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Existence of an Energy-Almost-Periodic Solution
to a Wave Equation Non-Linear in the Mixed Derivative (**)

Esistenza di una soluzione quasi periodica in energia
per un'equazione delle onde non lineare nella derivata mista

RASSUMERÒ. — Si considera un'equazione delle onde non lineare nella derivata mista e non omogenea, dove il termine non lineare ha una crescita di tipo polinomiale. Si dimostra che, se il termine forzante è Stepanov limitato, esiste una soluzione limitata rispetto ad una norma più forte della norma dell'energia. Ne segue, in base ad un risultato classico di A. Haraux, l'esistenza di una soluzione quasi periodica nello spazio dell'energia.

1. - INTRODUCTION

A number of works (Prouse, 1965; Biroli and Haraux, 1980; Biroli, 1981) deal with the existence and uniqueness of an energy-almost-periodic solution to a wave equation, which is nonlinear w.r.t. the first order derivative, u_t . Herewith we consider instead a kind of nonlinearity w.r.t. the mixed derivative i.e., the equation

$$(1.1) \quad u_{tt} - u_{xx} - [\beta(u_{xt})]_x = f(t, x), \quad (t, x) \in R_t^+ \times \Omega, \quad \Omega = (0, 1),$$

where

- β is a $C^1(R)$ monotonically increasing function s.t., $\beta(0) = 0$, which satisfies a polynomial growth condition;
- f , the forcing term, is chosen in the Stepanov space $J^p(R_t^+; W_0^{1,p}(\Omega))$.

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In considering Equation (1.1) Prestel (1982) proved the existence of at least one periodic *weak* solution, provided the forcing is periodic and β satisfies a sublinear growth condition.

Dang Dinh Hai (1989, to appear) proves that the Cauchy problem for (1.1), to which homogeneous boundary conditions apply, has a unique *weak* solution provided both the initial data

$$u(0, \cdot) = u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad u_t(0, \cdot) = v_0 \in H_0^1(\Omega)$$

and the forcing term $f \in L^q(R_t^+; W_0^{1,p}(\Omega))$ are « small ».

Our purpose is twofold:

i) by applying a method originally introduced by Prouse (1965) and later refined by Biroli and Haraux (1980), Biroli (1981), we prove the existence of a bounded trajectory in a stronger norm;

ii) by taking into account some results about nonlinear evolution equations, both of general (Amerio and Prouse, 1971; Benilan and Brezis, 1972; Brezis, 1973) and specific type (Haraux, 1978, 1981), we show an *energy-almost-periodic* solution exists.

2. - NOTATIONS AND FUNCTIONAL SETTING

Let X be a Banach space, then a *Sobolev space* of index $p > 1$ is defined by

$$S^p(R_t^+; X) = \left\{ f \in L_{loc}^p(R_t^+; X) \mid \sup \left\{ \int_0^{t+1} \|f(t)\|_X^p dt; t > 0 \right\} < \infty \right\}.$$

In the following we shall assume $f \in S^p(R_t^+; W_0^{1,p}(\Omega))$.

We now introduce the operators by which (1.1) will be recast into an equivalent vector form. Let

$$(b, b) \in \mathbb{R} \times \mathbb{R}, \quad b: \mathcal{D}(b) \rightarrow L^2(\Omega),$$

$$u \mapsto b(u) = [-\beta(u_a)]_x,$$

where

$$\mathcal{D}(b) = \{u \in H_0^1(\Omega) \mid \beta(u_a) \in H^1(\Omega) \subset H_0^1(\Omega)\},$$

$$\tilde{b}: H_0^1(\Omega) \rightarrow H^{-1}(\Omega),$$

$$u \mapsto \tilde{b}(u) = [-\beta(u_a)]_x.$$

Note that $\tilde{b} \supset b$.

Moreover

$$\begin{aligned} B: H_0^1(\Omega) \times \mathcal{D}(b) &\rightarrow H_0^1(\Omega) \times L^2(\Omega), \\ (u, v) &\mapsto B(u, v) = (0, b(v)), \end{aligned}$$

$$\begin{aligned} \tilde{B}: H_0^1(\Omega) \times H_0^1(\Omega) &\rightarrow L^2(\Omega) \times H^{-1}(\Omega), \\ (u, v) &\mapsto \tilde{B}(u, v) = (0, \tilde{b}(v)). \end{aligned}$$

Note that $\tilde{B} \supset B$.

$$\begin{aligned} L: (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) &\rightarrow H_0^1(\Omega) \times L^2(\Omega), \\ (u, v) &\mapsto L(u, v) = (-v, -u_{xx}), \\ A: \mathcal{D}(A) &\rightarrow H_0^1(\Omega) \times L^2(\Omega), \\ (u, v) &\mapsto A(u, v) = (-v, -u_{xx} + \tilde{b}(v)), \end{aligned}$$

where

$$\mathcal{D}(A) = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid u_{xx} + \tilde{b}(v) \in L^2(\Omega)\}.$$

By defining the vectors $U(t) = [u(t, \cdot), u_t(t, \cdot)] \in H_0^1(\Omega) \times L^2(\Omega)$ and $F(t) = [0, f(t, \cdot)]$ Equation (1.1) equipped with the homogeneous boundary conditions

$$(2.1) \quad u(t, 0) = u(t, 1) = 0, \quad \forall t \in \mathbb{R}^+,$$

transforms into the nonlinear evolution equation in vector form

$$(E) \quad \frac{dU}{dt}(t) + AU(t) = F(t), \quad t \in \mathbb{R}^+.$$

3. - EXISTENCE OF STRONG AND WEAK SOLUTIONS OF (E) AND (1.1)

The main results in this Section follow from the general theory of maximal monotone operators and corresponding nonlinear evolution equations (Benilan and Brezis, 1972; Brezis, 1973).

THM. 1: Let the function $\beta \in C^1(\mathbb{R})$ s.t., $\beta(0) = 0$, be monotonically increasing on \mathbb{R} . Then the operator $L + B$ is maximal monotone in $H_0^1(\Omega) \times L^2(\Omega)$ and $\overline{L + B} = A$.

PROOF: The proof can be carried out as in Prestel (1982) by taking into account some general properties of maximal monotone operators (Brezis, 1973).

THEM. 2: Consider the Cauchy problem for (E)

$$(3.1) \quad \begin{cases} \frac{dU}{dt}(t) + AU(t) = F(t), \\ U(0) = [u_0, v_0] \in H_0^1(\Omega) \times L^2(\Omega). \end{cases}$$

A) If we choose the initial datum $[u_0, v_0]$ in $\mathcal{D}(A)$ and assume that the forcing term $f \in L^1(0, T; L^2(\Omega))$, $T > 0$

- i) satisfies $f_t \in L^2(0, T; L^2(\Omega))$ and
- ii) is absolutely continuous on $[0, T]$,

then Problem (3.1) has a unique *strong* solution $U \in C([0, T]; H_0^1 \times L^2(\Omega))$.

B) If $[u_0, v_0] \in \overline{\mathcal{D}(A)}$ and $f \in L^1(0, T; L^2(\Omega))$, $T > 0$,

then Problem (3.1) has a unique *weak* solution $U \in C([0, T]; H_0^1(\Omega) \times L^2(\Omega))$.

The solution referred to is either strong or weak in the sense of Benilan and Brezis (1972).

PROOF: We refer the Reader to Proposition 3.4 of Brezis (1973) for strong solutions and to Benilan and Brezis (1972) for weak solutions of abstract evolution equations related to maximal monotone operators.

REM. 1: We recall that

A) *strong* solutions of (E), respectively (1.1), are represented by continuous trajectories in phase space

$$U: [0, T] \ni t \mapsto U(t) = [u(t, \cdot), u_t(t, \cdot)] \in H_0^1(\Omega) \times L^2(\Omega),$$

which additionally are in $\mathcal{D}(A)$, are differentiable and satisfy (E) in $H_0^1(\Omega) \times L^2(\Omega)$ for almost every $t \in [0, T]$, respectively satisfy (1.1) in $L^2(\Omega)$ for almost every $t \in [0, T]$;

B) on the other hand *weak* solutions of (E), respectively (1.1), are represented by continuous trajectories in phase space, so that the initial condition $U(0) = [u(0, \cdot), u_t(0, \cdot)] = [u_0, v_0]$ makes sense. Said trajectories satisfy (E) in the distribution sense i.e., in $L^2(\Omega) \times H^{-1}(\Omega)$ for almost every $t \in [0, T]$, respectively satisfy (1.1) in the distribution sense i.e., in $H^{-1}(\Omega)$ for almost every $t \in [0, T]$.

4. - EXISTENCE OF A WEAK SOLUTION TO (1.1),

WHICH IS BOUNDED IN A STRONGER NORM

We must now be more specific about the function β and the forcing term f in order to prove the existence of a *regular weak* solution to (1.1), which is a bounded trajectory both in phase space and in a stronger energy space.

Let $\beta \in C^1(\mathbb{R})$ s.t., $\beta(0) = 0$, satisfy a polynomial growth condition i.e., there are positive constants $\alpha, \epsilon_1, \epsilon_2, M$ s.t.

$$(4.1) \quad \epsilon_1 |u|^{q+1} < |\beta(u)| < \epsilon_2 (1 + |u|^{q+1}), \quad \forall u \in \mathbb{R},$$

$$(4.2) \quad 0 < \beta'(u) < M(1 + |\beta(u)|), \quad \forall u \in \mathbb{R}.$$

We let $p = 2 + \alpha, q = p/(p-1)$ and assume that f is in the Stepanov space $S^p(\mathbb{R}_t^+; W_0^{1,p}(\Omega))$ i.e., according to the notation introduced in Section 2,

$$f \in L_{loc}^\infty(\mathbb{R}_t^+; W_0^{1,p}(\Omega)), \\ \sup_t \left\{ \int_0^{t+1} \|f_s(t, \cdot)\|_{L^p(\Omega)}^p dt; t > 0 \right\} < +\infty.$$

Thm. 3: Let β satisfy (4.1), (4.2) and $f \in S^p(\mathbb{R}_t^+; W_0^{1,p}(\Omega))$, then the Cauchy problem related to Equation (1.1) subjected to homogeneous boundary conditions

$$(4.3) \quad \begin{cases} u_{tt} - u_{xx} - [\beta(u_{xx})]_x = f, \\ u(t, 0) = u(t, 1) = 0, \\ u(0, \cdot) = u_0 = 0, \\ u_t(0, \cdot) = u_0 = 0, \end{cases}$$

has a unique weak solution $u \in L^\infty(\mathbb{R}_t^+; H_0^1(\Omega) \cap H^3(\Omega))$ s.t.,

$$u_t \in L^\infty(\mathbb{R}_t^+; H_0^1(\Omega)) \cap S^p(\mathbb{R}_t^+; W_0^{1,p}(\Omega)),$$

which satisfies (1.1) in the distribution sense i.e., in $W^{-1,p}(\Omega)$ for almost every $t \in \mathbb{R}^+$.

REM. 2: We shall prove Theorem 3 in Section 6. To this end we recall after Biroli and Haraux (1980) the definition of the stronger energy $E(t)$ of solutions to (1.1). Indeed the proof will be based on some *a priori* estimates of E .

Let

$$(4.4) \quad E(t) = E(t) + E_1(t),$$

where

$$E(t) := \|u_x(t, \cdot)\|_{L^2(\Omega)}^2 + \|u_t(t, \cdot)\|_{L^2(\Omega)}^2 = \int_0^1 (u_x^2 + u_t^2) dx, \quad (\text{the usual energy}),$$

$$E_1(t) = \|u_{xx}(t, \cdot)\|_{L^2(\Omega)}^2 + \|u_{xt}(t, \cdot)\|_{L^2(\Omega)}^2 = \int_0^1 (u_{xx}^2 + u_{xt}^2) dx.$$

REM. 3: The above mentioned estimates of \tilde{E} could also apply to prove via a Faedo-Galerkin method the existence of a unique weak solution of the Cauchy problem for (1.1), where the initial data satisfy

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad v_0 \in H_0^1(\Omega).$$

A similar procedure is applied by Dang Dinh Hai (to appear, 1989), who in addition requires the initial data and the forcing term $f \in L^q(R_t^+; W_0^{1,q}(\Omega))$ to be « small ».

REM. 4: We note that the initial conditions in Problem (4.3) make sense, because the above mentioned « regular » weak solution u is represented by a continuous trajectory in phase space

$$[u, u_t] \in C^0([0, +\infty); H_0^1(\Omega) \times L^2(\Omega)).$$

Indeed this last statement is part of a Corollary which deals with the main topological properties of u .

Con. 1: The weak solution u of (1.1) defined by Theorem 3 when seen as a trajectory in phase space

$$[0, +\infty) \ni t \mapsto [u(t, \cdot), u_t(t, \cdot)] \in H_0^1(\Omega) \times L^2(\Omega)$$

is uniformly continuous on the half line $[0, +\infty)$ and has a relatively compact range in $H_0^1(\Omega) \times L^2(\Omega)$.

PROOF: Compactness in phase space follows immediately from the boundedness of $[u(t, \cdot), u_t(t, \cdot)]$ in the strong energy space. Now let us prove that the trajectory in $L^2(\Omega)$

$$[0, +\infty) \ni t \mapsto u_t(t, \cdot) \in L^2(\Omega)$$

is uniformly continuous. To this end we recall that $u_{tt} \in S^p(R_t^+; W^{-1,p}(\Omega))$ and

$$\|u_t(t+b, \cdot) - u_t(t, \cdot)\|_{W^{-1,p}} \leq \int_t^{t+b} ds \|u_{tt}\|_{W^{-1,p}} < c b^{1/p}.$$

Let us also recall that for every unit vector $v \in L^2(\Omega)$ there exists a sequence of test functions $\{v_n\} \subset C_0^\infty(\Omega)$ s.t., $\|v_n\|_{C_0^\infty} < \epsilon$ and $\|v - v_n\|_{H^{-1}(\Omega)} < K(n)$, where $K(n) \rightarrow 0^+$ as $n \rightarrow +\infty$.

We can estimate

$$\begin{aligned} \|u_t(t+b, \cdot) - u_t(t, \cdot)\|_{L^2(\Omega)} &= \sup_{\|v\|_{L^2(\Omega)}=1} |\langle u_t(t+b, \cdot) - u_t(t, \cdot), v \rangle| < \\ &< \sup_{\|v\|_{L^2(\Omega)}=1} (\|\langle u_t(t+b, \cdot) - u_t(t, \cdot), v_n \rangle\| + \|\langle u_t(t+b, \cdot) - u_t(t, \cdot), v_n - v \rangle\|) < \\ &\leq \alpha b^{1/p} + \sup_{n \geq 0} (\|u_t(t, \cdot)\|_{H^2(\Omega)}) K(n) < \tilde{\epsilon} (ab^{1/p} + K(n)). \end{aligned}$$

By choosing $\pi = b^{-1/\beta p}$ we get

$$\|u_\varepsilon(t + \delta, \cdot) - u_\varepsilon(t, \cdot)\|_{L^q(\Omega)} < \tilde{c}(b^{1/\beta p} + K(b^{-1/\beta p})) \rightarrow 0 \quad \text{as } b \rightarrow 0^+$$

uniformly w.r. to $t > 0$.

5. - EXISTENCE OF AN ENERGY—ALMOST—PERIODIC SOLUTION TO (1.1)

The last step is to show there exists an energy—almost—periodic solution to (1.1) in the strip $R_z \times \Omega$, which complies with homogeneous boundary conditions $u(t, 0) = u(t, 1) = 0, \forall t \in R_z$.

This result follows from Cor. 1 provided a general theorem by Haraux (1978, 1981) is taken into account, which holds for abstract non linear evolution equations related to maximal monotone operators. As a consequence we can state the following.

Thm. 4: Let β and f satisfy the same conditions as in Thm. 3. Then Equations (E) and (1.1) considered in the strip $R_z \times \Omega$ have at least one energy almost-periodic weak solution

$$\bar{U} = [\bar{u}, \dot{\bar{u}}] \in L^\infty(R_z; H_0^1(\Omega) \times L^2(\Omega)) \cap C^0(R_z; H_0^1(\Omega) \times L^2(\Omega)) .$$

6. - A-priori ESTIMATES OF THE ENERGY $\bar{E}(t)$

In this Section we prove Theorem 3. We assume that Problem (4.3) has a strong solution. If it has not, then a standard regularisation procedure applied to the forcing term f enables us to extend the results which follow to the general case. Moreover, we may assume that all integrals involved make sense. If they do not, first the Faedo-Galerkin approximation must be referred to, then a limit be evaluated.

The proof requires that we consider the boundedness of the «strong energy» $\bar{E}(t)$ of u defined by (4.4).

Let us examine the derivative $E'(t)$:

$$(6.1) \quad E'(t) = 2 \int_0^1 (u_x u_{xt} + u_t u_{tt} + u_{xx} u_{xxt} + u_{xt} u_{xxt}) dx .$$

By taking Equation (1.1) and the boundary conditions (2.1) into account we obtain

$$(6.2) \quad \frac{1}{2} E'(t) = \int_0^1 (-\beta(u_{xt}) u_{xt} - \beta'(u_{xt}) u_{xxt}^2 + f_x u_{xt} + f u_t) dx .$$

Let $I > 0$ be fixed and $T > 0$. Either of the following may occur:

- (a) $\tilde{E}(I+T) < \tilde{E}(I)$, or
- (b) $\tilde{E}(I+T) > \tilde{E}(I)$.

Let us examine case (b) and prove that $\sup_{i \in [I, I+T]} \{\tilde{E}(i)\} < \varepsilon$, where the constant ε does not depend on i . This result combined with (a) will imply the boundedness of $\tilde{E}(i)$ on R_i^+ . As the first step we have to show that the variation of $\tilde{E}(i)$ in the interval $[i, i+T]$ is uniformly bounded w.r.t. i . Indeed if (b) holds we obtain, by integrating (6.2) over $[i, i+T]$,

$$(6.3) \quad \begin{aligned} 0 &< \int\limits_i^{i+T} dt \int\limits_0^1 dx (\beta(u_{st}) u_{st} + \beta'(u_{st}) u_{st}^2) < \int\limits_i^{i+T} dt \int\limits_0^1 dx (f_s u_{st} + f u_s) < \\ &< \int\limits_i^{i+T} dt (\|f_s\|_{L^p(\Omega)} \|u_{st}\|_{L^p(\Omega)} + \lambda_p \lambda_q \|f_s\|_{L^q(\Omega)} \|u_{st}\|_{L^p(\Omega)}) < \\ &< (1 + \lambda_p \lambda_q) \int\limits_i^{i+T} dt \|f_s\|_{L^p(\Omega)} \|u_{st}\|_{L^p(\Omega)}, \end{aligned}$$

where λ_p , resp. λ_q , are the canonical imbedding constants of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ and of $W_0^{1,q}(\Omega)$ into $L^q(\Omega)$. Moreover, if we take the coercivity condition (4.1) on β into account and apply Hölder's inequality, we recast (6.3) into

$$(6.4) \quad \begin{aligned} \int\limits_i^{i+T} dt \int\limits_0^1 dx (|u_{st}|^p + \beta'(|u_{st}|) u_{st}^p) &= \int\limits_i^{i+T} dt \|u_{st}\|_{L^p(\Omega)}^p + \int\limits_i^{i+T} dt \int\limits_0^1 dx \beta'(|u_{st}|) u_{st}^p < \\ &< \max\{c_1, c_1^{-1}\} (1 + \lambda_p \lambda_q) \int\limits_i^{i+T} dt \|f_s\|_{L^p(\Omega)} \|u_{st}\|_{L^p(\Omega)} < c_1^2 \int\limits_i^{i+T} dt \|f_s\|_{L^p(\Omega)}^2 + c_1^2 \int\limits_i^{i+T} dt \|u_{st}\|_{L^p(\Omega)}^2, \end{aligned}$$

hence, if we choose $\varepsilon < 1$,

$$(6.5) \quad \int\limits_i^{i+T} dt \|u_{st}\|_{L^p(\Omega)}^p + \int\limits_i^{i+T} dt \int\limits_0^1 dx \beta'(|u_{st}|) u_{st}^p < k \int\limits_i^{i+T} dt \|f_s\|_{L^p(\Omega)}^2 < k(T).$$

We note that in the final term the quantity $k(T)$ does not depend on $i (> 0)$. Now we shall show that the energy variation in the interval $[i, i+T]$ is

uniformly bounded w.r. to \tilde{t} . Indeed by means of (6.5) we obtain

$$(6.6) \quad 0 < \tilde{E}(\tilde{t}, \tilde{t} + T) - \tilde{E}(\tilde{t}) < 2 \int_{\tilde{t}}^{\tilde{t}+T} dt \int dx (f_u u_{st} + f u_t) <$$

$$< 2(1 + \lambda_x \lambda_t) \int_{\tilde{t}}^{\tilde{t}+T} dt \|f_x\|_{L^p(\Omega)} \|u_{st}\|_{L^p(\Omega)} < \tilde{\lambda}(T),$$

where $\tilde{\lambda}(T)$ does not depend on how \tilde{t} (> 0) is chosen.

As a prerequisite for further estimates, let us look for upper bounds to $|\beta(u_{st})|$ and $|\beta'(u_{st})|$. By the mean value theorem there exists a suitable $x^* \in [0, 1]$ s.t.

$$\int_0^1 dx \beta(u_{st}) = \beta(u_{st}(t, x^*)) . \quad (1.1)$$

Therefore

$$|\beta(u_{st}) - \beta(u_{st}(t, x^*))| = \left| \int_{x^*}^t dx [\beta(u_{st})]_x \right| < \int_{x^*}^t dx |[\beta(u_{st})]_x|$$

whence by Hölder's inequality and (4.2) we get

$$(6.7) \quad |\beta(u_{st})| < \int_0^1 dx |[\beta(u_{st})]_x| + \int_0^1 dx |\beta(u_{st})| = \int_0^1 dx |\beta'(u_{st})| u_{stx} + \int_0^1 dx |\hat{\beta}(u_{st})| <$$

$$< \varepsilon \left(\int_0^1 dx |\beta'(u_{st})| u_{stx}^2 + \int_0^1 dx (\beta'(u_{st}) + |\beta(u_{st})|) \right) <$$

$$< \tilde{\varepsilon} \left(\int_0^1 dx |\beta'(u_{st})| u_{stx}^2 + \int_0^1 dx |\beta(u_{st})| + M \right).$$

Again by taking (4.2) into account a similar estimate for $|\beta'(u_{st})|$ is obtained

$$(6.8) \quad |\beta'(u_{st})| < M(1 + |\beta(u_{st})|) < \tilde{\varepsilon} \left(\int_0^1 dx |\beta'(u_{st})| u_{stx}^2 + \int_0^1 dx |\beta(u_{st})| + M \right).$$

Now let us multiply Equation (1.1) by u_{st}

$$(6.9) \quad (u_{tt} - u_{xx} - [\beta(u_{st})]_x | u_{st})_{L^p(\Omega)} = (f | u_{st})_{L^p(\Omega)}$$

and estimate every term from above. Since the following holds by induction

$$\begin{aligned} \left| \int_0^1 dx [\beta(u_{st})]_{st} u_{st} \right| &\leq \int_0^1 dx |\beta'(u_{st})| |u_{st}| \leq c \int_0^1 dx |\beta'(u_{st})| u_{st}^2 + \epsilon \int_0^1 dx |\beta'(u_{st})| u_{st}^2 \\ &\leq c \int_0^1 dx |\beta'(u_{st})| u_{st}^2 + \epsilon \sup_{\substack{s \in [0, t] \\ i \in \{I, I+T\}}} \{|\beta'(u_{si})|\} \sup_{t \in [I, I+T]} \{\|u_{st}\|_{L^2(\Omega)}\} \\ [\text{by (6.8)}] \quad &< c \int_0^1 dx |\beta'(u_{st})| u_{st}^2 + \\ &\quad + k \epsilon \left(\int_0^1 dx |\beta(u_{st})| u_{st}^2 + \int_0^1 dx |\beta(u_{st})| + M \right) \sup_{t \in [I, I+T]} \{\|u_{st}\|_{L^2(\Omega)}\}, \end{aligned}$$

then integration over $[I, I+T]$ yields

$$\begin{aligned} (6.10) \quad & \left| \int_I^{I+T} \int_0^1 dx [\beta(u_{st})]_{st} u_{st} \right| \leq c \int_I^{I+T} \int_0^1 dx |\beta'(u_{st})| u_{st}^2 + k \epsilon \sup_{t \in [I, I+T]} \{\|u_{st}\|_{L^2(\Omega)}\} \cdot \\ & \cdot \left(\int_I^{I+T} \int_0^1 dx |\beta'(u_{st})| u_{st}^2 + \int_I^{I+T} \int_0^1 dx |\beta(u_{st})| + M T \right) < [\text{by (6.5) and (6.7)}] < \\ & < c \int_I^{I+T} \int_0^1 dx |\beta'(u_{st})| u_{st}^2 + \eta \sup_{t \in [I, I+T]} \{E_1(t)\}. \end{aligned}$$

Now let us estimate the term $\left| \int_I^{I+T} \int_0^1 dx u_{st} u_{st} \right|$. Hereinafter $\|\cdot\|_2$ shall stand for the $L^2(\Omega)$ -norm.

$$\begin{aligned} (6.11) \quad & \left| \int_I^{I+T} \int_0^1 dx u_{st} u_{st} \right| = \left| \int_I^{I+T} \int_0^1 dx \langle u_{st} \rangle_t u_{st} \right| < \\ & < \int_I^{I+T} \int_0^1 dx u_{st}^2 + \int_I^{I+T} \int_0^1 dx |u_{st}(I+T, \cdot)| u_{st}(I+T, \cdot) + \int_I^{I+T} \int_0^1 dx |u_{st}(I, \cdot)| u_{st}(I, \cdot) < \\ & < \int_I^{I+T} \int_0^1 dx u_{st}^2 + \|u_{st}(I+T, \cdot)\|_2 \|u_{st}(I+T, \cdot)\|_2 + \|u_{st}(I, \cdot)\|_2 \|u_{st}(I, \cdot)\|_2 < \\ & < \int_I^{I+T} \int_0^1 dx u_{st}^2 + \lambda_0^2 (\|u_{st}(I+T, \cdot)\|_2 \|u_{st}(I+T, \cdot)\|_2 + \|u_{st}(I, \cdot)\|_2 \|u_{st}(I, \cdot)\|_2) < \\ & < \int_I^{I+T} \int_0^1 dx u_{st}^2 + 2 \lambda_0^2 \sup_{t \in [I, I+T]} \{E_1(t)\}, \end{aligned}$$

where λ_0 is the canonical imbedding constant of $H_0^1(\Omega)$ into $L^2(\Omega)$.

Finally we need to consider the term containing f . The following holds

$$(6.12) \quad \left| \int_i^{i+T} dt \int_0^1 dx f u_{xx} \right| < \int_i^{i+T} dt \int_0^1 dx |f| |u_{xx}| < c_\eta \int_i^{i+T} dt \|f\|_{L^2(\Omega)}^2 + \eta \int_i^{i+T} dt \int_0^1 dx |u_{xx}|^2.$$

From (6.9), by taking (6.10), (6.11) and (6.12) into account we arrive at the following estimate for u_{xx}

$$(6.13) \quad \begin{aligned} \int_i^{i+T} dt \int_0^1 dx u_{xx}^2 &< (1+\delta) \left(\int_i^{i+T} dt \int_0^1 dx u_x^2 + \varepsilon \int_i^{i+T} dt \int_0^1 dx \beta'(u_{xx}) u_{xx}^2 \right) \\ &+ (1+\delta)(2J_0^2 + \eta) \sup_{t \in [i, i+T]} \{E_1(t)\} + k_0 \Big) < [\text{by (6.5)}] \\ &< C + (1+\delta)(2J_0^2 + \eta) \sup_{t \in [i, i+T]} \{E_1(t)\}, \end{aligned}$$

then, by recalling (6.5),

$$(6.14) \quad \int_i^{i+T} dt E_1(t) < C_1 + (1+\delta)(2J_0^2 + \eta) \sup_{t \in [i, i+T]} \{E_1(t)\}.$$

Let $T = 4J_0^2$ and the constants δ and η be chosen in a way that $(T/2 + \eta) \cdot (1+\delta) = \frac{2}{3}T$. By virtue of the mean value theorem there exists $t^* \in [i, i+T]$ s.t.,

$$(6.15) \quad E_1(t^*) = \frac{1}{T} \int_i^{i+T} dt E_1(t) < C_1 + \frac{2}{3} \sup_{t \in [i, i+T]} \{E_1(t)\}.$$

We now recall that, according to (6.6), the variation of $E_1(t)$ is bounded in $[i, i+T]$

$$(6.16) \quad |E_1(t) - E_1(i)| < K \quad \forall t, i \in [i, i+T]$$

and deduce

$$(6.17) \quad \sup_{t \in [i, i+T]} \{E_1(t)\} < E_1(t^*) + K < C_1 + \frac{2}{3} \sup_{t \in [i, i+T]} \{E_1(t)\}.$$

Finally

$$(6.18) \quad \sup_{t \in [i, i+T]} \{E_1(t)\} < C,$$

where the constant C does not depend on i .

The boundedness of $E_1(t)$ in the interval $[i, i+T]$ implies the boundedness both of the usual energy $E(t)$ (this follows from $\|u_t\|_2 \leq J_0 \|u_{xx}\|_2$ and from $\|u_x\|_2 \leq J_0^2 \|u_{xx}\|_2$) and of the total energy $\tilde{E}(t)$ in the same interval.

The latter property holds, as we stated at the beginning of this Section, in every interval where (b) occurs. Since the alternative is condition (a), then $\tilde{E}(t)$ is bounded in R_t^+ . In other words the solution of the Cauchy problem (4.1) has a bounded trajectory in the strong energy space

$$R_t^+ \ni t \mapsto [u, u_t] \in L^\infty(R_t^+; H_0^1(\Omega) \cap H^2(\Omega)) \times L^\infty(R_t^+; H_0^1(\Omega)).$$

Finally we point out that (6.5) alone implies the Stepanov boundedness of u_t in $W_0^{1,p}(\Omega)$, for every interval $[t, t+T]$ where (b) occurs. When (6.18) is also taken into account said boundedness property extends to those intervals where (a) occurs i.e.,

$$u_t \in S^p(R_t^+; W_0^{1,p}(\Omega)).$$

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