x \notin Rp$ and $x' = sp \in Rp$. Then $rx + tp = 1$ for some $r, t \in R$ and so $p^k = p^k rx + p^k tp = (r's + t')p^{k+1}$ which yields a contradiction: $Rp^k = Rp^{k+1}$. The rest is similar.

(ii) Let $a = rp^k$, $b = sp^h$, where $r, s \notin P$. Then $ab = rs'p^{k+h}$ and $s' \notin P$ by (i). Denote $m = k + h$ and suppose that $ab = tp^{m+1}$ for some $t \in R$. Since $rs' \notin P$ by Lemma 2.1 (i), there are $u, v \in R$ such that $urs' + vp = 1$ and we get the contradiction: $p^m = urs'p^m + vp^{m+1} = (ut + v)p^{m+1}$. □

We give now a sufficient condition in order that our rings R have non zero divisors.

PROPOSITION 2.3: *Let $P = Rp = pR$ be a maximal left ideal of R. If $\bigcap_n P^n = 0$ and R contains a maximal left ideal $J \neq P$, then R is a domain.*

PROOF: By Lemma 2.1 (ii), $p^n \notin J$ for every $n \in \mathrm{N}$ and so $Rp^n \neq 0$ and $Rp^{n+1} < Rp^n$. Thus, if $a, b \in R$ are non-zero elements of R, then $a \in Rp^k \setminus Rp^{k+1}$, $b \in Rp^h \setminus Rp^{h+1}$ for some $k, h \in \mathrm{N}$, so $ab \neq 0$ by Lemma 2.2. □

*The Dickson torsion theory in R-*Mod.

In the rest of this section we will consider the Dickson torsion theory τ in R-Mod; recall that the τ-torsion modules are those with the property that each non-zero quotient has non-zero socle; and the τ-torsion-free modules are those with zero socle.

Proposition 2.4: *Assume that $P = Rp = pR$ for each maximal left ideal of R and that $\bigcap\limits_n P^n = 0$. If either R is not τ-torsion, or it contains at least two different maximal left ideals, then R is a τ-torsion-free domain.*

Proof: If R is not τ-torsion, then obviously $P^n \neq 0$ for each $n \in \mathrm{N}$ so $P^{n+1} < P^n$ and R is a domain by Lemma 2.2 (ii). The ring R is τ-torsion-free, for otherwise $0 \neq a \in T_\tau(R)$ gives $(0:a) = 0 \in \mathscr{L}_\tau(R)$, a contradiction. If R contains two maximal left ideals, then it is a domain by Proposition 2.3, and if R is not τ-torsion-free, then $0 \neq a \in \operatorname{Soc} R$ has zero annihilator ideal which is the only maximal left ideal of R, again a contradiction. □

If I is a left ideal of R, we denote $\bigcap\limits_n I^n$ by I^ω.

Lemma 2.5: *Let R be a τ-torsion-free ring such that $P = Rp = pR$ for each maximal left ideal P of R. Then R/P^ω is τ-torsion-free.*

Proof: Assume, by way of contradiction, that $0 \neq x + P^\omega \in \operatorname{Soc}(R/P^\omega)$. Then there is a maximal left ideal Rq of R with $qx \in P^\omega$. If $q \notin Rp$ then, by Lemma 2.1 (iii), for every $n \in \mathrm{N}$ it is: $rq + sp^n = 1$ for suitable $r, s \in R$, and so $x = rqx + sp^n x$. But $qx \in P^\omega$ gives $qx = zp^n$, which yields $x = rzp^n + sp^n x = (rz + sx')p^n$ $(x' \in R)$ and consequently $x \in P^\omega$, a contradiction. If $q = p$, then for each $n \in \mathrm{N}$ it is $px = p^{n+1} z \in P^\omega$. Then $x = p^n z$, R being τ-torsion-free, and so $x \in P^\omega$ gives a contradiction again. □

The following proposition characterizes, among the rings considered in this section, those which are τ-cocritical.

Proposition 2.6: *Let R be a τ-torsion-free ring such that $P = Rp = pR$ for each maximal left ideal P of R. Then R is τ-cocritical if and only if $P^\omega = 0$ for each maximal left ideal P. In this case R is a domain.*

Proof: If R is τ-cocritical, then $P^\omega = 0$ by Lemma 2.5. Conversely, let the condition be satisfied and let $L \neq 0$ be any proper left ideal of R. Then $L \leq P$ for some maximal left ideal P of R. By the hypothesis, the sequence $(L \cap P^n)_{n \in N}$ tends to 0, and so $L \cap P^k > L \cap P^{k+1}$ for some $k \in \mathrm{N}$. Taking $x \in L \cap P^k \setminus L \cap P^{k+1}$ we have $x = p^k z$, $z \in R \setminus P$. Consequently there is $1 \leq b \leq k$ such that $p^b y = x$ for a $y \notin L$ and $py \in L$. In this case $(L:y) = P$ and $\langle y + L \rangle$ is a simple submodule of R/L; therefore R is τ-cocritical. R is a domain by Proposition 2.4. □

P-height and τ-cocritical modules.

Let R be a τ-torsion-free ring, $P = Rp = pR$ a maximal left ideal of R; let A be a τ-torsion-free R-module, where τ denotes the Dickson torsion theory, as before. We define, as usual, the *P-height* of an element $a \in A$ as:

$$h_P(a) = \sup \{k \in \mathrm{N} : p^k x = a \text{ is solvable in } A\}.$$

Obviously, we set $h_P(a) = \infty$ if $p^k x = a$ is solvable in A, for each $k \in \mathbf{N}$, and $\infty = \infty + \infty = \infty + k$, for each $k \in \mathbf{N}$.

LEMMA 2.7: *Let A be a τ-torsion-free module and let $a \in A$ and $r \in R$. Then $h_P(ra) = h_P(r) + h_P(a)$.*

PROOF: We can clearly restrict ourselves to the case $h_P(a) = h < \infty$ and $h_P(r) = k < \infty$. Then $p^h x = a$ $(x \in A)$ and $p^k s = r$ $(s \in R)$ with $h_P(x) = h_P(s) = 0$. Thus $ra = p^k s p^h x = p^{h+k} s' x$, where $h_P(s') = 0$, by Lemma 2.2 (i). Denote $m = h + k$ and suppose that $p^{m+1} y = ra$ for some $y \in A$. Then $py = s'x$, A being τ-torsion-free, and from $us' + vp = 1$ for suitable $u, v \in R$ we get the contradiction: $x = us'x + vpx = p(u'y + v'x)$ $(u', v' \in R)$. $\square$

We collect in the next lemma some properties of the P-height in τ-cocritical modules.

LEMMA 2.8: *Let A be a τ-cocritical module.*

(i) *If $a, b \in A$ are non-zero elements, then there are elements $r, s \in R$ such that $rb = sa \neq 0$.*

(ii) *If A contains an element of finite P-height, then all non-zero elements of A have finite P-height.*

(iii) *If $a, b \in A$ are non-zero elements such that $h_P(a) = 0$, then there are elements $r, s \in R$ such that $h_P(r) = 0$ and $rb = sa \neq 0$.*

PROOF: (i) $A/\langle a \rangle$ is τ-torsion, so $J = (\langle a \rangle : b) \in \mathscr{L}_\tau(R)$, and $K = (0 : b) \notin \mathscr{L}_\tau(R)$. Hence $J \not\leq K$ and for some $r \in J \setminus K$ we have $rb = sa \neq 0$.

(ii) For an element $a \in A$ of finite P-height, consider the natural mapping $\varphi \colon R \to Ra/P^\omega a$. For $r \in \operatorname{Ker} \varphi \setminus P^\omega$, r is of finite P-height and the same property has ra by Lemma 2.7. Then $ra \notin P^\omega a$ and so $\operatorname{Ker} \varphi = P^\omega$. Thus $R/P^\omega \cong Ra/P^\omega a$ is τ-torsion-free by Lemma 2.5 and so $P^\omega a = 0$, Ra being τ-cocritical as a submodule of A. Now let $0 \neq b \in A$ be arbitrary. By (i), there are elements $r, s \in R$ with $rb = sa \neq 0$. But then $s \notin P^\omega$ gives that rb, and consequently b, is of finite P-height.

(iii) By (i) we have $rb = sa \neq 0$ for some $r, s \in R$. By (ii) rb is of finite P-height and so, by Lemma 2.7, we have $h_P(s) = h_P(sa) = h_P(rb) \geq h_P(r) = k$. Hence $r = p^k r'$, $r' \notin Rp = P$, $s = p^k s'$ and $r'b = s'a \neq 0$, A being τ-torsion-free. $\square$

3. - Butler modules in the Dickson torsion theory

Throughout all this section R will denote, as in the preceding section, an associative ring with 1, such that every maximal left ideal P is singly generated on the left and on the right by the same element; τ will denote the Dickson torsion theory. In order to guarantee the existence of τ-cocritical R-modules, we shall also assume that R is not τ-torsion. The goal of this section is to show that $\mathscr{P}_\tau(R) = \mathscr{F}_\tau(R)$; the first step is to reduce the investigation to the case of R τ-torsion-free.

The τ-torsion submodule $T_\tau(R)$ of R is a two-sided ideal; let us denote by S the factor ring $R/T_\tau(R)$. It is clear that, if A is a τ-torsion-free R-module, then it is a τ-torsion-free S-module in the natural way; conversely, every S-module is an R-module via the canonical projection of R onto S. The proof of the following lemma is straightforward and it is left to the reader.

Lemma 3.1: *Let A be an R-module. Then*

(i) *A is τ-torsion-free of finite length as an R-module if and only if it is τ-torsion-free of the same τ-length as an S-module.*

(ii) *A is τ-torsion as an R-module if and only if it is τ-torsion as an S-module.* □

Using Lemma 3.1 it is immediate to prove the following

Proposition 3.2: *If $\mathscr{P}_\tau(R/T_\tau(R)) = \mathscr{F}_\tau(R/T_\tau(R))$, then*

$$\mathscr{P}_\tau(R) = \mathscr{F}_\tau(R). \quad \square$$

In view of Proposition 3.2 we shall always assume, in the rest of this section, that *R is τ-torsion-free.*

Lemma 3.3: *Let A be a τ-torsion-free R-module of the form $A = A_1 + \dots + A_m$ with the A_i's τ-cocritical τ-pure submodules. Suppose that $a_i \in A_i$, $i = 1, \dots, m$, are elements such that $h_P(a_1) = \dots = h_P(a_k) = \infty$ and $h_P(a_{k+1}) = \dots = h_P(a_m) = 0$ for a $k < m$. Then for each $a \in A$ there is an $r \in R \setminus P$ such that $ra = c_1 + \dots + c_k + r_{k+1}a_{k+1} + \dots + r_m a_m$ for suitable elements $c_i \in A_i$ and $r_j \in R$.*

Proof: By hypothesis, $a = b_1 + \dots + b_m$, $b_i \in A_i$, $i = 1, \dots, m$. By Lemma 2.8 (iii) there are $t_m \in R \setminus P$ and $s_m \in R$ such that $t_m b_m = s_m a_m$, and so $t_m a = t_m(b_1 + \dots + b_{m-1}) + s_m a_m$. Similarly, there are $t_{m-1} \in R \setminus P$ and $s_{m-1} \in R$ such that

$$t_{m-1} t_m b_{m-1} = s_{m-1} a_{m-1}$$

and so

$$t_{m-1} t_m a = t_{m-1} t_m (b_1 + \dots + b_{m-2}) + s_{m-1} a_{m-1} + t_{m-1} s_m a_m .$$

Continuing in this way, after $m - k$ steps we get the desired expression, with r the product of the t_j's, which is not in P by Lemma 2.1 (i). □

We can now prove the first inclusion: $\mathscr{P}_t(R) \subseteq \mathscr{F}_\tau(R)$, by showing that the hypothesis in Proposition 1.2 is satisfied. Notation is an in Section 1.

PROPOSITION 3.4: *Let B be a τ-pure submodule of $A_1 \oplus \ldots \oplus A_n$, where each A_i is τ-cocritical. If* $\operatorname{len}_\tau B > 2$, *then* $B = \sum_i B^i$.

PROOF: From Lemma 1.1 there follows that the quotient $B/\sum_i B^i$ is τ-torsion. Assume, by way of contradiction, that $\sum_i B^i < B$; then there exists a $b = b_1 + \ldots + b_n \in B \setminus \sum_i B^i$ ($b_i \in A_i$) such that $pb \in \sum_i B^i$, where $pR = Rp = P$ is a maximal left ideal of R. Take elements $a_i \in A_i$, $i = 1, \ldots, n$, such that, possibly after a permutation of the A_i's, $h_P(a_1) = \ldots = h_P(a_k) = \infty$, $h_P(a_{k+1}) = \ldots = h_P(a_n) = 0$, where $k \leq n$. Note that necessarily $k < n$, since otherwise every element of A is of infinite P-height, and the same property has $\sum_i B^i$ by its definition, so the τ-torsion-freeness of B gives $b \in \sum_i B^i$, which is a contradiction. Now we shall show that for some $i = 1, \ldots, n$ there is an element $0 \neq b' \in B^i \setminus (A_1 \oplus \ldots \oplus A_k)$. If not, then clearly $\sum_i B^i$ is contained into $A_1 \oplus \ldots \oplus A_k$, and is suffices to show that from this inclusion the inclusion $B < A_1 \oplus \ldots \oplus A_k$ follows, since in this case the equality $B^m = B$ gives the contradiction. But if there is a $c \in B$, $c = c_1 + \ldots + c_m$ with some $c_i \neq 0$ for $i = k+1, \ldots, m$, then $J = \left(\sum_i B^i : c\right) \in \mathscr{L}_\tau(R)$ and $K = (0 : c_i) \notin \mathscr{L}_\tau(R)$, so for $r \in J \setminus K$ we get $rc \in \sum_i B^i$ and $rc \notin A_1 \oplus \ldots \oplus A_k$, rc_i being non-zero. By Lemma 3.3 there is an $r \in R \setminus P$ such that

$$rb' = d_1 + \ldots + d_k + r_{k+1}a_{k+1} + \ldots + r_m a_m \qquad (d_i \in A_i\,,\; r_j \in R)\,,$$

and at least one of the r_j's is zero. Moreover, we can suppose that at least one of the r_j's, say r_m, is not in P, for otherwise we can divide rb' by p in $\sum_i B^i$ to get an element with this property. Now, if $b \in B \setminus \sum_i B^i$ is the element above, then all the b_i's are not zero. By Lemma 2.8 (iii), there are $r \in R \setminus P$ and $s \in R$ such that $rb_m = sr_m a_m \neq 0$, so that $rb - sb' \in B^m$, and consequently $rb \in \sum_i B^i$. But $ur + vp = 1$, for some $u, v \in R$, which yields $b = urb + vpb \in \sum_i B^i$, which is a contradiction. $\square$

To prove the inclusion $\mathscr{F}_\tau(R) \subseteq \mathscr{P}_t(R)$, we need some more results and the following definition: if M is a subset of the τ-torsion-free module A, and $P = Rp = pR$ is a maximal left ideal, then the *P-pure closure* of M in A is the submodule of A:

$$\langle M \rangle_P = \{x \in A : p^k x \in \langle M \rangle \text{ for some } k \in \mathbb{N}\}\,.$$

Throughout the rest of this section, A will always denote a τ-torsion-free R-module of the form $A = A_1 + \ldots + A_m$, where the A_i are τ-cocritical τ-pure submodules.

LEMMA 3.5: *If $a_i \in A_i$ are elements of 0 P-height for all $i \leq m$ then*

$$\langle a_1, \ldots, a_m \rangle_P = \langle a_1, \ldots, a_m \rangle .$$

PROOF: Let $a \in \langle a_1, \ldots, a_m \rangle_P$; then $p^k a = r_1 a_1 + \ldots + r_m a_m$ ($k \in \mathbf{N}$, $r_i \in R$); by Lemma 3.3, there is an $s \in R \setminus P$ such that $sa = s_1 a_1 + \ldots + s_m a_m$. By Lemma 2.1 (iii), $us + vp^k = 1$ for some $u, v \in R$, and so

$$a = usa + vp^k a = \sum_{i \leq m} (us_i + vr_i) a_i \in \langle a_1, \ldots, a_m \rangle . \qquad \square$$

If A is as above, then A is τ-semicocritical, by [G, Prop. 16.10], so it contains a τ-dense submodule B which is a finite direct sum of $n = \mathrm{len}_\tau A$ τ-cocritical τ-pure submodules of A; we can assume, without loss of generality, that $B = A_1 \oplus \ldots \oplus A_n$; in this notation we have the following

LEMMA 3.6: *If no non-zero element of A_i is of infinite P-height for all $i \leq n$, then $\langle A_1 \oplus \ldots \oplus A_n \rangle_P / (A_1 \oplus \ldots \oplus A_n)$ is bounded by p^s for some $s \in \mathbf{N}$.*

PROOF: Select elements $a_i \in A_i$ of 0 P-height for each $i \leq n$; then it is easy to find elements a_j, $n+1 \leq j \leq m$, of 0 P-height too, such that $p^s a_j = \sum_{i \leq n} \lambda_{ji} a_i$ for all $n+1 \leq j \leq m$, where $s \in \mathbf{N}$. If $a \in \langle A_1 \oplus \ldots \oplus A_n \rangle_P$, then we can write $p^t a = b_1 + \ldots + b_n$ ($b_i \in A_i$) with t as small as possible. By Lemmas 3.3 and 2.2 (i), there are elements $r, r' \in R \setminus P$ such that $r' p^t a = p^t ra = \sum_{i \leq n} \lambda_i a_i$, where $\lambda_i \in R \setminus P$ for some $i \leq n$, owing to the choice of t; hence by Lemma 3.5, $ra \in \langle a_1, \ldots a_n \rangle_P \subseteq \langle a_1, \ldots, a_m \rangle_P = \langle a_1, \ldots, a_m \rangle$ and consequently, $ra = \sum_{i \leq m} \mu_i a_i$. Supposing $t > s$ we have:

$$\begin{aligned} p^t ra = p^{t-s} p^s ra = \\ &= p^{t-s}\Big(p^s \mu_1 a_1 + \ldots + p^s \mu_n a_n + \mu'_{n+1} \sum_{i \leq n} \lambda_{n+1,i} a_i + \ldots + \mu'_m \sum_{i \leq n} \lambda_{mi} a_i\Big) = \\ &= p^{t-s}\Big(p^s \mu_1 + \sum_{n+1 \leq j \leq m} \mu'_j \lambda_{j1}\Big) a_1 + \ldots + \Big(p^s \mu_n + \sum_{n+1 \leq j \leq m} \mu'_j \lambda_{jn}\Big) a_n = \sum_{i \leq n} \lambda_i a_i . \end{aligned}$$

Comparing the corresponding terms we obtain:

$$\lambda_i a_i = p^{t-s}\Big(p^s \mu_i + \sum_{n+1 \leq j \leq m} \mu'_j \lambda_{ji}\Big) a_i .$$

However, since at least one λ_i is of 0 P-height, from Lemma 2.7 there follows that $t = s$, so the bound is p^s. $\square$

LEMMA 3.7: *If $a_i \in A_i$ is an element of 0 P-height for each $i \le m$, then $A/PA =$* $= \langle a_1 + PA, \ldots, a_m + PA \rangle$.

PROOF: Let $a \in A \setminus PA$; by Lemma 3.3, there exists an $r \in R \setminus P$ such that $ra = r_1 a_1 + \ldots + r_m a_m$ $(r_i \in R)$. Then $ur + vp = 1$ for suitable $u, v \in R$, and consequently $a = ura + vpa = \sum_{i \le m} ur_i a_i + pv'a$. □

LEMMA 3.8: *If $n = \operatorname{len}_\tau A$ and no non-zero element of the A_i's $(i \le m)$ is of infinite P-height, then $A/PA \simeq (R/P)^n$.*

PROOF: Let $B = A_1 \oplus \ldots \oplus A_n$ be τ-dense in A and choose elements a_i in A_i $(i \le n)$ of 0 P-height. Suppose that $p^t x = r_1 a_1 + \ldots + r_{i-1} a_{i-1} + a_i$ for some $i \le n$, and let p^s be the bound of Lemma 3.6. Then $x \in \langle A_1 \oplus \ldots \oplus A_n \rangle_\tau$ and so $p^s x \in A_1 \oplus \ldots \oplus A_n$. Thus, for some $r \in R \setminus P$ we have $rp^s x = \mu_1 a_1 + \ldots + \mu_n a_n$, by Lemma 3.3. If $p^t r = r' p^t$, then

$$r' p^{s+t} x = r' p^s r_1 a_1 + \ldots + r' p^s r_{i-1} a_{i-1} + r' p^s a_i =$$
$$= p^t \mu_1 a_1 + \ldots + p^t \mu_i a_i + \ldots + p^t \mu_n a_n .$$

Then obviously $s = t + h_P(\mu_i)$, hence $t \le s$, and consequently $\infty > k_i = h_P(a_i + \langle a_1, \ldots, a_{i-1} \rangle)$ $(i \le n)$. In a suitable enumeration of $A_1, \ldots, A_n$, we can assume that $k_1 \le k_2 \le \ldots \le k_n$; so there are elements $x_1, \ldots, x_n$ in A such that

$$(*) \qquad p^{k_i} x_i = a_i + \sum_{j \le i-1} r'_j a_j , \qquad h_P(x_i) = 0 \ (i \le n) .$$

First we show that $\bigoplus_{i \le n} \langle x_i + PA \rangle \subseteq A/PA$. Let $\sum_{i \le n} r_i x_i = px$ for some $x \in A$. Then

$$p^{k_n+1} x = \sum_{i \le n} p^{k_n - k_i} r'_i \Big(a_i + \sum_{j \le i-1} r'_j a_j \Big) = r'_n a_n + \sum_{i \le n-1} s_i a_i .$$

If $r'_n \notin P$, then by Lemma 2.1 (iii) it is $ur'_n + vp^{k_n+1} = 1$ for some elements $u, v \in R$. So we get

$$a_n = (ur'_n + vp^{k_n+1}) a_n = u \Big(p^{k_n+1} x \sum_{i \le n-1} s_i a_i \Big) + vp^{k_n+1} a_n$$

and consequently

$$p^{k_n+1}(u'x + v'a_n) = a_n + \sum_{i \le n-1} us_i a_i$$

which contradicts the choice of k_n; hence, by Lemma 2.2 (i), $r_n = p\bar{r}_n \in Rp$. Now we can continue similarly for $p(x - \bar{r}_n x_n) = \sum_{i \le n-1} r_i x_i$ and after n steps we clearly get $r_i \in Rp$ for each $i \leq n$. To finish the proof it suffices to show that $A/PA = \bigoplus_{i \leq n} \langle x_i + PA \rangle$. Let $a \in A \setminus PA$; from Lemma 3.3 it easily follows the existence of $r \in R \setminus P$ such that $p^s ra = r_1 a_1 + \ldots + r_n a_n$ for some $s > 1$. Using the equalities (*) we easily get that $p^s ra = \sum_{i \leq n} \mu_i x_i$. However, from the preceding part of the proof it immediately follows that $\mu_i = p^s \beta_i$ $(i \leq n)$, and consequently $ra = \sum_{i \leq n} \beta_i x_i$, A being τ-torsion-free. Now $ur + sp = 1$ for suitable $u, v \in R$, so that $a = ura + spa = \sum_{i \leq n} u\beta_i x_i + ps'a$, and we are through. $\square$

We can now prove that $\mathscr{F}_\tau(R) \subseteq \mathscr{P}_\tau(R)$, by showing that the hypothesis in Proposition 1.4 is satisfied.

Proposition 3.9: *Let $A = A_1 + \ldots + A_m$ be a τ-torsion-free sum of τ-pure τ-cocritical different submodules A_i. Then $\pi(A)$ is τ-pure in $\bigoplus_i \bar{A}_i$.*

Proof: If $\operatorname{Im} \pi$ is not τ-pure in $\bigoplus_i \bar{A}_i$, then there is an element

$$b = (b_1 + A_1, \ldots, b_m + A_m) \in \bigoplus_i \bar{A}_i \setminus \operatorname{Im} \pi$$

such that $pb = (a + A_1, \ldots, a + A_m)$, where $b_1, \ldots, b_m, a \in A$ and $P = Rp = pR$ is a maximal left ideal of R. So we have, for each $i \leq m$, $a - pb_i = a_i$ $(a_i \in A)$. If, for some $i \leq m$, $a_i \in PA_i < PA$, then $a_i = py$ $(y \in A)$, and so

$$a = p(b_i + y) = px \qquad (x \in A).$$

In this case it is

$$\begin{aligned} pb = (pb_1 + A_1, \ldots, pb_m + A_m) = \\ = (a + A_1, \ldots, a + A_m) = p(x + A_1, \ldots, x + A_m), \end{aligned}$$

and so $b = \pi(x)$, $\bigoplus_i \bar{A}_i$ being τ-torsion-free. This contradiction shows that $a_i \notin PA_i$ for each $i \leq m$, and consequently $A/PA \simeq (R/P)^m$ by Lemma 3.8. On the other hand, by Lemma 3.7 and the equalities: $a - pb_i = a_i$ $(i \leq m)$, we have $A/PA = \langle a_1 + PA, \ldots, a_m + PA \rangle = \langle a + PA \rangle \simeq R/P$; this shows that $\operatorname{len}_\tau A = 1$, which is clearly a contradiction. $\square$

The main theorem of this section is now an immediate consequence of Propositions 1.2, 1.4, 3.2, 3.4 and 3.9.

Theorem 3.10: *Let R be a ring which is not τ-torsion, where τ denotes the Dickson torsion theory in R-Mod, and let $P = Rp = pR$ for a $p \in P$, for each maximal left ideal P of R. Then $\mathscr{P}_\tau(R) = \mathscr{F}_\tau(R)$.* $\square$

4. - Butler modules over Prüfer domains

Let R be a commutative integral domain and $Q \neq R$ its field of quotients; τ will denote, from now on, the usual torsion theory in R-Mod, which coincides both with the Lambek and the Goldie torsion theories. An R-module is τ-cocritical exactly if it is isomorphic to a submodule of Q, and an R-module is τ-torsion-free of finite τ-length exactly if it is torsion-free of finite rank in the usual sense. A submodule N of a torsion-free module M is τ-pure if it is RD-pure, i.e. if $rM \cap N = rN$ for each $r \in R$ (see [FS2]).

The first result in this section shows that the sufficient conditions of Propositions 1.2 and 1.4 can be tested in the local case; we leave to the reader the straightforward proof, just recalling that, if M is a torsion-free R-module, then $M = \bigcap_P M_P$, where P ranges over the maximal spectrum Max R of R, and M_P denotes the localization of M at the maximal ideal P.

Notation is as in Section 1.

Lemma 4.1: *Let R be a commutative integral domain. Then*

1) *if B is RD-pure in $\bigoplus_1^n A_i$, where each A_i is a rank-one torsion-free R-module, then $B = \sum_i B^i$ if and only if $B_P = \sum_i (B^i)_P$ for each $P \in \mathrm{Max}\, R$;*

2) *if $C = C_1 + \ldots + C_m$ is a torsion-free sum of rank-one RD-pure submodules, then $\pi(C)$ is RD-pure in $\bigoplus_1^n \bar{C}_i$ if and only if $\pi_P(C_P)$ is RD-pure in $\bigoplus_1^n (\bar{C}_i)_P$ for all $P \in \mathrm{Max}\, R$.* □

In the preceding lemma π_P obviously denotes the unique extension of π to the localization at P.

We assume now that R is a Prüfer domain; a characteristic property for R is that R_P is a valuation domain for each $P \in \mathrm{Max}\, R$. RD-purity is equivalent to the purity in the sense of Cohn. We will show that the two sufficient conditions of Propositions 1.2 and 1.4 are satisfied by modules over valuation domains; therefore, by Lemma 4.1, they are satisfied by modules over Prüfer domains. Recall that the rank-one torsion-free modules over valuation domains are uniserial modules, i.e. modules with linearly ordered set of submodules.

Lemma 4.2: *Let R be a valuation domain and B a pure submodule of rank >2 of $A = \bigoplus_1^n A_i$, where each A_i is rank-one torsion-free. Then $B = \bigoplus_1^n B_i$, where each B_i is a rank-one submodule of some A^i. Hence $B = \sum_i B^i$.*

Proof: In view of Lemma 1.1, we have $B^1 \neq 0$, so we can choose B_1 to be a rank-one pure submodule of B^1. It is evidently pure in A, hence it is a summand of A by [FS2, IX.5.6]. Apply the Exchange Property of B_1 to conclude that $A = B_1 \oplus A_1' \oplus \ldots \oplus A_n'$ with A_i' a summand of A_i. As each A_i is of rank one, the comparison of ranks shows that some A_i', say A_j' equals 0, while $A_i' = A_i$ for $i \neq j$. Therefore $A = B \oplus A^j$ for some j which is evidently $\neq 1$. Now $B = B_1 \oplus B'$ with $B' = B \cap A^j$. If rank $B' = 1$, we are done; otherwise a simple induction concludes the proof. □

Lemma 4.3: *Let R be a valuation domain. If $C = C_1 + \ldots + C_m$ is a torsion-free irredundant sum of rank-one pure submodules C_i $(i = 1, \ldots, m)$, then $C = \bigoplus_1^m C_i$.*

Proof: C is an epic image of the R-module $A = C_1 \oplus \ldots \oplus C_m$, say with kernel K. The kernel is pure, and so a summand of A. The Exchange Property of K yields $A = K \oplus C_{i_1} \oplus \ldots \oplus C_{i_k}$ for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$. But then $C = C_{i_1} \oplus \ldots \oplus C_{i_k}$, and irredundancy implies $k = m'$. □

Lemma 4.4: *Let R be a valuation domain and $C = C_1 + \ldots + C_m$ a torsion-free sum of rank-one pure submodules C_i, with $rk\, C > 2$. Then $\pi(C)$ is pure in $\bigoplus_1^m \bar{C}_i$.*

Proof: Without loss of generality we can assume that the sum of the C_i's is irredundant, with $m > 2$. By Lemma 4.3, $C = \bigoplus_1^m C_i$, so $\pi_i \colon C \to \bar{C}_i$ induces an isomorphism between $\bar{C}_i$ and C^i for each $i < m$. The proof that $\pi(C)$ is pure in $\bigoplus_1^m \bar{C}_i \cong \bigoplus_1^m C^i$ is easy and it is left to the reader. □

From Propositions 1.2 and 1.4 and from the preceding lemmas we deduce the

Theorem 4.5: *Over Prüfer domains, the class of Butler modules in the usual torsion theory coincides with the class of purely finitely generated torsion-free modules.* □

If we look at the Dickson torsion theory over Prüfer domains, we must preliminarily take care of the existence of τ-cocritical modules. In the local case we have the following

Proposition 4.6: *Let R be a valuation domain and let τ be the Dickson torsion theory in R-Mod. Then there exist τ-cocritical R-modules if and only if the maximal ideal P of R is principal. In this case, a τ-cocritical R-module is, in the natural way, a rank-one torsion-free R/P^ω-module.*

Proof: If P is not principal, then R/rR has zero socle for each $rR < R$, hence R/I is not τ-cocritical for each ideal $I < P$. Thus there are no τ-cocritical R-modules, by [G, 14.3]. Conversely, if $P = pR$, then R/P^∞ is obviously τ-cocritical. If M is a τ-cocritical R-module, then $(0:a) = P^\infty$ for each $0 \neq a \in M$. This shows that M is canonically a torsion-free R/P^∞-module; it has rank-one, otherwise M has a non-zero torsion-free quotient as an R/P^∞-module, which is a τ-torsion-free quotient as an R-module. $\square$

We leave the global case as an open

Question 4: Characterize Prüfer domains R such that there exist τ-cocritical R-modules, where τ denotes the Dickson torsion theory.

A first relevant consequence of Proposition 4.6 is that the study of Butler modules in the Dickson torsion theory over a valuation domain can be reduced to that of Butler modules in the usual torsion theory over a discrete rank-one valuation domain, which are, as is well known, direct sums of rank-one modules. A second remark following from Proposition 4.6 is that the condition imposed to the rings in Sections 2 and 3, namely that the maximal left ideals are principal, is a necessary condition for a valuation domain for the existence of τ-cocritical modules in the Dickson torsion theory.

On the other hand, we must remark that in the global case the above condition is no more necessary: in fact, any Dedekind domain which is not a principal ideal domain gives a counterexample, since the Dickson torsion theory coincides with the usual one.

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