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Tori and Continuous Dualities (**)

Tori e dualità continue

SUNTO. — Dikranjan e Orsatti [2], usando metodi dovuti a Prodanov, hanno studiato le autodualità nella categoria ${}_R\mathfrak{C}$ dei moduli localmente compatti sull'anello commutativo R; ad ogni tale dualità D è canonicamente associato un modulo compatto ${}_RT$, detto un *toro* e questo modulo ${}_RT$ definisce una autodualità su ${}_R\mathfrak{C}$ che coincide con D se e solo se D è « continua ». In questo lavoro, generalizzando la definizione di toro al caso non commutativo, si dimostra che ogni toro ${}_RT$ sull'anello R definisce una dualità (continua) fra ${}_R\mathfrak{C}$ e $\mathfrak{C}_A$, dove A è l'anello degli endomorfismi continui di ${}_RT$.

INTRODUCTION

In a recent paper [2] Dikranjan and Orsatti, using methods due to Prodanov, studied self-dualities of the category ${}_R\mathfrak{C}$ of all locally compact modules over a discrete commutative ring R. The most important result of Prodanov is the « representation » of dualities by means of a compact module ${}_RT$ and of a module automorphism M of ${}_RT$, such that M^2 is a topological automorphism. It comes from this a classification of this type of dualities into two classes, *continuous* and *discontinuous* dualities: a duality is called continuous if the automorphism M is continuous (hence a topological isomorphism) and discontinuous otherwise.

The compact module ${}_RT$ is an injective cogenerator of the category of compact modules over R such that the ring of continuous endomorphisms $\mathrm{Chom}_R(T, T)$ of ${}_RT$ is canonically isomorphic to R; such a module is called a *torus*. Then it can be proved that these properties are equivalent to the fact

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that the Pontrjagin dual of ${}_RT$ is a rank 1 progenerator of Mod-R. In particular ${}_RT$ is without small submodules.

If ${}_RT$ is a torus, then it defines a continuous self-duality of ${}_R\mathfrak{L}$: if ${}_RM$ is a locally compact module, its dual is the R-module $\mathrm{Chom}_R(M, T)$ of continuous morphisms of M into T, endowed with the compact-open topology, i.e. the topology of uniform convergence on the compact subsets of M (we can regard this as a generalized Pontrjagin duality). It turns out that every continuous duality is naturally equivalent to a duality of the above type.

We want to extend some of these results on continuous dualities to the non-commutative case (at present discontinuous dualities are being investigated by Colpi). However we must modify the context: we have to consider dualities between the categories of locally compact modules over two rings and it is natural to consider one category as consisting of right modules and the other one of left modules. Moreover the definition of torus has to be modified; hence we take as definition the following: a compact left module ${}_RT$ over a discrete ring R is called a (*generalized*) *torus* if it is an injective cogenerator without small submodules of the category of compact left R-modules: this means (see Corollary 1.2) that the Pontrjagin dual of ${}_RT$ is a progenerator of Mod-R.

The main result of the paper is the following: let ${}_RT$ be a generalized torus and let $A = \mathrm{Chom}_R(T, T)$ be the ring of all continuous endomorphisms of ${}_RT$. Then R is canonically isomorphic to $\mathrm{Chom}_A(T, T)$ and the functors that associate to every locally compact left (resp. right) R-module (resp. A-module) the module of continuous morphisms into T, endowed with the compact-open topology, give a duality (Theorem 2.7).

The main tool used in this paper is the observation that a progenerator P_R, with endomorphism ring A, defines an equivalence between the categories of all *topological* modules over R and over A, which preserves locally compact modules (Theorem 2.3 and Corollary 2.2). Then, by combining this equivalence with the Pontrjagin duality over R, we get the result.

All rings considered in the paper have 1 and are endowed with the discrete topology; all modules are unitary. We shall write morphisms on the opposite side of the scalars. If a module over a ring is to be regarded as an abelian group (i.e. a module over the ring $\boldsymbol{Z}$ of integers), we shall consider $\boldsymbol{Z}$ as acting on the opposite side to the given ring.

We shall denote by Mod-R, TM-R and $\mathfrak{L}_R$ respectively the categories of abstract, topological and locally compact right modules over R. Analogous notations will hold for left modules.

The *standard torus* is the additive group $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ of real numbers modulo 1, endowed with the quotient topology of the usual topology on $\boldsymbol{R}$: then $\boldsymbol{T}$ is a compact group.

We assume that the reader is familiar with the concepts of Morita equivalence of rings as explained, for example, in [3]. Moreover we shall use Pontrjagin duality in an extensive way, so we refer to the classical book [5]; the paper [4] can be used for a short account of Pontrjagin duality over a ring.

1. - Tori

Definition: Let R be a discrete ring and let ${}_RT$ be a topological module over R.

We say that ${}_RT$ is *without small submodules* if there exists a neighbourhood of zero U in T (which we shall call a *small neighbourhood* of zero) containing no non zero submodules of T.

We say that ${}_RT$ is a (*generalized*) *torus* if ${}_RT$ is compact and is an injective cogenerator without small submodules of the category ${}_R\mathcal{C}$ of all compact left R-modules. This means that:

(1) for any non zero continuous morphism $f\colon X \to Y$ of compact left R-modules, there is a continuous morphism $g\colon Y \to T$ such that $fg \neq 0$;

(2) for any topological embedding $f\colon X \hookrightarrow Y$ of compact left R-modules and any continuous morphism $g\colon X \to T$, there exists a continuous morphism $h\colon Y \to T$ with $fh = g$.

We also frequently use Pontrjagin duality over the ring R, which is defined as follows.

If G is a locally compact abelian group, then its *Pontrjagin dual* is the locally compact group $\Gamma(G) = \mathrm{Chom}_{\mathbf{Z}}(G, R/\mathbf{Z})$, of all continuous group morphisms of G into the *standard torus* $\mathbf{T} = R/\mathbf{Z}$, endowed with the so-called *compact-open topology* i.e. topology of uniform convergence on the compact subsets of G. We can define the evaluation map

$$\omega_G\colon G \to \Gamma\Gamma(G)$$

and classical results from [5] give that ω_G is a topological isomorphism, hence Γ defines a duality of the category of locally compact abelian groups into itself.

If ${}_RM$ is a locally compact *left* module over the discrete ring R, then the Pontrjagin dual of M is the locally compact group $\Gamma(M)$ with the natural structure of *right* R-module defined by

$$\chi r(x) = \chi(rx)$$

for $\chi \in \mathrm{Chom}_{\mathbf{Z}}(G, \mathbf{T})$, $r \in R$ and $x \in M$.

It can now easily be shown that $\Gamma(M)$ becomes a topological module over R and that the functors

$$\Gamma\colon {}_R\mathcal{C} \to \mathcal{C}_R, \qquad \Gamma\colon \mathcal{C}_R \to {}_R\mathcal{C}$$

define a duality. To distinguish the two functors we shall use indices in the following way: Γ_1 denotes the functor $\Gamma\colon {}_R\mathcal{C} \to \mathcal{C}_R$, while Γ_2 denotes the functor $\Gamma\colon \mathcal{C}_R \to {}_R\mathcal{C}$.

1.1. Proposition: *Let ${}_RT$ be a compact left R-module and let $P_R = \Gamma_1(T)$ be its Pontrjagin dual. Then ${}_RT$ is without small submodules if and only if P_R is finitely generated.*

Proof: Follows immediately from [3, Lemma 5.3] as it is shown in [4, Corollary 1.12]. ∎

1.2. Corollary: *Let ${}_RT$ be a compact module. Then ${}_RT$ is a torus if and only if $P_R = \Gamma_1(T)$ is a progenerator.*

Let ${}_RT$ be a torus and set $A = \mathrm{Chom}_R(T, T) = \mathrm{Hom}_R(\Gamma_1(T), \Gamma_1(T))$: then $P_R = \Gamma_1(T)$ is a progenerator and so ${}_AP$ is also a progenerator and $\mathrm{End}({}_AP)$ is canonically isomorphic to R. Thus we have the following

1.3. Proposition: *Let ${}_RT$ be a torus and let $A = \mathrm{Chom}_R(T, T)$ be discrete. Then T_A is a torus, $\mathrm{Chom}_A(T, T)$ is canonically isomorphic to R and the rings R and A are Morita equivalent.*

The following lemma will be useful as it shows the role of small neighbourhoods in compact modules.

1.4. Lemma: *Let ${}_RM$ be a compact R-module without small submodules and let U be a small neighbourhood of zero in M. For $r \in R$ and $X \subseteq M$ we put $r^{-1}(X) = \{r \in M \mid rx \in X\}$. Then the family $\{r^{-1}(U) \mid r \in R\}$ is a subbasis of neighbourhood of zero in M.*

Proof: It is clear that, for any $r \in R$ and any neighbourhood of zero U in M, $r^{-1}(U)$ is a neighbourhood of zero in M.

The thesis will follow from the compactness of M if we show that

$$U_0 = \bigcap_{r \in R} r^{-1}(U) = 0 .$$

Let then $x \in U_0$: then $rx \in U$ for all $r \in R$ and so $Rx \subseteq U$; since U is small, we have $Rx = 0$ and so $x = 0$. ∎

1.5. Theorem: *Let ${}_RT$ be compact without small submodules and let ${}_RM$ be locally compact. Then the group $\mathrm{Chom}_R(M, T)$, endowed with the compact-open topology, is locally compact. Moreover, if we set $A = \mathrm{Chom}_R(T, T)$, then $\mathrm{Chom}_R(M, T)$ is a locally compact right module over the discrete ring A.*

Proof: The proof follows closely the pattern of the analogous one by Pontrjagin [5]. We include it since our setting is much more general than his.

Denote by $M^\vee$ the group $\mathrm{Chom}_R(M, T)$ with the compact-open topology.

Let K be a symmetric compact neighbourhood of zero in M and U be a closed neighbourhood of zero in T such that $U + U$ is small. We shall prove that the neighbourhood of zero $\mathfrak{W}(K, U)$ in $M^\vee$ is compact.

(1.5.1) *Assume that* $\xi \in \mathrm{Hom}_R(M, T)$ *and* $(K)\xi \subseteq U$. *Then* ξ *is continuous.*

Let $r \in R$: then $r^{-1}(K)$ is a neighbourhood of zero in M; it is clear that $(r^{-1}(K))\xi \subseteq r^{-1}(K)$ and, by Lemma 1.4, we are done.

The following assertion is clear:

(1.5.2) *The group* $\bar{M} = \mathrm{Hom}_R(M, T)$ *endowed with the topology of pointwise convergence is compact.*

If $X \subseteq M$ and $Y \subseteq T$ we put

$$\mathfrak{W}(X, Y) = \{\xi \in \mathrm{Chom}_R(M, T) \mid (X)\xi \subseteq Y\},$$

$$\mathfrak{W}'(X, Y) = \{\xi \in \mathrm{Hom}_R(M, T) \mid (X)\xi \subseteq Y\}.$$

Then we can express (1.5.1) by saying that $\mathfrak{W}'(K, U) = \mathfrak{W}(K, U)$.

(1.5.3) *The topology* τ *induced on* $\mathfrak{W}(K, U)$ *by* $\bar{M}$ *coincides with the topology* σ *induced by* M'.

It is clear that τ is coarser than σ. Let $\alpha \in \mathfrak{W}(K, U)$: a basic neighbourhood of α with respect to σ is of the form

$$(\alpha + \mathfrak{W}(F, V)) \cap \mathfrak{W}(K, U),$$

where F is compact in M and V is a neighbourhood of zero in T. Take a neighbourhood of zero V' in T with $V' + V' \subseteq V$; since $U + U$ is small, by Lemma 1.5 there exist elements $r_1, \ldots, r_n \in R$ such that

$$\bigcap_{i=1}^{n} r^{-1}(U + U) \subseteq V'.$$

Put $K' = \bigcap_{i=1}^{n} r^{-1}(K)$: K' is a neighbourhood of zero in M and so, by the compactness of F, there exists a finite subset X of F such that $F \subseteq X + K'$.

Let $\xi \in \mathfrak{W}'(X, V')$ be such that $\alpha + \xi \in \mathfrak{W}(K, U)$; then $\xi = (\alpha + \xi) - \alpha$ is continuous by (1.5.1) and moreover

$$(K)\xi = (K)((\xi + \alpha) - \alpha) \subseteq (K)(\xi + \alpha) + (K)\alpha \subseteq U + U.$$

If $y \in K'$, it is $r_i y \in K$ $(i = 1, \ldots, n)$, so that $(r_i y)\xi = r_i(y)\xi \in U + U$, hence $(y)\xi \in V'$. Thus $(K')\xi \subseteq V'$ and

$$(F)\xi \subseteq (X + K')\xi \subseteq (X)\xi + (K')\xi \subseteq V' + V' \subseteq V.$$

Hence $\xi \in \mathfrak{W}(F, V)$, so that

$$(\alpha + \mathfrak{W}(F, V)) \cap \mathfrak{W}(K, U) \supseteq (\alpha + \mathfrak{W}'(X, V')) \cap \mathfrak{W}(K, U)$$

and $\tau \supset \sigma$.

(1.5.4) *$\mathfrak{W}(K, U)$ is compact in $M^{\sharp}$.*

Since $\mathfrak{W}(K, U)$ is closed in $\overline{M}$, it is compact. Now (1.5.3) ends the proof. ∎

When ${}_RM$ and ${}_RT$ are topological modules over the discrete ring R, we shall denote by $\mathrm{Chom}_R^{\infty}(M, T)$ the group of continuous morphisms of M into T, endowed with the compact-open topology, and by $\mathrm{Chom}_R^{s}(M, T)$ the same group with the topology of pointwise convergence.

2. - The duality Δ^T

In this section we want to prove that every torus defines a duality between categories of locally compact modules and that this duality is strictly linked to Morita equivalence. In what follows R is always a discrete ring.

2.1. Lemma: *Let ${}_RM$ be a locally compact module and ${}_RT$ a compact module without small submodules. Then there is a topological isomorphism*

$$\mathrm{Chom}_R^{\infty}(M, T) \cong \mathrm{Hom}_R^{s}(\Gamma_1(T), \Gamma_1(M))$$

which is natural in M.

Proof: Put $P_R = \Gamma_1(T)$ and $N_R = \Gamma_1(M)$ and define

$$\Phi_M\colon \mathrm{Chom}_R^{\infty}(M, T) \to \mathrm{Hom}_R^{s}(P, N)$$

by $\Phi_M(\xi) = \Gamma(\xi)$. It is obvious that Φ_M is a group isomorphism natural in M.

Let us consider a basic neighbourhood of zero in $\mathrm{Hom}_R^{s}(P, N)$ of the form $V = \mathfrak{W}(x, \mathfrak{W}(K, U))$, where $x \in P$, K is compact in M and U is a neighbourhood of zero in T. Then it is plain that $\Phi_M^{-1}(V) = \mathfrak{W}(K, x^{-1}(U))$ is a neighbourhood of zero in $\mathrm{Chom}_R^{\infty}(M, T)$.

To find the inverse morphism of Φ_M we identify M with $\Gamma_2(N)$ and T with $\Gamma_2(P)$ and we define

$$\Phi_M\colon \mathrm{Hom}_R^{s}(P, N) \to \mathrm{Chom}_R^{\infty}(M, T)$$

by $\Phi_M(f) = \Gamma(f)$. Then Φ_M is the desired inverse morphism.

A basis of neighbourhoods of zero in $\mathrm{Chom}_R^{\infty}(\Gamma_2(N), T)$ is the family of all subsets of the form $\mathfrak{W}(K, V)$, where V is a neighbourhood of zero in T and K is a compact neighbourhood of zero in $\Gamma_2(N)$. The proof of Theorem 1.5 makes possible to assume that $K = \mathfrak{W}(K', U)$, where K' is a symmetric compact neighbourhood of zero in N and U is a closed neighbourhood of zero in T such that $U + U$ is small. Moreover we can assume that V is of the form $V = \mathfrak{W}(F, U)$, with F a finite subset of P.

Put $W = \mathfrak{W}(\mathfrak{W}(K', U), \mathfrak{W}(F, U))$. Then

$$\Phi_M^{-1}(W) = \mathfrak{W}(F, \cap \{\xi^{-1}(U) \mid \xi \in \mathfrak{W}(K', U)\}) \subset K'$$

and Φ_M is continuous. ■

2.2. Corollary: *If P_R is a (discrete) finitely generated module and N_R is a locally compact module, then the group* $\mathrm{Hom}_R^c(P, N)$ *is locally compact.*

Proof: Easy by 2.1 and 1.1. ■

The following theorem can be regarded as an extension of the classical results of Morita equivalence to the categories of topological modules.

2.3. Theorem: *Let P_R be a progenerator and set $A = \mathrm{End}(P_R)$. Then the functor* $\mathrm{Hom}_R^c(P, -)$: TM-$R \to$ TM-A *is an equivalence.*

Proof: Let $Q = \mathrm{Hom}_R(P, R)$: then Q is an R-A-bimodule, Q_A is a progenerator, the topology of pointwise convergence on Q is discrete and, for any $M \in$ Mod-R there is an isomorphism

$$\varrho_M\colon M \to \mathrm{Hom}_A(Q, \mathrm{Hom}_R(P, M)),$$
$$m \mapsto \tilde{m},$$

where, for $q \in Q$, we put $\tilde{m}(q)\colon p \in P \mapsto m(q(p))$. We shall prove that, for every $M \in$ TM-R, the morphism $\varrho_M\colon M \to \mathrm{Hom}_A^c(Q, \mathrm{Hom}_R^c(P, M))$ is a topological isomorphism.

If $x \in Q$ and $y \in P$, then, for every neighbourhood of zero U in M we have

$$\varrho_M^{-1}(\mathfrak{W}(x, \mathfrak{W}(y, U))) = r^{-1}(U)$$

where $r = x(y) \in R$, so that ϱ_M is continuous.

Fix now $F = \{q_1, \ldots, q_n\} \subset Q$ and $G = \{p_1, \ldots, p_n\} \subset P$ such that

$$\sum_{i=1}^{n} q_i(p_i) = 1 \in R.$$

If U is a neighbourhood of zero in M, we consider a neighbourhood of zero V in M such that

$$\underbrace{V + \ldots + V}_{n \text{ times}} \subset U.$$

Let $m \in M$ be such that $\tilde{m} \in \mathfrak{W}(F, \mathfrak{W}(G, V))$: then, for $i = 1, 2, \ldots, n$, it is

$mq_i(p_i) \in V$ and so

$$m = m1 = \sum_{i=1}^{n} mq_i(p_i) \in \underbrace{V + \ldots + V}_{n \text{ times}} \subseteq U$$

and so ϱ_R^{-1} is continuous. ∎

The following lemma is the fundamental result to prove the existence of the duality defined by a torus.

2.4. Lemma: *Let P_R be a progenerator. For every locally compact module M_R there is a topological isomorphism of groups*

$$\Gamma(\mathrm{Hom}_R^c(P, M)) \cong \mathrm{Hom}_R^c(\mathcal{Q}, \Gamma_R(M)),$$

where $\mathcal{Q} = \mathrm{Hom}_R(P, R)$, which is natural in M.

Proof: Let $\mathcal{A} = \mathrm{End}(P_R)$. The canonical morphisms

$$\mu\colon \mathcal{Q} \otimes_{\mathcal{A}} P \to R,$$

$$\nu\colon P \otimes_R \mathcal{Q} \to \mathcal{A}.$$

are bimodule isomorphisms for which the following associativity properties hold:

$$[q, p]q' = q(p, q'),$$

$$p'[q, p] = (p', q)p$$

for $p, p' \in P$ and $q, q' \in \mathrm{G}$, where we put $[q, p] = \mu(q \otimes p)$ and $(p, q) = \nu(p \otimes q)$. In all this proof we fix $p_1, \ldots, p_n \in P$ and $q_1, \ldots, q_n \in \mathcal{Q}$ such that

$$\sum_{i=1}^{n} q_i(p_i) = 1 \in \mathcal{A}.$$

Let M_R be a locally compact module and define a morphism

$$\zeta_M\colon \Gamma(\mathrm{Hom}_R^c(P, M)) \to \mathrm{Hom}_R^c(\mathcal{Q}, \Gamma_R(M)),$$

$$\alpha \mapsto \zeta_M(\alpha) = \hat{\alpha},$$

by putting, for $q \in \mathrm{G}$,

$$(q)\hat{\alpha}\colon m \in M \mapsto (\tilde{m} \circ q)\alpha$$

where $\tilde{m}\colon R \to M$ is the multiplication by m, so $\tilde{m} \circ q \in \mathrm{Hom}_R(P, M)$.

(1) Let us verify first that ζ_M is well-defined, i.e., for any α belonging to $\Gamma(\mathrm{Hom}^n_R(P, M))$ and any $q \in Q$, $(q)\alpha \in \mathrm{Chom}_R(M, T)$.

If U is a neighbourhood of zero in T, there exist a finite subset F of P and a neighbourhood of zero V in M such that $\mathcal{W}(F, V) \subseteq \alpha^{-1}(U)$. Then $V' = \bigcap_{p \in F} (q(p))^{-1}(V)$ is a neighbourhood of zero in M and, for all $m \in V'$, it is $\tilde{m} \circ q \in \mathcal{W}(F, V)$ and so $(m)(q)\hat{\alpha} = (\tilde{m} \circ q)\alpha \in U$ or $((q)\hat{\alpha})^{-1}(U) \supseteq V'$.

(2) Define a morphism

$$\vartheta_M \colon \mathrm{Hom}^n_R(Q, \Gamma_R(M)) \to \Gamma(\mathrm{Hom}^n_R(P, M)),$$

$$\beta \mapsto \vartheta_M(\beta) = \check{\beta},$$

by $(f)\check{\beta} = \sum_{i=1}^{n} (f(p_i))((q_i)\beta$.

This group morphism is well-defined since, if U and U' are neighbourhoods of zero in T with $\underbrace{U' + \ldots + U'}_{n \text{ times}} \subseteq U$, we have

$$\check{\beta}^{-1}(U) \supseteq \mathcal{W}\Big(\{p_1, \ldots, p_n\}, \bigcap_{i=1}^{n} ((q_i)\beta)^{-1}(U')\Big)$$

and so $\check{\beta} \in \mathrm{Chom}_R(\mathrm{Hom}^n_R(P, M), T)$.

(3) ϑ_M is continuous.

In fact, let U be a neighbourhood of zero in T and K a compact subset in $\mathrm{Hom}^n_R(P, M)$. If $x \in P$ we set $K(x) = \{f(x) \mid f \in K\}$: then $K(x)$ is compact in M and so also $K' = \bigcup_{i=1}^{n} K(p_i)$ is compact in M. Let U and U' be neighbourhoods of zero in T with $\underbrace{U' + \ldots + U'}_{n \text{ times}} \subseteq U$. It is immediate to verify that, for $\beta \in \mathcal{W}(\{q_1, \ldots, q_n\}, \mathcal{W}(K', U'))$, we have $\check{\beta} \in \mathcal{W}(K, U)$.

(4) It is now easy to see, by using the associativity properties of μ and ν, that ϑ_M is the inverse of ζ_M. ∎

Let $T \in R$-TM and set $A = \mathrm{Chom}_R(T, T)$. Then, for any $M \in R$-TM we can define a morphism of left R-modules

$$\psi_M \colon M \to \mathrm{Chom}_A(\mathrm{Chom}^m_R(M, T), T)$$

in the usual way as the evaluation: $\psi_M(x) = \hat{m}$ (for $m \in M$), where, for $\xi \in \mathrm{Chom}_R(M, T)$, $\hat{m}(\xi) = (m)\xi$.

In the same way we have, for any $N \in$ TM-A, a morphism of right A-modules

$$\chi_N \colon N \to \mathrm{Chom}_A(\mathrm{Chom}^m_A(N, T), T).$$

Of course the morphisms ψ_M is not necessarily continuous, when we endow $\mathrm{Chom}_R(\mathrm{Chom}^m_A(N, T), T)$ with the compact-open topology.

The important result is that they are topological isomorphisms whenever M is a locally compact modules and ${}_RT$ is a torus, as we state in the following

2.5. THEOREM: *Let ${}_RT$ be a torus and put $A = \mathrm{Chom}_R(T, T)$. Then, for every locally compact module ${}_RM$ the morphism*

$$\psi_M\colon M \to \mathrm{Chom}^m_A(\mathrm{Chom}^m_R(M, T), T)$$

is a topological isomorphism.

PROOF: Put $P_R = \Gamma_1(T)$ and $Q = \mathrm{Hom}_R(P, R)$. Then P_R is a progenerator with $A = \mathrm{End}({}_RQ)$ and $P = \mathrm{Hom}_R(Q, R)$, so that 2.3 gives, *mutatis mutandis*, a topological isomorphism

$$\varrho_M\colon M \to \mathrm{Hom}^c_A(P, \mathrm{Hom}^c_R(Q, M)),$$

so that we can construct the chain of topological isomorphisms

$$\begin{aligned} M \xrightarrow{\varrho_M} & \mathrm{Hom}^c_R(P, \mathrm{Hom}^c_R(Q, M)) \to \\ \xrightarrow{\mathrm{Hom}^c_A(P, \theta_M)} & \mathrm{Hom}^c_A\big(P, \Gamma(\mathrm{Hom}^c_R(P, \Gamma_1(M)))\big) \to \\ \xrightarrow{\Phi^{-1}_{\mathrm{Hom}^c_R(P, \Gamma_1(M))}} & \mathrm{Chom}^m_A\big(\mathrm{Hom}^c_R(P, \Gamma_1(M)), T\big) \to \\ \xrightarrow{\mathrm{Chom}^m_A(\Psi_M, T)} & \mathrm{Chom}^m_A(\mathrm{Chom}^m_R(M, T), T) \end{aligned}$$

and it is easy, but tedious, to verify that ψ_M is just the composition of these morphisms. ∎

2.6. REMARK: Under the hypotheses of Theorem 2.5, we have, by 1.3 that also T_A is a torus, so that we could have proved the same statement as in 2.5 with R and A interchanged. Hence it follows that, for every locally compact right A-module N_A, the morphism

$$\chi_N\colon N \to \mathrm{Chom}^m_R(\mathrm{Com}^m_A(N, T), T)$$

is a topological isomorphism too.

Thus we have the following result, which summarizes all that we have done.

Given a torus ${}_RT$, and having set $A = \mathrm{Chom}_R(T, T)$, we can define two functors

$$\Delta^T_1\colon {}_R\mathfrak{L} \to \mathfrak{L}_A,$$

$$\Delta^T_2\colon \mathfrak{L}_A \to {}_R\mathfrak{L},$$

which to every locally compact module associate the module of its continuous morphisms into T and to every continuous morphism its transpose.

2.7. THEOREM: *Let* ${}_RT$ *be a torus and set* $A = \mathrm{Chom}_R(T, T)$. *Then* T_A *is a torus,* $R \simeq \mathrm{Chom}_A(T, T)$ *canonically and*

$$\Delta^T = (\Delta_1^T : {}_R\mathcal{L} \to \mathcal{L}_A,\ \Delta_2^T : \mathcal{L}_A \to {}_R\mathcal{L})$$

is a duality.

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