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A Stability Theorem for a Class of Stochastic Differential Equations «With Memory» (***) (****)

ABSTRACT. — On the basis of a result contained in [1], we give an actual improvement of the theorem in [3].

Un teorema di stabilità per una classe di equazioni differenziali stocastiche «con memoria»

RASSINTO. — Utilizzando un risultato contenuto in [1], qui si ottiene un effettivo miglioramento del teorema dato in [3].

INTRODUCTION

In the paper [3] we have given a stability theorem for the solutions of a sequence of stochastic differential equations: let $(X^n)_n$ be a sequence of processes that are (strong) solutions of the equations

$$dX^n(t) = r_n(t, X^n_s(\omega)) dA^n(t) + \sigma_n(t, X^n_s(\omega)) dM^n(t)$$

where, for every n , A^n is a non-negative increasing process and M^n is a square integrable martingale (defined on a given stochastic basis) and the coefficients $r_n(t, X^n_s(\omega))$, $\sigma_n(t, X^n_s(\omega))$, at every time t , may depend on the whole past of the solution-process before time t (equations «with memory»).

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The theorem in [3] generalizes a K. Yamada's result (see [2]) and gives sufficient conditions under which, as n tends to infinity, the laws $(A^n)_n$ of the processes $(X^n)_n$ converge weakly to the law A of the solution of an Ito equation:

$$dX(t) = r(t, X_s(\omega)) dt + \sigma(t, X_s(\omega)) dW(t),$$

for which existence and uniqueness holds in the weak sense.

Roughly speaking, this result asserts that sequence $(A^n)_n$ converges weakly to A under the following assumptions: as n tends to infinity, the sequences $(r_n)_n$, $(\sigma_n)_n$ converge to r , σ , respectively, in a suitable sense; sequence $(A^n)_n$ converges in a usual way to the deterministic process H such that $H(t) = t$ (for all t); about processes $(M^n)_n$ we have conditions sufficient to guarantee the weak convergence of their laws to the Wiener measure.

As to the coefficients $(\sigma_n)_n$, in [3] we have made the assumption that, for every n , σ_n is everywhere the limit of a given sequence of (D_t) -predictable step functions, $(D_t)_t$ denoting the « canonical » filtration of the Skorokhod space D (formed by all càdlàg mappings from \mathbb{R}_+ to \mathbb{R}).

This hypothesis, although often verified in applications (cf. [2], § 3), was only linked with the possibility of applying to our problem the Skorokhod representation theorem.

Now a result on stochastic integration subsequently proved in [1], makes it possible to avoid such assumption.

The present paper is in fact devoted to give an actual improvement of the theorem in [3] on the basis of such a property of stochastic integrals.

Here we show indeed that the result in [3] still holds without any regularity restriction on the coefficients $(\sigma_n)_n$: it suffices that, for every n , σ_n be measurable (exactly, (D_t) -predictable).

1. - BASIC RESULT

Everywhere in the following we will make use of the terminology, notations and conventions of [3].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ be a stochastic basis satisfying the usual hypotheses and let W be an (\mathcal{F}_t) -Wiener process defined on it.

Suppose also that the space (Ω, \mathcal{F}, P) be endowed with a sequence of filtrations $(\mathcal{F}^n_t)_{t \in \mathbb{R}_+}$, $n \geq 1$, such that, for every n , the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}^n_t)_{t \in \mathbb{R}_+}, P)$ verifies the usual hypotheses.

We are given two sequences $(A^n)_n$, $(M^n)_n$, of càdlàg, real-valued and (\mathcal{F}^n_t) -adapted processes, where, for every $n \geq 1$, M^n is a square integrable (\mathcal{F}^n_t) -martingale (i.e., for every t , the random variable $M^n(t)$ is square integrable with respect to P and moreover $\sup_n E_t[M^n(t)]^2 < +\infty$) and A^n is an increasing process.

We assume that $A^n(0) = M^n(0) = 0$ for all n .

We shall denote by $\mu_n(\omega, dt, dx)$ the random measure of jumps of martingale M^n

$$\mu_n(\omega, dt, dx) = \sum_{s>0} \mathbf{1}_{\{\Delta M^n(s, \omega) \neq 0\}}(s) \delta_{(s, \Delta M^n(s, \omega))}(dt, dx)$$

where $\delta_{(s, x)}$ is the Dirac measure concentrated at point (s, x) and $\Delta M^n(s, \omega) = M^n(s, \omega) - M^n(s^-, \omega)$.

We will write $r_n(\omega, dt, dx)$ for the compensator i.e. the dual (\mathcal{F}_t^n) -predictable projection of measure μ_n .

We are also given two sequences $(r_n)_n, (\sigma_n)_n, n \geq 1$ of (\mathcal{D}_t) -predictable processes on the basis $(\mathcal{D}, \mathcal{D}, (\mathcal{D}_t)_{t \in \mathbb{R}_+})$ where $\mathcal{D} = \mathcal{D}(\mathbb{R}_+, \mathbb{R})$ is the Skorokhod space of all càdlàg mappings from \mathbb{R}_+ into \mathbb{R} , \mathcal{D} denotes the Borel σ -algebra of \mathcal{D} endowed with the Skorokhod topology and $(\mathcal{D}_t)_{t \in \mathbb{R}_+}$ is the « canonical » filtration of $(\mathcal{D}, \mathcal{D})$.

Now let us set the assumptions of our stability theorem.

(1.1) Let $(X^n)_n$ be a given sequence of càdlàg and (\mathcal{F}_t^n) -adapted processes such that, for every $n \geq 1$ and all $t \in \mathbb{R}_+$, we have

$$X^n(t) = X^n(0) + \int_{[0,t]} r_n(s, X_s^n(\omega)) dA^n(s) + \int_{[0,t]} \sigma_n(s, X_s^n(\omega)) dM^n(s).$$

Moreover we assume that

$$\sup_n E_T[\|X^n(0)\|^2] < +\infty.$$

(1.2) Let $\langle M^n \rangle$ denote the Meyer process of M^n . Then, for any $t \geq 0$, the sequence of random variables $(\langle M^n \rangle(t))_n$ converges to the constant t in the norm of $L^2(\Omega, \mathcal{F}, P)$ when n tends to infinity.

(1.3) For any $t > 0$ and $\varepsilon > 0$, the sequence of random variables

$$\left(\int_{[0,t]} \int_{|x|>\varepsilon} x^2 r_n(ds, dx) \right)_n$$

converges in probability to 0, as n tends to infinity.

(1.4) Functions r_n, σ_n are bounded uniformly in n . There exist two (\mathcal{D}_t) -predictable mappings r, σ from $\mathbb{R}_+ \times \mathcal{D}$ into \mathbb{R}_+ to which sequences $(r_n)_n, (\sigma_n)_n$ converge in the following sense, when n tends to infinity:

For each path f in \mathcal{D} , there exists a Lebesgue nullset N , such that one has

$$\lim_n r_n(f_n, f_n) = r(f, f), \quad \lim_n \sigma_n(f_n, f_n) = \sigma(f, f)$$

whenever $t \in \mathbb{R}_+$, N_t and $(t_n)_{n \in \mathbb{N}}$, $(f_n)_{n \in \mathbb{N}}$ are arbitrary sequences of real numbers, of paths in D , respectively, verifying the conditions

$t_n \xrightarrow{n} t$ in the usual topology of \mathbb{R}_+ ,

$f_n \xrightarrow{n} f$ in the topology (on D) of uniform convergence on compact sets.

Moreover r_n, σ_n are both bounded.

(1.5) As n tends to infinity, the sequence of random variables $(A^n(t))_n$ converges in probability to the constant t , for every t in \mathbb{R}_+ .

Moreover, for every $n \geq 1$, process A^n is (\mathcal{F}_t^n) -predictable.

(1.6) Let λ be a given probability measure on \mathbb{R} . We suppose that, for the Itô stochastic differential equation

$$X(t) = X(0) + \int_0^t r(s, X_s(\omega)) ds + \int_0^t \sigma(s, X_s(\omega)) dw(s)$$

existence and uniqueness holds in the sense of probability law and in connection with the initial distribution λ .

(1.7) NOTATION: For every n , we denote by Λ_n the law on (D, \mathcal{D}) of process X^n defined in (1.1) and by λ_n the law of the real random variable $X^n(0)$.

The (unique) law of any solution of the equation in (1.6) with λ as initial distribution, is denoted by Λ .

The theorem we want to prove is the following:

(1.8) THEOREM: Let $(\lambda_n)_n$ be the sequence of probability measures on \mathbb{R} introduced above and suppose that, as n tends to infinity, λ_n converges weakly to λ , λ being the probability measure defined in (1.6).

Then under conditions (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6), the sequence of the above defined laws $(\Lambda_n)_n$ converges weakly to the law Λ , when n tends to infinity.

(1.9) REMARK: The above theorem is the same as Theorem (1.10) in [3] with the difference that there, for every n , the coefficient σ_n is everywhere the limit of a given sequence of (\mathcal{D}_t) -predictable step functions. (This assumption is verified, for example, when the mappings $t \mapsto \sigma_n(t, f)$ are left-continuous, cfr. [3], Remark (1.12)).

The fact that such assumption is unessential for the problem we study, is a consequence of a result in [1].

2. - AUXILIARY RESULTS

(2.1) LEMMA: Let n be an integer ≥ 1 . There exists a set L of probability laws on \mathcal{D}^n for which the following property holds:

Consider any probability space (Ω, \mathcal{F}, P) and any \mathbb{R}^n -valued càdlàg process (X^1, \dots, X^n) defined on it; then the following conditions are equivalent:

(a) The law of the random variable

$$\omega \mapsto (X_\omega^1, \dots, X_\omega^n)$$

belongs to L .

(b) Let m be any integer with $1 \leq m \leq n$. On the basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ the process X^m is a square integrable (\mathcal{F}_t) -martingale, $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ denoting the usual P -augmentation of the filtration generated by the given process (X^1, \dots, X^n) .

PROOF: See [3], Prop. (2.4). \square

(2.2) LEMMA: Let X, Y denote two (càdlàg adapted) semimartingales both defined on a generic stochastic basis satisfying the usual hypotheses; let C be a version of their mutual variation $[X, Y]$.

Then there exists a measurable mapping b from (D^2, \mathcal{D}^2) into (D, \mathcal{D}) such that one has almost surely

$$C_\cdot = b(X_\cdot, Y_\cdot).$$

Moreover this mapping b is uniquely determined by the law of the random variable

$$\omega \mapsto (X_\omega(\omega), Y_\omega(\omega)).$$

PROOF: See [1], Coroll. (2.5). \square

(2.3) LEMMA: Let σ be a bounded measurable function from

$$(\mathbb{R}_+ \times D, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{D}) \quad \text{into } (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

Then there exists a measurable mapping g_σ from (D^2, \mathcal{D}^2) into (D, \mathcal{D}) , for which the following property holds:

If we consider a generic measurable space (Ω, \mathcal{F}) and two càdlàg processes X, Z defined on it, process Z being a (raw) finite variation process, one has (for all ω)

$$Y_\omega(\omega) = g_\sigma(X_\omega(\omega), Z_\omega(\omega))$$

Y denoting the process $\int_0^\cdot \sigma(t, X_t(\omega)) dZ_t(\omega)$.

PROOF: See [3], Th. (2.3). \square

(2.4) THEOREM: Let σ be a real-valued, bounded process which is defined on the basis $(D, \mathcal{D}, (\mathcal{D}_t)_{t \in \mathbb{R}_+})$ and (\mathcal{D}_t) -predictable.

There exists a set M_σ of probability laws on \mathcal{D}^0 for which the following property holds:

Consider any stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P)$ satisfying the usual hypotheses and let X, Y, Z be càdlàg adapted processes defined on it and such that Y, Z both are square integrable (\mathcal{F}_t) -martingales.

Then the following conditions are equivalent:

(a) The law of the random variable

$$\omega \mapsto (X_\sigma(\omega), Y_\sigma(\omega), Z_\sigma(\omega))$$

belongs to M_σ .

(b) Y is a version of the stochastic integral

$$\int_0^\cdot \sigma(s, X_s(\omega)) dZ_s(\omega).$$

PROOF: It's an easy consequence of Lemmas (2.2) and (2.3). Cf. the proof of Theorem (3.1) in [1]. \square

3. - THE PROOF

To prove Theorem (1.8), we will show that for every subsequence of the given sequence $(A_n)_n$, there exists a further subsequence converging weakly to A (see (1.7)).

For every n , set

$$(3.1) \quad I_t^n = \int_{[0,t]} r_n(s, X_s^n) dA^n(s), \quad J_t^n = \int_{[0,t]} \sigma_n(s, X_s^n) dM^n(s)$$

and let V^n be the random element (of $(\mathcal{D}^0, \mathcal{D}^0)$) defined as follows:

$$(3.2) \quad V^n = (X_\sigma^n, I_\sigma^n, J_\sigma^n, A_\sigma^n, M_\sigma^n).$$

We have the following

(3.3) LEMMA: The laws $(\Gamma_n)_n$ of the random elements $(V^n)_n$ constitute a tight sequence of probability measures on $(\mathcal{D}^0, \mathcal{D}^0)$.

PROOF: It suffices to prove that the Aldous-Rebolledo conditions are verified by each sequence of processes appearing as components in the definition of $(V^n)_n$.

See [3], Lemma (3.3). \square

Now let $(A_n)_n$ be an arbitrary subsequence of the given sequence of laws (1.7) and consider the corresponding subsequence $(\Gamma_n)_n$ of laws of the random elements $(V^n)_n$.

In view of the preceding Lemma, there exists a law Γ' on (D^0, \mathcal{D}^0) and a further subsequence (still indexed by n) $(\Gamma'_n)_n$ converging weakly to Γ' .

Then in the proof one always considers the latter subsequence $(\Gamma'_n)_n$ and one shows that $\pi_1(\Gamma') = A$, π_1 denoting the projection from D^0 to the first factor D .

Since $\pi_1(\Gamma'_n) = A_n$ for every n , conclusion of Theorem (1.8) will follow.

By virtue of the Skorokhod representation theorem, on a complete probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ there exist random elements $(\bar{V}^n)_n$, \bar{V} of (D^0, \mathcal{D}^0) , such that

$$(3.4) \quad \begin{cases} \bar{V}^n(\bar{P}) = \Gamma'_n \text{ for every } n, \\ \bar{V}(\bar{P}) = \Gamma', \text{ and} \\ \lim_n \bar{V}^n(\bar{\omega}) = \bar{V}(\bar{\omega}) \text{ in } D^0, \text{ for } \bar{P}\text{-almost all } \bar{\omega} \in \bar{\Omega}. \end{cases}$$

Let us set, for every $n \geq 1$

$$(3.5) \quad \begin{cases} \bar{V}^n = (\bar{X}_t^n, \bar{I}_t^n, \bar{J}_t^n, \bar{A}_t^n, \bar{M}_t^n), \\ \bar{V} = (\bar{X}_t, \bar{I}_t, \bar{J}_t, \bar{A}_t, \bar{M}_t), \end{cases}$$

and denote by X_t, I_t, J_t, \dots etc. (resp. $\bar{X}_t^n, \bar{I}_t^n, \bar{J}_t^n, \dots$ etc.) the random elements $\bar{X}_t, I_t, J_t, \dots$ etc. (resp. $\bar{X}_t^n, \bar{I}_t^n, \bar{J}_t^n, \dots$ etc., $n \geq 1$) considered as processes on $\mathbb{R}_+ \times \bar{\Omega}$.

(3.6) DEFINITION: We denote by $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t \in \mathbb{R}_+}, \bar{P})$ (resp. $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t^*)_{t \in \mathbb{R}_+}, \bar{P})$) the usual \bar{P} -augmentation of the basis generated by \bar{V} (resp. by \bar{V}^n , for every n).

(3.7) LEMMA: For every $n \geq 1$, the process \bar{A}^n is \bar{P} -almost surely increasing and the processes \bar{M}^n, J^n both are square integrable $(\bar{\mathcal{F}}_t^*)$ -martingales.

PROOF: Since A^n is càdlàg, the property of being \bar{P} -almost surely increasing is an easy consequence of the fact that its law is the same as the law of A^n (cf. [3], proof of Lemma (3.12)).

What stated about \bar{M}^n, J^n follows from Lemma (2.1) and the definition of \bar{V}^n . \square

As a consequence of assumption (1.1) and of the definition of \bar{V}^n , the relation

$$\bar{X}^n(t) = \bar{X}^n(0) + \bar{I}^n(t) + \bar{J}^n(t)$$

is \bar{P} -a.s. verified for any $t \in \mathbb{R}_+$ and any $n \geq 1$.

Because of Lemmas (2.2), (3.7) and of Theorem (2.4), the above relation

can be written in the following way

$$(3.8) \quad X^*(t) = X^*(0) + \int_{[0,t]} r_n(s, X_s^*) dA^*(s) + \int_{[0,t]} \sigma_n(s, X_s^*) dM^*(s).$$

Lastly, to complete the proof of Theorem (1.8) one proceeds exactly in the same way as in the sections 3 and 4 of [3].

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