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On the Motion of a String Vibrating Against a Gluing Obstacle (**) (***)

SUMMARY. — We study the motion of a vibrating string, subject to an unilateral condition given by a plane, gluing obstacle. The impact is supposed to be inelastic. The peeling follows two possible models (constant force or constant energy).

Sul moto di una corda vibrante contro un ostacolo adesivo

SUNTO. — Si studia il moto di una corda vibrante in presenza di un ostacolo piano adesivo. L'urto è supposto anelastico; lo strappo segue due possibili modelli (forza costante o energia costante).

0. - INTRODUCTION

This paper concerns the motion of a string in presence of a plane, gluing obstacle. Then two phenomena can occur: an impact against the obstacle (in some cases also a soft contact), and a subsequent detachment, impeded by the glue. For the modelling of the impact we follow the method of the influence lines (see [3]); the detachment is treated, on the contrary, following the schemes given in [4]. For an up-to-date survey about these topics, see [5].

1. - THE IMPACT AND PEELING LAWS

Let us consider in the plane (x, y) a string, vibrating in presence of a plane wall, covered by glue.

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In this situation, the string may touch the wall with an impact, which we shall consider as completely inelastic. A reaction J will then take place in the impact points. The « a priori » information about this reaction is that its support is contained in the contact set, that it is directed towards the string, and that after the impact the speed of the string must vanish. After the impact, some part of the string will adhere to the wall, because of the glue. By effect of some external force, or by the tension given by the free portion of the string, the glued part may be peeled off. In this case the reaction J is directed towards the wall. Then the motion is described by the following system of (in)equalities:

$$\mu\, \partial^2 y/\partial t^2 - T\, \partial^2 y/\partial x^2 = F(x, t, y, \partial y/\partial x, \partial y/\partial t) + J \quad \text{in } Z = \{a<x<b,\ t>0\},$$

with

— $y(a, t) = p(t) > 0$, $y(b, t) = q(t) > 0$;

— $y(x, 0) = u(x) > 0$, $\partial y/\partial t(x, 0) = v(x)$;

— $y(x, t) \geq 0$;

— $J \geq 0$ in the sense of distributions on every open « impact arc »,

— $J \leq 0$ in the sense of distributions on every open « peeling arc »,

— $\operatorname{supp} J \subseteq \{(x, t) \mid y(x, t) = 0\}$;

where μ is the linear density of the string, T its tension, p, q, u and v are given (continuous) functions.

We suppose obviously $p(0) = u(a)$, $q(0) = u(b)$. We shall put, moreover, $c = \sqrt{T/\mu}$.

It is well known how to solve the free vibrating string equation by the method of characteristics: the strip $\{a<x<b,\ t>0\}$ is decomposed into a sequence of triangles and rectangles (in characteristic coordinates), of equation ξ = constant or η = constant, which can be arbitrarily choosen, and the problem is reduced to a sequence of « elementary » (Cauchy, Darboux and Goursat) problems (see fig. 1)

The characteristic variables

$$(1.1) \qquad \left|\begin{array}{l} \xi = t + x/c\,, \\ \eta = t - x/c\,, \end{array}\right.$$

define a transformation Φ of the plane, whose jacobian is $|\Phi| = 2/c$. The equation then takes the form:

$$(1.2) \qquad \partial^2 y/\partial \xi\, \partial \eta = f(\xi, \eta, y, \partial y/\partial \xi, \partial y/\partial \eta) + \Upsilon$$

(we employ the same letter y to indicate $y(\xi, \eta)$ as well as $y(x, t)$, because there is no ambiguity), where

— $f(\xi, \eta, y, \partial y/\partial \xi, \partial y/\partial \eta) = F(x, t, y, \partial y/\partial x, \partial y/\partial t)/4\mu$ and

— $\Upsilon = \Phi \circ J/4\mu$.

If the external force is independent of y, there are explicit formulae for the solutions of these problems; if it depends on y and/or on its derivatives, the same formulae represent integral equations, which can be solved under wide assumptions (e.g. if f is Lipschitz continuous with respect to the unknowns). In the following, we shall always suppose $f = f(\xi, \eta) \in C^{\infty}(Z)$ for the sake of simplicity.

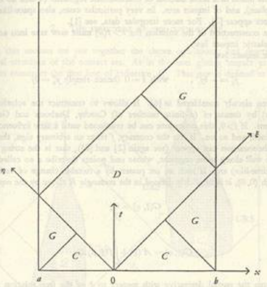

Fig. 1.

For the Darboux problem in the rectangle $R = [0, a] \times [0, b]$

$$(1.3)\qquad \left|\begin{array}{ll} y(\xi, 0) = A(\xi) & 0 < \xi < a \\ y(0, \eta) = B(\eta) & 0 < \eta < b \end{array}\right.$$

(with $A(0)=B(0)$ for consistency), the solution is given by

$$y(\xi,\eta)=A(\xi)+B(\eta)-A(0)+\int_0^{\xi}dx\int_0^{\eta}f(\alpha,\beta)\,d\beta\,. \tag{1.4}$$

Let us sketch briefly the results which hold for the non gluing wall.

Since $u(x)>0$, for small t the string does not touch the wall. Then a contact can take place at some time $t=t(x)$ (either as an impact, or as a soft leaning). These contact arcs belong to the first influence line Λ_1, defined as in [3]. We remember that Λ_1 is given by an equation $t=t(x)$, where $t(x)$ is Lipschitz continuous with constant $1/c$; if the data are smooth enough, Λ_1 is the union of a finite number of characteristic segments (where no impact takes place), and of impact arcs. In very particular cases, also space-like support arcs appear [2]. For more irregular data, see [1].

The construction of the solution for $t>t(x)$ must now take into account the inelastic impact law.

The impact law

$$y_t^+=-by_t^-\,,\qquad \text{with } b=0 \text{ (hence simply } y_t^+=0)$$

has been already considered in [6]. It allows to construct the solution for $t>t(x)$ by means of (a finite number of) Cauchy, Darboux and Goursat problems. If $f>0$, this procedure can be continued until a new influence line is attained, and so on. If, on the contrary, f takes an arbitrary sign, the support phenomenon can appear (see again [2] and [8]), that is the string leans on the wall along some segment, whose end points describe a so called support (time-like) arc. If such an arc passes, by a suitable change of variables, through $(0,0)$, it is implicitly defined in the rectangle R above by the equation

$$G(\xi,\eta)=0\,,$$

where

$$G(\xi,\eta)=A'(\xi)+\int_0^{\eta}f(\xi,\beta)\,d\beta \tag{1.5}$$

represents the partial derivative with respect to ξ of the free solution given in (1.4). Obviously $A'(0)=G(0,0)=0$ is a necessary condition for the support phenomenon.

On the other hand, we recall the results of [4] for the peeling. For the linear vibrating string equation, the authors consider two kinds of peeling conditions. The first one is that the normal force at the point of separation between glued and free part of the string takes some constant value F; alternatively, the gluing energy per unity length is some given constant γ. These approaches lead to the following conditions: either

$$(T-\mu\sigma'^2)[y_x] = F \tag{1.6a}$$

or

$$(T-\mu\sigma'^2)[y_x^2] = \pm 2\gamma\,, \tag{16b}$$

where $x = \sigma(t)$ is the equation of the moving boundary (peeling arc), and $[u] = u(x^+, t) - u(x^-, t)$ denotes the jump of a function u across this boundary. In (1.6*b*) we must choose the sign according to the sign of $[y_x^2]$. If $\sigma' = 0$, we must substitute (1.6*a*) and (1.6*b*) with the corresponding inequalities (see Sect. 4).

2. - The structure of the contact set

In this section we put together the above conditions, and describe the general structure of the contact set. As in the non gluing impact problem, we can construct the first line of influence Λ_1. This one is defined as in [3],

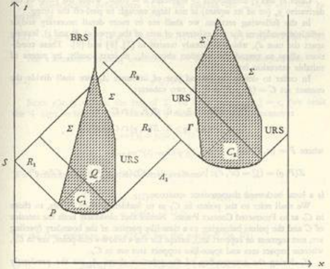

Fig. 2.

except for the fact that instead of the condition $y > 0$ for $t < t(x)$ we must impose the stronger one $y > 0$. Indeed if at some point P it is $y(P) = 0$, the effect of the glue may change the structure of the solution.

If the data are smooth enough (so that, for instance, the free solution is piecewise C^1), Λ_1 is again constituted by a finite number of space-like arcs (possibly reduced to a single point), joined by characteristic segments. Above Λ_1, the solution is computed as follows. First of all, we put $y(x, t) = 0$ in the curved triangles C_i (see fig. 2), bounded by arcs where $y(x, t(x)) = 0$ and $y_t(x, t(x)^+) = 0$, and by characteristic segments.

Then we take into account the characteristic rectangles R_i. It is $P \in \Lambda_1$, and $y(P) = 0$. In most cases, $y = 0$ in the whole side PQ (or PS). In general the boundary between glued and free part of the string is constituted by a time-like arc, which can be of four types:

a) Peeling arc (Σ).

b) Unilateral rest segment (URS).

c) Bilateral rest segment (BRS).

d) Support arc (Γ).

Cases *b*) and *c*) correspond to segments $x =$ const, where the jump of the derivative y_x (or of its square) is not large enough to peel-off the string.

In the following sections we shall see in more detail necessary and/or sufficient conditions for the existence of arcs of the types *a*), *b*) and *c*), leaving apart the case *d*), which is already treated in [2], [8] and [9]. These conditions allow to extend the solution above Λ_1, at least locally, by means of suitable extension laws.

In order to define the second line of influence Λ_2, we shall divide the contact set $C = \{P : y(P) = 0\}$ in two subsets:

$$C_s = \{P \in C : \exists \varrho > 0 : Z(P, \varrho) \cap C = \emptyset\},$$
$$C_p = \{P \in C : \forall \varrho > 0 : Z(P, \varrho) \cap C \neq \emptyset\},$$

where $P = (x, t)$ and

$$Z(P, \varrho) = \{Q = (x', t') : |x - x'| < c(t - t'), (x - x')^2 + c^2(t - t')^2 < \varrho^2\}$$

is a local backward characteristic semicone.

We shall refer to the points in C_s as to Sudden Contact Points, to those in C_p as to Protracted Contact Points. Notice that obviously both the interior of C and the points belonging to a time-like portion of the boundary (peeling arc, rest segment or support arc), except for the « below » end-point, are in C_p, whereas impact arcs and space-like support arcs are in C_s.

If, during the construction of the solution above Λ_1 via the previously quoted problems, we find new Sudden Contact Points, then we must construct a second influence line Λ_2. Unlike in [3], Λ_2 and Λ_1 may be joined together by Protracted Contact Points, so that the definition of Λ_2 must be given in a slightly different manner, to avoid ambiguities. Precisely we shall put:

$$t_2(x) = \sup\{t : Q \in Z_{(x,t)} \cap C_2 \Rightarrow Q \in \Lambda_1\},$$

where

$$Z_{(x,t)} = \{Q = (x', t') : |x - x'| < c(t - t')\}$$

is a global backward characteristic semicone.

The procedure can be continued until a third influence line is reached, and so on. Notice that the points of the boundary ∂C of C which $\in C_2$ are on some Λ_i.

3. - Peeling arcs

Let us study now the peeling phenomenon in a characteristic rectangle R as in sect. 2, with the Darboux conditions (1.3). Let us firstly suppose that the left part of the string is glued, so that $B(\eta) = 0$.

In characteristic coordinates (1.1), the peeling arc $\Sigma : x = \sigma(t)$ will be written as

$$\eta = \psi(\xi) \Leftrightarrow \xi = \phi(\eta) \Rightarrow \sigma'(t) = c(1 - \psi')/(1 + \psi') = c(\phi' - 1)/(\phi' + 1). \tag{3.1}$$

The « subsonic » peeling condition $-c < \sigma'(t) < 0$ is transformed into

$$\psi'(\xi) > 1. \tag{3.2}$$

Since $y(x, t) = 0$ on the left of Σ, we have simply $[y_x] = y_x$ (we omit the $+$ sign to indicate the values of y on the right of Σ); moreover

$$y_x = (y_\xi - y_\eta)/c,$$
$$T - \mu\sigma'^2(t) = 4T\psi'/(1 + \psi')^2.$$

With reference to fig. 3, the Goursat problem

$$\left|\begin{array}{ll} y(\xi, 0) = A(\xi), & \qquad A(0) = 0, \\ y|_\Sigma & = y(\phi(\eta), \eta) = 0, \end{array}\right. \tag{3.3}$$

for the free equation $y_{\xi\eta} = f(\xi, \eta)$ has the solution (in Ω):

$$y(\xi, \eta) = A(\xi) - A(\phi(\eta)) + \int_0^\eta d\beta \int_{\phi(\beta)}^{\xi} f(\alpha, \beta)\, d\alpha. \tag{3.4}$$

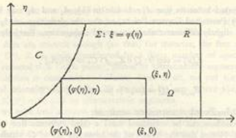

Fig. 3.

It follows (see (1.5)):

$$y_\xi(\xi,\eta) = A'(\xi) + \int_0^\eta f(\xi,\beta)\,d\beta = G(\xi,\eta)\,, \tag{3.5}$$

and in particular on Σ:

$$y_\xi(\phi(\eta),\eta) = G(\phi(\eta),\eta)\,. \tag{3.6}$$

Similarly:

$$y_\eta(\phi(\eta),\eta) = -\phi'(\eta)\,G(\phi(\eta),\eta)\,. \tag{3.7}$$

Hence:

$$y_\xi(\phi(\eta),\eta) - y_\eta(\phi(\eta),\eta) = G(\phi(\eta),\eta)\,(1+\phi'(\eta)) = \\ = G(\xi,\psi(\xi))\,(1+\psi'(\xi))/\psi'(\xi)\,. \tag{3.8}$$

The condition (1.6*a*) of [4] is easily transformed into:

$$G(\xi,\psi(\xi)) = C(1+\psi'(\xi)) \qquad \text{with } C = \epsilon F/4T\,. \tag{3.9}$$

Hence $\psi(\xi)$ is a solution of the differential equation

$$\psi'(\xi) = \{G(\xi,\psi(\xi)) - C\}/C \qquad \text{with } \psi(0) = 0\,. \tag{3.10a}$$

The condition (1.6*b*) leads to the similar equation:

$$\psi'(\xi) = G(\xi,\psi(\xi))^2/D \qquad \text{with } \psi(0) = 0 \;\; (D = \gamma/2\mu)\,. \tag{3.10b}$$

Now we can state the *first extension law* relative to the peeling problem with Darboux data:

$$y(\xi, 0) = A(\xi), \qquad 0<\xi<a,$$
$$y(0, \eta) = 0, \qquad 0<\eta<b,$$

(with $A(0) = 0$, $A(\xi) > 0$ in $(0, a]$, $A \in C^1[0, a]$).

If in some rectangle $R' = [0, a'] \times [0, b'] \subset R$ the following conditions hold:

1) there exists a solution Σ of (3.10*a*) (or (3.10*b*)), with $\psi'(\xi) > 1$ for $\xi \in [0, a']$;

2) the function y given by (3.4) is > 0 in $\Omega' = \{(\xi, \eta): 0<\eta<b', \phi(\eta) < \xi < a'\}$,

then Σ is a peeling arc and the solution of the peeling problem is given by (3.4) in Ω' and is $= 0$ in $R' \setminus \Omega'$ (see fig. 3).

Remark: If $\psi'(\xi) = 1$ in some subinterval $[\xi', \xi'']$ of $[0, a']$, the corresponding arc would be more properly designed as a rest segment (see next section). However the extension law above continues to hold.

A *second* quite obvious *extension law* is to be stated if $A(\xi) = 0$ in $[0, a]$, $B(\eta) = 0$ in $[0, b]$, in which case there is no peeling and we put $y = 0$ in R.

Notice that this law holds independently of the sign of the external force f because in the model of [4] the glue is supposed strong enough to withstand to every distributed force.

Now we give some simple conditions for the existence of a peeling arc. We note firstly that the hypotheses on f and A imply that the function $G(\xi, \eta)$ is continuous in R as well as its partial derivative $G_\eta = f(\xi, \eta)$, so that the Cauchy problem (3.10*a*) (or (3.10*b*)) has always a unique solution $\psi \in C^1([0, a''])$ for a suitable a''. If $a'' < a$, it must be $\psi(a'') = b$. Hence we have only to impose that $\psi' > 1$ and $y > 0$ in Ω.

Proposition 1: Independently of the sign of f, the conditions:

(3.11*a*) $\qquad A'(0) \geq 2C \qquad$ (resp. (3.11*b*) $A'(0) \geq \sqrt{D}$)

are necessary,

(3.12*a*) $\qquad A'(0) > 2C \qquad$ (resp. (3.12*b*) $A'(0) > \sqrt{D}$)

are sufficient for the existence of a solution of the peeling problem in a suitable rectangle $R' \subseteq R$.

Proof: By (3.5) it is $G(0, 0) = A'(0)$, hence $\psi'(0) = (A'(0) - C)/C$ (resp. $= A'(0)^2/D$). The condition $\psi'(0) \geq 1$ implies then the (3.11).

On the contrary, if (3.12*a*) holds, there exists $\delta > 0$ such that $A'(\xi) > 2C + \delta > 2C$ in some interval $[0, \xi']$; moreover $f(\xi, \eta) > -m$, so that

$$\psi'(\xi) = \{G(\xi, \psi(\xi)) - C\}/C > (2C + \delta - m\eta - C)/C = 1 + (\delta - m\eta)/C .$$

Hence for $\eta < b' = \delta/m$ $(\Rightarrow \xi < a' = \phi(\delta/m))$ it is $\psi'(\xi) > 1$. In the same interval, for $\eta < \psi(\xi)$ it is also, by (3.4):

$$y(\xi, \eta) > (2C + \delta)(\xi - \phi(\eta)) - m\eta(\xi - \phi(\eta)) = (2C + \delta - m\eta)(\xi - \phi(\eta)) > 0 .$$

In case *b*), from $A'(\xi) > \sqrt{D} + \delta$ we get again $\psi'(\xi) > 1$ and $y(\xi, \eta) > 0$ in the same region. ■

With suitable hypotheses of f and A we get the existence in the large.

PROPOSITION 2: If $f(\xi, \eta) > 0$ in R and $A'(\xi) > 2C$ (resp. $A' > \sqrt{D}$) in $[0, a]$, the solution is defined in the whole of R.

PROOF: It is indeed $G(\xi, \eta) > 2C$ (resp. $> \sqrt{D}$) in R, hence $\psi' > 1$ in its interval of definition $[0, a']$, and $y_\xi = G > 0 \Rightarrow y > 0$ in Ω. ■

In the *limit case* $A'(0) = 2C$ (resp. $\sqrt{D}$), if $A \in C^2$, we state without proof the following simple

PROPOSITION 3: Independently of the sign of f, the condition:

$$A'(0) + f(0, 0) \geq 0 \tag{3.13}$$

is necessary,

$$A'(0) + f(0, 0) > 0 \tag{3.14}$$

is sufficient for the existence of a solution of the peeling problem in a suitable rectangle $R' \subseteq R$.

REMARK: For the Darboux problem

$$y(\xi, 0) = 0, \qquad y(0, \eta) = B(\eta),$$

(that is when the glued part is on the right), we introduce the function

$$G^*(\xi, \eta) = B'(\eta) + \int_0^\xi f(\alpha, \eta)\, d\alpha .$$

Then the peeling arc Σ^*: $\xi = \psi^*(\eta) \Leftrightarrow \eta = \phi^*(\xi)$ satisfies the differential equation

$$\psi^{*\prime}(\eta) = \{G^*(\psi^*(\eta), \eta) - C\}/C , \qquad \text{with } \psi^*(0) = 0 .$$

4. - Rest segments

If the force exerted by the free part of the string is not enough to peel off the glued part (that is if

(4.1*a*) $$[y_x] < F/T$$

or

(4.1*b*) $$|[y_x^2]| < 2\gamma/T)\,,$$

and obviously if no support phenomenon takes place, the free boundary is constituted by a segment $x = \text{const}$, which we shall denote as Rest Segment.

We can state a *third extension law* relative to the peeling problem with Darboux data:

$$y(\xi, 0) = A(\xi) \qquad 0 < \xi < a\,,$$
$$y(0, \eta) = 0 \qquad 0 < \eta < b\,,$$

(with $A(0) = 0$, $A(\xi) > 0$ in $(0, a]$, $A \in C^2(0, a]$).

If in some rectangle $R' = [0, a'] \times [0, b'] \subset R$ the following conditions hold:

1) there exists a segment S: $\eta = \xi$ for $\eta \in [0, b']$ where $0 < G(\eta, \eta) < 2C$ (resp. $< \sqrt{D}$);

2) the function y given by (3.4) with $\phi(\eta) = \eta$ is > 0 in $\Omega' = \{(\xi, \eta)$: $0 < \eta < b', \eta < \xi < a'\}$,

then S is a rest segment and the solution of the peeling problem is given by (3.4) in Ω' and is $= 0$ in $R' \setminus \Omega'$.

Then we can state the following

Proposition 4: The conditions for the existence of a Rest Segment issuing from the point $(0, 0)$ in the characteristic rectangle R of the previous section (and with the same Darboux data) are

(4.2) $$0 < A'(0) < 2C \qquad (\text{resp.} < \sqrt{D}): \text{ necessary}\,,$$

and

(4.3) $$0 < A'(0) < 2C \qquad (\text{resp.} < \sqrt{D}): \text{ sufficient}\,.$$

The proof reduces simply to verify that the solution $y(\xi, \eta)$, given for $\xi < \eta$ by the (3.4) with $\xi = \phi(\eta) = \eta$ is strictly positive (at least locally). This is very similar to Prop. 1 or Prop. 2 above.

The two cases (peeling arc and rest segment) may be treated in a unified manner by introducing the function

(4.4) $H(\xi,\eta) = \max\{[G(\xi,\eta) - C]/C, 1\}$ (resp. $= \max\{G^*(\xi,\eta)/D, 1\}$)

and solving the differential equation

(4.5) $\psi' = H(\xi, \psi)$ with $\psi(0) = 0$.

Then the first and third extension laws assume the common form: If

1) (4.5) has a solution Σ: $\eta = \psi(\xi)$ $(\Leftrightarrow \xi = \phi(\eta))$ in some interval $[0, a']$,
2) $G(\xi, \psi(\xi)) > 0$,
3) the function y given by (3.4) is > 0 in $\Omega' = \{(\xi,\eta)\colon 0 < \eta < b' = \psi(a')$ $\phi(\eta) < \xi < a'\}$,

then Σ is the boundary of the contact set and the solution of the peeling problem is given by (3.4) in Ω' and is $= 0$ in $R' \setminus \Omega'$.

Condition 2) above is required in order to avoid that a support phenomenon occurs; indeed if $G(\xi', \eta') = 0$ for some point $(\xi', \eta') \in \Sigma$, then the equation $G(\xi,\eta) = 0$ may implicitly define a line Γ: $\eta = \gamma(\xi)$. If $0 < \gamma'(\xi) < 1$ in $[\xi', \xi'']$, then Γ is a support arc, as defined in [2], and the free boundary is given by Σ in $[0, \xi']$, and by Γ in $[\xi', \xi'']$.

Remark: *In contrast with* [2], if $\gamma'(\xi) > 1$, the line Γ is not to be accepted as a support arc, because the glue does not allow a soft detachment of the string. In this case the boundary is again constituted by a rest segment.

Let us now consider the Darboux problem:

$$y(\xi, 0) = A(\xi), \qquad y(0, \eta) = B(\eta),$$

with $A(0) = B(0) = 0$, $A(\xi) > 0$ for $\xi > 0$, $B(\eta) > 0$ for $\eta > 0$.

In this case, as we have already observed, even if the free solution $z(\xi,\eta) = A(\xi) + B(\eta)$ is > 0 in $R \setminus \{(0,0)\}$, the contact with the gluing wall may modify the solution, so that we can have a contact set issuing from the origin, and whose structure will be described below. If in $(0,0)$ condition (4.1) holds, that is, if the force exerted by both parts of the string is not enough to peel off this point, then it is possible that only this point adheres (for some time) to the wall, so that the contact set reduces to a segment $x = \text{const}$, which which will be denoted as a Bilateral Rest Segment (BRS).

Under suitable hypotheses on the data, it is possible to discuss the various cases which can arise. The parameters to be considered are:

$$G_0 = G(0,0) = A'(0) > 0, \qquad G_0^* = G^*(0,0) = B'(0) > 0,$$

$$S_0 = G(0,0) + G^*(0,0) > 0.$$

In the case of a BRS, S_0 is proportional to the jump of the derivative y_x across the segment, so that the condition $S_0 < 2C$ will be necessary for the existence of such BRS.

If $f(\xi, \eta) > 0$, then no support phenomenon may exist, so that the condition $S_0 < 2C$ is sufficient for the (local) existence of a BRS. On the contrary, the condition $S_0 > 2C$ implies the detachment of the string. The limit case $S_0 = 2C$ requires a further discussion.

If $f(\xi, \eta) < 0$, but $G_0 > 0$, $G_0^* > 0$, the same discussion applies. In general, however, if a support arc Γ^* (or Γ) can issue from $(0, 0)$, we must compare the relative positions of Γ^*, Σ (the possible peeling arc) and of the segment $\eta = \xi$.

The following schema summarizes the discussion:

1) $S_0 = 0 \Leftrightarrow G_0 = G_0^* = 0$:
   - 1.1) $\gamma^{*\prime}(\eta) < 1$, $\gamma'(\xi) < 1$ $\Rightarrow \exists$ two SAs Γ^* and Γ;
   - 1.2) $\gamma^{*\prime}(\eta) < 1$, $\gamma'(\xi) > 1$ $\Rightarrow \exists$ a SA Γ^* and a URS;
   - 1.3) $\gamma^{*\prime}(\eta) > 1$, $\gamma'(\xi) > 1$ $\Rightarrow \exists$ a BRS;

2) $G_0^* = 0$, $0 < G_0 < 2C$:
   - 2.1) $\gamma^{*\prime}(\eta) < 1$ $\Rightarrow \exists$ a SA Γ^* and a URS;
   - 2.2) $\gamma^{*\prime}(\eta) > 1$ $\Rightarrow \exists$ a BRS;

3) $G_0^* = 0$, $2C < G_0$:
   - 3.1) $\gamma^*(\eta) > \varphi(\eta)$ $\Rightarrow$ detachment;
   - 3.2) $\gamma^*(\eta) < \varphi(\eta)$, $\gamma^{*\prime}(\eta) < 1$ $\Rightarrow \exists$ a SA Γ^* and a PA Σ;
   - 3.3) $\gamma^*(\eta) = \varphi(\eta)$ $\Rightarrow \exists$ a SPA $\Gamma^* = \Sigma$.

(Support-Peeling arc: along Σ the string leans on one hand and is peeled off on the other, so that the contact set reduces to a moving point).

The limit case $S_0 = 2C$ requires a further investigation and is not studied here.

Examples (see corresponding figures): Let:

$$f(\xi, \eta) = -1;$$

$$A(\xi) = a\xi^2/2 + c\xi \Rightarrow A'(\xi) = a\xi + c, \qquad G(\xi, \eta) = a\xi + c - \eta;$$

$$B(\eta) = b\eta^2/2 + d\eta \Rightarrow B'(\eta) = b\eta + d, \qquad G^*(\xi, \eta) = b\eta + d - \xi;$$

$$S_0 = c + d.$$

Then:

1) $c = d = 0$:
   - 1.1) $0 < a < 1$, $0 < b < 1$ $\Rightarrow \xi = b\eta$ and $\eta = a\xi$ are SAs;

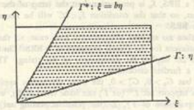

Fig. 4. - Case 1.1.

1.2) $a>1$, $0<b<1$ $\Rightarrow \xi = b\eta$ is a SA, and $\eta = \xi$ is a URS;

1.3) $a>1$, $b>1$ $\Rightarrow \eta = \xi$ is a BRS;

2) $d=0$, $0<c<2C$:

2.1) $0<b<1$ $\Rightarrow \xi = b\eta$ is a SA, and $\eta = \xi$ is a URS;

2.2) $b>1$ $\Rightarrow \eta = \xi$ is a BRS;

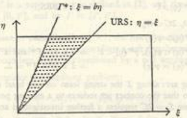

Fig. 4. - Case 1.2 = 2.1.

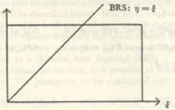

Fig. 4. - Case 1.3 = 2.2.

3) $d=0,\ a>1,\ 0<b,\ c=(a+1)C \qquad (\Rightarrow S_0=(a+1)C>2C)$.

The equation (3.10*a*) reduces to $\psi'=(a\xi+c-\psi-C)/C,\ \psi(0)=0 \Rightarrow \Sigma$ is possibly given by $\eta=a\xi$. Hence:

3.1) $a>1/b \qquad \Rightarrow$ detachment;

3.2) $a<1/b \qquad \Rightarrow \xi=b\eta$ is a SA, and $\eta=a\xi$ is a PA;

3.3) $a=1/b \qquad \Rightarrow \xi=b\eta$ is a SPA.

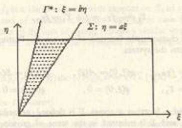

Fig. 4. - Case 3.2.

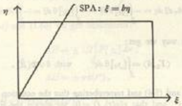

Fig. 4. - Case 3.3.

5. - Energy relations

In this section we state the formulae giving the expression of the reaction of the constraint and of the loss of energy in a characteristic rectangle, due to the peeling phenomenon, and we contrast the two approaches *a*) and *b*).

For the sake of simplicity, we consider the case $B(\eta)=0$, as in the first extension law of sect. 3. With reference to the same fig. 3, the peeling arc Σ

divides the rectangle R in two parts: the contact set C (to the left of Σ; $\Sigma \subset C$), and the set $\Omega = R \setminus C$.

The reaction $\Upsilon = \Phi \circ J/4\mu$ (see (1.2)) is a distribution, which will be expressed via its crochet $\langle \Upsilon, \theta \rangle$, with $\theta \in \mathfrak{D}(\mathring{R})$.

Υ is the sum of two contributions: a first one, Υ_1, simply contrasts the external force f in the interior of the contact set (on the left of the peeling arc Σ in our case); the second, Υ_2, is a negative distribution with support on Σ, hence a measure which can be expressed as a second mixed derivative of some continuous function $q(\xi, \eta)$.

Hence

$$\Upsilon_1 = -f\chi_C \Rightarrow f + \Upsilon_1 = f\chi_\Omega, \tag{5.1}$$

where χ_E is the characteristic function of the set E. Putting $y = z + q$, we can split (1.2) into the system

$$\left| \begin{array}{lll} z_{\xi\eta} = f\chi_\Omega, & z(\xi, 0) = A(\xi), & z(0, \eta) = B(\eta) = 0, \\ q_{\xi\eta} = \Upsilon_2, & q(\xi, 0) = 0, & q(0, \eta) = 0. \end{array} \right. \tag{5.2}$$

If the jump of a function u across Σ is defined, according with Sect. 1, as $[u] = u_\Omega - u_C$ and Σ is oriented in the sense of growing times, then an easy calculation via Green formulae gives:

$$\begin{aligned} \langle \Upsilon_2, \theta \rangle &= \langle q_{\xi\eta}, \theta \rangle = -\langle q_\xi, \theta_\eta \rangle = \\ &= -\int_{C \cup \Omega} q_\xi \theta_\eta \, d\xi \, d\eta = -\int_\Sigma [q_\xi] \theta \, d\xi = -\int_\Sigma [y_\xi] \theta \, d\xi = -\int_\Sigma G\theta \, d\xi \text{ with } \theta \in \mathfrak{D}(\mathring{R}). \end{aligned} \tag{5.3}$$

In analogous way we get:

$$\langle \Upsilon_2, \theta \rangle = \int_\Sigma [y_\eta] \theta \, d\eta \qquad \text{with } \theta \in \mathfrak{D}(\mathring{R}). \tag{5.4}$$

Adding (5.3) and (5.4) and remembering that the equation of the PA Σ is: $x = \sigma(t)$, $0 < t < t^*$, and that $y(\sigma(t), t) = 0$, we obtain the following expression:

$$\begin{aligned} \langle \Upsilon_2, \theta \rangle &= \int_\Sigma \theta \{ [y_\eta] \, d\eta - [y_\xi] \, d\xi \}/2 = -\int_\Sigma \theta \{ \mu [y_t] \, dx + T[y_x] \, dt \}/(2\mu c) = \\ &= -\int_\Sigma \theta \{ \mu [y_t] \sigma'(t) + T[y_x] \} \, dt/(2\mu c) = -\int_\Sigma \theta \{ T - \mu \sigma'^2(t) \} [y_x] \, dt/(2\mu c). \end{aligned} \tag{5.5}$$

Hence in (x, t) coordinates the corresponding reaction $J_2 = 4\mu \; \Phi^{-1} \circ \Upsilon_2$ is given by

$$\langle J_2, \theta \rangle = -\int_\Sigma \theta \{ T - \mu \sigma'^2(t) \} [y_x] \, dt.$$

If the model corresponds to choice (1.6*a*), we get at once:

$$\langle J_2, \theta\rangle = -\int_\Sigma \theta F\, dt\,; \tag{5.6a}$$

in the case (1.6*b*), on the contrary, we obtain:

$$\langle J_2, \theta\rangle = -\int_\Sigma \theta 2\gamma/[y_x]\, dt\,.$$

In both cases J_2 is a Dirac measure with support on Σ, of constant strength $-F$ in case *a*), and of variable strength $-2\gamma/[y_x]$ in case *b*).

The computation of the loss of energy follows a similar way. We remember that the energy of the string may be expressed by the integral of the differential form

$$dE = (1/2)(\mu y_x^2 + Ty_t^2)\, dx + Ty_x y_t\, dt = \sqrt{T\mu}\,(y_\xi^2\, d\xi - y_\eta^2\, d\eta) \tag{5.7}$$

(usually on a line $t = \text{const}$).

Then along the peeling arc Σ: $x = \sigma(t)$, $0 < t < t^*$, a simple application of Green formulae gives the loss of energy due to the reaction J_2 (concentrated on Σ):

$$\Delta E = \int_\Sigma \{(1/2)(T - \mu\sigma'^2(t))[y_x^2]\sigma'(t)\}\, dt = \int_\Sigma \{(1/2)(T - \mu\sigma'^2(t))[y_x^2]\}\, dx < 0\,. \tag{5.8}$$

In the cases (1.6*a*) and (1.6*b*) we get henceforth:

$$\Delta E = \pm 1/2 \int_\Sigma F[y_x]\, dx\,, \tag{5.9a}$$

$$\Delta E = \pm \gamma\sigma(t^*)\,, \tag{5.9b}$$

where $\pm = \operatorname{sgn}\{[y_x^2]\} = -\operatorname{sgn}\{\sigma'\}$.

Let us now contrast the two approaches *a*) and *b*) to the problem, with the condition

$$D = 4C^2 \tag{5.10}$$

so that the (3.11*a*) and (3.11*b*) agree. If (5.10) holds, the peeling begins at the same time in both models.

If ψ_a and ψ_b satisfy (3.10*a*) and (3.10*b*) respectively, by subtracting we get at once:

$$\psi_b' - \psi_a' = \frac{G^2(\xi, \psi_b)}{D} - \frac{G(\xi, \psi_a) - C}{C}\,. \tag{5.11}$$

Now we suppose $\psi_b(\xi) = \psi_a(\xi)$ at some point ξ^* (in particular this is true for $\xi^* = 0$). Then (5.10) and (5.11) imply:

$$\psi_b'(\xi^*) - \psi_a'(\xi^*) = \frac{(G(\xi^*, \psi_a(\xi^*)) - 2C)^2}{4C^2} > 0 . \tag{5.12}$$

Then, if (3.12) hold, in a right neighborhood of 0 it is $\psi_b(\xi) > \psi_a(\xi)$. Moreover, if the PAs Σ_a and Σ_b cross in some other point ξ^*, it must be $\psi_b'(\xi^*) < \psi_a'(\xi^*)$; hence, by (5.12), $\psi_b'(\xi^*) = \psi_a'(\xi^*)$ and $G(\xi^*, \psi_a(\xi^*)) = 2C$, and finally $\psi_b'(\xi^*) = \psi_a'(\xi^*) = 1$.

Let us now study the case $\psi_b'(\xi^*) = \psi_a'(\xi^*) = 1$, with $\xi^* = 0$ for the sake of simplicity, supposing the data $A(\xi)$ and $f(\xi, \eta)$ regular enough, and that the sufficient condition (3.14) holds. A straightforward calculation shows that

$$\psi_a''(0) = \psi_b''(0) = \frac{A'(0) + f(0,0)}{C} > 0 ,$$

but

$$\psi_b'''(0) - \psi_a'''(0) = \frac{(A'(0) + f(0,0))^2}{2C^2} > 0 .$$

Hence $\psi_b(\xi) > \psi_a(\xi)$ again in a right neighborhood of 0.

We can conclude by observing that, under the hypothesis (5.12), the model *b*) «peels off» the string more rapidly than model *a*), at least locally.

6. - Examples

In this section we give firstly an example, where a complete discussion with respect to parameters is carried out; then we provide some cases, which show the possible occurrence of infinitely many Peeling or/and Support arcs in a finite time. In both cases we adopt model *a*).

Example 1: Let $y(x, t)$ satisfy the initial-boundary value conditions:

$$\begin{aligned} y(x, 0) &= b + 1 - 2|x| , \qquad -1/2 < x < 1/2 , \\ y_t(x, 0) &= 0 , \\ y(-1/2, t) &= y(1/2, t) = b , \end{aligned}$$

with $0 < b < 1$ (symmetric pickled string). Then the following cases can arise.

1) $b = 0$.

At $t = 1/2$ the whole string hits the wall, then it adheres forever, independently of the value of F.

2) $b = 1$.

At $t = 1$ the middle point of the string touches the wall. Then

2.1) If $F > 4$ a BRS arises, which does never terminate: the string remains at rest, and for $t > 1$ the solution is $y(x, t) = 2|x|$.

2.2) If $F < 4$ this point is an isolated contact point; the motion of the string is periodic, and the solution agrees with the free one.

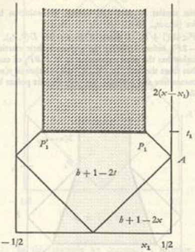

Fig. 5. - Case 3.1.

3) $0 < b < 1$.

At $t = t_1 = (1 + b)/2$ an impact takes place along the segment $|x| < x_1 = (1 - b)/2$.

3.1) $F > 2$. From $P_1(x_1, t_1)$ and $P_1'(-x_1, t_1)$ two endless URS start, dividing the string in three parts: the central one, $|x| < x_1$, adheres permanently, whereas the side portions remain at rest, the solution being $y(x, t) = 2(|x| - x_1)$ for $|x| > x_1$, $t > t_1$.

3.2) $F < 2$. We obtain a PA Σ starting from P_1 (and symmetrically Σ' from P'), whose equation is $x = x_1 + (F/2 - 1)(t - t_1)$. The solution in AP_1B is again $y(x, t) = 2(|x| - x_1)$.

3.2.1) $4b/(1 + b) < F < 2$. The ξ-characteristic issuing from P_1 is reflected by the boundary $x = 1/2$ in a point B $(1/2, 1/2 + b)$. Σ crosses the

η-characteristic starting from B in a point $P_2(x_2, t_2)$, with

$$x_2 = (F + bF - 4b)/2F > 0\,, \qquad t_2 = (F + bF + 4b)/2F\,.$$

In the triangle P_1BP_2 the solution is

$$y(x, t) = [(4 - 2F)t + 4x - 4 + F + bF]/(4 - F)\,.$$

In P_2 a situation similar to case 3.1) occurs, the solution being $y(x, t) = 2(|x| - x_2)$ for $|x| > x_2$, $t > t_2$.

3.2.2) $F < 4b/(1 + b)$. Σ meets Σ' in a point $D(0, t_D)$, with $t_D = 1 + F(1 - b)/(4 - 2F)$, before crossing the η-characteristic starting from B. In P_1BCD the solution has the same expression as in P_1BP_2 of case 3.2.1). In D the string detaches from the wall; in $DCEC'$ the solution is $y(x, t) = 2(t - t_D)$. The motion becomes free and periodic, the only contact points being $D_n(0, t_n)$, with $t_n = t_D + 2n$.

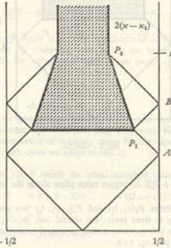

Fig. 5. - Case 3.2.1.

Remark: This example leads to a conjecture: if a string is subject to no external force, then there exists a time T after which either some internal segment (possibly reduced to a point) of the string remains permanently glued to the wall, whereas the side parts vibrate freely, or the whole string does no longer touch the wall, except for isolated points (see sect. 4).

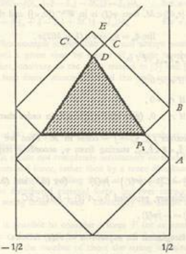

Fig. 5 - Case 3.2.2.

Example 2: In this example we construct a solution, for which the boundary of the contact set is an « a priori » chosen line $\eta = \psi(\xi)$ formed by an infinite number of Peeling and/or Support arcs, joined by Rest Segments. Let us consider in a characteristic rectangle $R = [0, a] \times [0, b]$ a sequence of points $P_n = (\bar{\xi}_n, \bar{\eta}_n)$, with $\bar{\xi}_n \to a$, $\bar{\eta}_n \to b$, chosen as follows.

Put $\delta_n = a/2^{n+1}$, $\bar{\xi}_0 = 0$, $\bar{\eta}_0 = 0$, and by recursion:

$$\begin{aligned}
\bar{\xi}_{n+1} &= \bar{\xi}_n + \delta_n \,, \\
\xi_n &= \xi - \bar{\xi}_n \qquad (0 < \xi_n < \delta_n) \,, \\
\psi_n(\xi_n) &= \xi_n - (3/2) r_n \delta_n \xi_n^2 + r_n \xi_n^3 \,, \qquad \Rightarrow \psi_n'(0) = \psi_n'(\delta_n) = 1 \,, \\
\bar{\eta}_{n+1} &= \bar{\eta}_n + \psi_n(\delta_n) = \bar{\eta}_n + \delta_n - (1/2) r_n \delta_n^3 \,,
\end{aligned}$$

where r_n are arbitrary provided that $\sum_{n=0}^{\infty} r_n \delta_n^3$ converges.

Then a C^1 line l: $\eta = \psi(\xi)$ is defined in $[0, a]$ by

$$\eta = \bar{\eta}_n + \psi_n(\xi - \bar{\xi}_n), \qquad \bar{\xi}_n < \xi < \bar{\xi}_{n+1}.$$

If moreover $|r_n \delta_n^2| < M$, then $\psi(\xi)$ is in $W^{2,\infty}([0, a])$ and there exists

$$\lim \bar{\eta}_n = \bar{\eta}_\infty = b < (1 + M/2)a.$$

A single arc $P_n P_{n+1}$ of l is

(6.1*a*) a PA if $r_n < 0$,

(6.1*b*) a RS if $r_n = 0$,

(6.1*c*) a SA if $r_n > 0$, (provided $r_n \delta_n^2 < 4/3$, in order that $\psi'(\xi) > 0$).

Suppose moreover $f(\xi, \eta) = \bar{f} = \text{const}$ in R. Then we can choose the datum $A(\xi)$ in $\bar{\xi}_n < \xi < \bar{\xi}_{n+1}$, starting from ψ, according respectively to the formulae:

(6.2*a*) $A'(\xi) = C[1 + \psi'(\xi)] - \bar{f}\psi(\xi)$ (by (3.5) and (3.9)),

(6.2*b*) A' arbitrary, provided $0 < A'(\xi) + \bar{f}\psi(\xi) < 2C$,

(6.2*c*) $A'(\xi) = -\bar{f}\psi(\xi)$,

from which by integration the expression of $A(\xi)$ follows.

Case 2.1 (PAs joined by RSs). If $a = 1$, and

$$r_n = \begin{cases} -2^{n+2} & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}, \end{cases}$$

we can choose $A'(\xi)$ on the RSs (n odd) in a very arbitrary manner. For instance, if we put in $\bar{\xi}_n < \xi < \bar{\xi}_{n+1}$ (n odd):

$$A'(\xi) = 2C - \bar{f}(\bar{\eta}_n + \xi - \bar{\xi}_n) + s_n(\xi - \bar{\xi}_n)(\xi - \bar{\xi}_{n+1}),$$

we obtain a function $A'(\xi) \in C^0([0, 1])$. The constant s_n must only satisfy the inequality $0 < s_n < 8C/\delta_n^2$ in order that (6.2*b*) holds.

Case 2.2 (PAs and SAs alternately joined by RSs). If $a = 1$, $\bar{f} < 0$, and

$$r_n = \begin{cases} -2^{n+2} & \text{for } n = 4k, \\ 0 & \text{for } n \text{ odd} \\ 2^{n+2} & \text{for } n = 4k + 2 \end{cases}$$

condition (6.1c) is satisfied, and we can simply choose, in order to get continuity of $A'(\xi)$:

$$A'(\xi) = 2C - \gamma(\bar{\eta}_n + \xi - \xi_n) - 2C(\xi - \xi_n)/\delta_n \qquad \text{for } n = 4k+1,$$

$$A'(\xi) = -\gamma(\bar{\eta}_n + \xi - \xi_n) + 2C(\xi - \xi_n)/\delta_n \qquad \text{for } n = 4k+3.$$

Remark: This example shows that it is not always possible to construct the solution until a given time by means of a finite number of elementary problems. In fact, contrary to the cases treated in [3] and [1], but similarly to case in [2], here forces directed toward the wall are present.

7. - Final remarks

The model of [4] we presented in this paper is subject to some criticism for possible refinements.

First of all, it seems not completely satisfactory to model the effect of the glue by a concentrated force, rather then by a force distributed in some small segment. In fact, only the peeling from an endpoint of the glued part is allowed, and no distributed external force, however strong, could detach a segment of the string from the wall. Obviously this model results in a great simplification.

Secondly, it is possible to consider a force F (or an energy γ) depending on x and/or on t: for example, the efficiency of the glue may decrease with time or by effect of the number of times the string has already been peeled off. This would not give rise to an essentially different model, and the calculations could be carried out in a very similar manner, provided $F(x, t)$ is a smooth enough given positive function. If $F(x, t)$ vanishes somewhere, only the impact model of [3] applies, with the inelastic impact law already considered in [6].

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