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On Generalized Morita Bimodules and their Dualities (**)(***)

SUMMARY. — In a recent paper J. Kraemer has considered faithfully balanced bimodules which are quasi-injective and finitely cogenerated on both sides and the related concept of modules with a quasi-duality, in an algebraic setting. In this paper we give a topological characterization of these modules and consequently we prove that a ring R has a quasi-duality if and only if R is linearly compact and semiperfect. Related questions are studied.

Sui moduli di Morita generalizzati e le loro dualità

SUNTO. — In un lavoro recente J. Kraemer ha considerato i bimoduli fedelmente bilanciati, quasi-iniettivi e finitamente generati da ambo le parti, nonchè l'associato concetto di moduli aventi una quasi-dualità, in un contesto algebrico. In questo lavoro noi diamo una caratterizzazione topologica di questi bimoduli e di conseguenza troviamo che un anello R ha una quasi-dualità se e solo se R ha una topologia linearmente compatta ed è semiperfetto. Questioni collegate alle precedenti sono studiate.

1. - INTRODUCTION AND NOTATIONS

Let A and R be two rings with a non-zero identity. We call a bimodule ${}_RK_A$ a *Morita bimodule* if it is faithfully balanced and both ${}_RK$ and K_A are injective cogenerators. This terminology is motivated by the following well known result of Morita [Mo] (see also [AF, Theorem 24.1]): The bimodule ${}_RK_A$ has the above properties if and only if both ${}_RR$ and A_A are K-reflexive and every submodule and every factor module of a K-reflexive module is K-reflexive.

A ring R has a *left Morita duality* if there exists a ring A and a Morita bimodule ${}_RK_A$. The structure of rings having a Morita duality is not very clear; nevertheless a number of remarcable results was obtained in this topic.

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First of all B. Osofsky [O] proved that a ring having a Morita duality must be semiperfect. Subsequently B. Müller [Mü$_1$] pointed out a deep relation between Morita duality and linear compactness, by showing the following facts:

a) A ring R has a left Morita duality if and only if both $_RR$ and the minimal cogenerator $_RU$ of R-Mod are linearly compact in the discrete topology.

b) If $_RK_A$ is a Morita bimodule, then the K-reflexive modules are exactly the linearly compact discrete modules.

Moreover Müller obtained in [Mü$_2$] the following result:

c) Let A and R be two (discrete) rings and let $\mathfrak{B}_A$ and $_R\mathfrak{B}$ be two categories of linearly topologized Hausdorff right A-modules and left R-modules respectively. If both $\mathfrak{B}_A$ and $_R\mathfrak{B}$ contain all discrete modules, then every duality between $\mathfrak{B}_A$ and $_R\mathfrak{B}$ is performed by a Morita bimodule $_RK_A$ via the functors $\mathrm{Chom}_A(-, K)$ and $\mathrm{Chom}_R(-, K)$, up to equivalent topologies.

In a very recent paper [K], J. Kraemer gave a generalization of the concept of a Morita bimodule; he considered bimodules $_RK_A$ which are faithfully balanced with both K_A and $_RK$ quasi-injective and finitely cogenerated. We call such a bimodule a *Generalized Morita bimodule* (GM-bimodule for short). Following Kraemer we say that the ring R has a *left quasi-duality* if there is a ring A and a GM-bimodule $_RK_A$. Using GM-bimodules Kraemer obtained a number of interesting results concerning rings having a quasi-duality, with applications to tensor rings.

The present work is a topological investigation on rings having a quasi-duality. Sections 2 and 3 deal with some preliminary facts. In Section 4 we give our first main result. Let (A, σ) and (R, τ) be respectively a right and a left linearly topologized Hausdorff ring and let LT-A_σ (resp. R_τ-LT) be the category of all linearly topologized right A-modules (resp. left R-modules) over the topological ring (A, σ) (resp. (R, τ)) and continuous module homomorphisms. Assume that two subcategories $\mathfrak{B}_A \subset$ LT-A_σ, $_R\mathfrak{B} \subset R_\tau$-LT are given such that:

1) $(A, \sigma) \in \mathfrak{B}_A$ and $(R, \tau) \in {_R\mathfrak{B}}$;

2) $\mathfrak{B}_A$ contains all simple right A-modules and those finitely cogenerated right A-modules, each element of which is annihilated by a σ-open right ideal of A, and the same properties for $_R\mathfrak{B}$;

3) A duality $H = (H_1 : \mathfrak{B}_A \to {_R\mathfrak{B}},\ H_2 : {_R\mathfrak{B}} \to \mathfrak{B}_A)$ is given.

Then we prove that there is a GM-bimodule $_RK_A$ together with natural equivalences $H_1 \simeq \mathrm{Chom}_A(-, K)$ and $H_2 \simeq \mathrm{Chom}_R(-, K)$ (up to equivalent topologies); moreover the topological rings (A, σ) and (R, τ) are automatically linearly compact. Conversely, every GM-bimodule gives rise to a duality of the kind just described.

If in the above situation the topologies σ and τ are discrete, then ${}_RK_A$ turns out to be a Morita bimodule and we get the above quoted result *c*) of Müller in a more general form.

Section 5 deals with K-reflexive modules, where ${}_RK_A$ is a GM-bimodule. We show that a module M is K-reflexive if and only if M is complete and Hausdorff in its K-topology. If K_A is either injective or a cogenerator, then it turns out that K_A is an injective cogenerator of Mod-A. In this case a right A-module is reflexive if and only if it is complete in the cofinite topology, while a left R-module is K-reflexive if and only if it is linearly compact in the K-topology. Thus, if ${}_RK_A$ is a Morita bimodule, we get result *b*) of Müller as a particular case.

In section 6 we give the second main result of our work. We prove that a ring R has a left quasi-duality if and only if R satisfies the following two conditions:

1) R is semiperfect,

2) R has a left linear topology τ such that (R, τ) is linearly compact.

By means of an example we show that such a ring need not be linearly compact in the discrete topology. Müller's result *a*) follows easily as a corollary.

Finally we prove that if a commutative ring R has a quasi-duality, then R has a quasi-duality with itself. This generalize Müller's result concerning commutative rings with a Morita duality (see [Mü$_1$, Theorem 3]).

We conclude this introduction with some words concerning notations and terminology. All ring considered are associative with a non-zero identity and all modules are unital. Given a ring R, Mod-R (resp. R-Mod) will be the category of all right (resp. left) R-modules. Morphisms between modules will be vritten on the side opposite to the scalars. Categories and functors are understood to be additive and all subcategories are full and closed by isomorphic objects. All ring and module topologies will be linear and Hausdorff. If (R, τ) is a right (resp. left) topologized ring, then LT-R_τ (resp. R_τ-LT) will denote the category of Hausdorff linearly topologized right (resp. left) modules over (R, τ) and continuous R-homomorphisms. If $L, M \in R_\tau$-LT, then $\mathrm{Chom}_R(L, M)$ will denote the group of continuous R-homomorphisms from L into M. We say that $\mathrm{Chom}_R(L, M)$ *separates points* of L if for each $x \in L$ with $x \neq 0$, there exists $f \in \mathrm{Chom}_R(L, M)$ such that $f(x) \neq 0$. By a topological submodule of a given topological module we will always mean a submodule equipped with the induced topology, while with the term « topological isomorphism » we shall mean a module isomorphism which is an homeomorphism. Whenever speaking of a ring or a module without specifying any topology over them, we shall intend they are equipped with the discrete topology. Given a left R-module M, if $x \in M$ and $r \in R$, then $\mathrm{Ann}_R(x)$ (resp. $\mathrm{Ann}_M(r)$) will denote the annihilator of x in R (resp. of r in M). For all other undefined module-theoretical terms and notations we shall refer to [AF].

2. - Preliminaries on dualities between categories of topological modules

In this section we list some results (taken from [MO_2]) concerning dualities between categories of topological modules, which will be extensively used in the sequel.

2.1. Throughout, (A, σ) and (R, τ) are fixed Hausdorff respectively right and left linearly topologized rings. By a *topological bimodule* we mean a bimodule ${}_RK_A$ endowed with two topologies χ_A and ${}_R\chi$ such that: (K_A, χ_A) and $({}_RK, {}_R\chi)$ are topological modules over the topological rings (A, σ) and (R, τ), respectively, and the left (resp. right) multiplications by elements of R (resp. A) are continuous endomorphisms of K_A (resp. ${}_RK$).

Assume now that ${}_RK_A$ is a topological bimodule, faithfully balanced as an (R, A)-bimodule. Given a topological module $M = (M, \varepsilon) \in \text{LT-}A_\sigma$, we denote by $\bar{\varepsilon}$ the weak topology of $\text{Chom}_A(M, K)$ and write $\bar{M} = (M, \bar{\varepsilon})$. Then M is called *K-completely regular* if M is topologically isomorphic to a submodule of a topological product K_A^X, for some set X, with the induced topology. It is not difficult to see that this is the case exactly when $\text{Chom}_A(M, K)$ separates points of M and $\varepsilon = \bar{\varepsilon}$. We denote with $\mathfrak{B}(K_A)$ (resp. $\mathfrak{B}({}_RK)$) the subcategory of $\text{LT-}A_\sigma$ (resp. $R_\tau\text{-LT}$) of all K-completely regular topological modules.

2.2. Given $M \in \text{LT-}A_\sigma$, let us denote with M^* the left R-module $\text{Chom}_A(M, K)$ equipped with the topology of pointwise convergence, that is the topology induced by the inclusion of $\text{Chom}_A(M, K)$ as an R-submodule of the topological product ${}_RK^M$. It is clear that $M^* \in \mathfrak{B}({}_RK)$ and the assignment $M \mapsto M^*$ defines a contravariant functor from $\text{LT-}A_\sigma$ to $\mathfrak{B}({}_RK)$. In a similar way we define the contravariant functor $N \mapsto N^*$ from $R_\tau\text{-LT}$ to $\mathfrak{B}(K_A)$.

Given $M \in \text{LT-}A_\sigma$ and $x \in M$, let $\bar{\bar{x}}$ denote the restriction to M^* of the map $f \mapsto f(x)$ from $\text{Hom}_A(M, K)$ to K. Then $\bar{\bar{x}} \in \text{Chom}_R(M^*, K)$, since $\bar{\bar{x}}$ is the restriction to M^* of the x-th projection of ${}_RK^M$ onto ${}_RK$. We may then define the canonical morphism $\omega_M \colon M \to M^{**}$ by setting $\omega_M(x) = \bar{\bar{x}}$ for all $x \in M$. It turns out that ω_M is natural in M. Of course we may define the appropriate canonical morphisms $\omega_N \colon N \to N^{**}$ for each $N \in R_\tau\text{-LT}$.

Throughout the present paper $D_1 \colon \mathfrak{B}(K_A) \to \mathfrak{B}({}_RK)$ and $D_2 \colon \mathfrak{B}({}_RK) \to \to \mathfrak{B}(K_A)$ will be the contravariant functors defined by $D_1(M) = M^*$ and $D_2(N) = N^*$ for each $M \in \mathfrak{B}(K_A)$ and $N \in \mathfrak{B}({}_RK)$. We shall be often concerned with the case in which the pair $D_K = (D_1, D_2)$ is a duality, that is ω_M and ω_N are topological isomorphisms for all $M \in \mathfrak{B}(K_A)$ and $N \in \mathfrak{B}({}_RK)$ (see Theorem 2.7 below).

2.3. For the remaining part of this section we assume that two subcategories $\mathfrak{B}_A$, ${}_R\mathfrak{B}$ of $\text{LT-}A_\sigma$ and $R_\tau\text{-LT}$ are given satisfying the following conditions:

a) $(A, \sigma) \in \mathfrak{B}_A$ and $(R, \tau) \in {}_R\mathfrak{B}$;

b) A duality $H = (H_1 \colon \mathfrak{B}_A \to {}_R\mathfrak{B},\ H_2 \colon {}_R\mathfrak{B} \to \mathfrak{B}_A)$ is given.

Then, according to [MO$_2$, Section 1], there is a faithfully balanced topological bimodule ${}_RK_A$ such that:

1) ${}_RK \simeq H_1(A)$ and $K_A \simeq H_2(R)$ as objects of ${}_R\mathfrak{B}$ and $\mathfrak{B}_A$ respectively;

2) For every $M \in \mathfrak{B}_A$ (resp. $N \in {}_R\mathfrak{B}$) there is an R-module (resp. A-module) isomorphism

$$H_1(M) \simeq \mathrm{Chom}_A(M, K) \qquad (\text{resp. } H_2(N) \simeq \mathrm{Chom}_R(N, K))$$

natural in M (resp. in N).

The above topological bimodule K turns out to be uniquely determined, up to topological isomorphisms of topological bimodules, by the duality H; it will be called the *topological bimodule associated to* H.

2.4. Proposition [MO$_2$, Proposition 2.8]: *If* $M \in \mathfrak{B}_A$ *and* $N \in {}_R\mathfrak{B}$, *then the following properties hold*:

a) *The canonical morphisms* ω_M *and* ω_N *are continuous (module) isomorphisms*;

b) ω_M *is a topological isomorphism if and only if* $M = \overline{M}$. *Similarly for* ω_N. ///

2.5. It follows from the last Proposition that if $M \in \mathfrak{B}_A$, then $\overline{M} \in \mathfrak{B}(K_A)$. It is easily seen that the assignment $M \mapsto \overline{M}$ defines a covariant functor $T_A \colon \mathfrak{B}_A \to \mathfrak{B}(K_A)$ wich leaves unchanged the morphisms; similarly we define the functor ${}_RT \colon {}_R\mathfrak{B} \to \mathfrak{B}({}_RK)$. Let us write $\overline{\mathfrak{B}}_A = T_A(\mathfrak{B}_A)$ and ${}_R\overline{\mathfrak{B}} = {}_RT({}_R\mathfrak{B})$; it is clear that $(K_A, \chi_A) \in \overline{\mathfrak{B}}_A$ and $({}_RK, {}_R\chi) \in {}_R\overline{\mathfrak{B}}$. If $\overline{D}_1$ and $\overline{D}_2$ denote the restrictions of D_1 and D_2 to $\overline{\mathfrak{B}}_A$ and ${}_R\overline{\mathfrak{B}}$ respectively, then we have the following result.

2.6. Theorem [MO$_2$, Theorem 2.13]: *With the above notations the following properties hold*:

1) $D_1(\overline{\mathfrak{B}}_A) \subseteq {}_R\overline{\mathfrak{B}}$ *and* $D_2({}_R\overline{\mathfrak{B}}) \subset \overline{\mathfrak{B}}_A$;

2) *The diagram of functors and categories*

$$\text{(1)} \qquad \begin{tikzcd} \mathfrak{B}_A \arrow[r, shift left, "D_1"] \arrow[d, "T_A"'] & {}_R\mathfrak{B} \arrow[l, shift left, "D_2"] \arrow[d, "{}_RT"] \\ \overline{\mathfrak{B}}_A \arrow[r, shift left, "\overline{D}_1"] & {}_R\overline{\mathfrak{B}} \arrow[l, shift left, "\overline{D}_2"] \end{tikzcd}$$

is commutative;

3) $(\overline{D}_1, \overline{D}_2)$ *defines a duality between* $\overline{\mathfrak{B}}_A$ *and* ${}_R\overline{\mathfrak{B}}$. ///

Finally we recall the following theorem.

2.7. Theorem [MO_2, Theorem 5.3]: *Let* A, R *be (discrete) rings and let* ${}_RK_A$ *be a (discrete) bimodule, faithful on both sides. Then the pair* $D_K = (D_1, D_2)$ *is a duality between* $\mathfrak{B}(K_A)$ *and* $\mathfrak{B}({}_RK)$ *if and only if* ${}_RK_A$ *is faithfully balanced and both* K_A *and* ${}_RK$ *are quasi-injective.* ///

3. - Generalized Morita bimodules

3.1. Given a left linearly topologized ring (R, τ), we denote by $\mathcal{F}_\tau$ the filter of all open left ideals of R and we set

$$\mathfrak{T}_\tau = \{{}_RM\colon \mathrm{Ann}_R(x) \in \mathcal{F}_\tau \text{ for all } x \in M\}.$$

It is well known that $\mathfrak{T}_\tau$ is closed by submodules, homomorphic images and arbitrary direct sums, that is $\mathfrak{T}_\tau$ is a hereditary pretorsion class with associated left exact preradical t_τ defined by

$$t_\tau(M) = \{x \in M\colon \mathrm{Ann}_R(x) \in \mathcal{F}_\tau\}$$

for every left R-module M. We observe that, given a module ${}_RM$ with discrete topology δ, then $(M, \delta) \in R_\tau\text{-LT}$ if and only if $M \in \mathfrak{T}_\tau$.

The category $\mathfrak{T}_\tau$ is a Grothendiek category; if we set $E_\tau(M) = t_\tau(E(M))$, then for every $M \in \mathfrak{T}_\tau$ $E_\tau(M)$ is an injective envelope of M in $\mathfrak{T}_\tau$.

The proof of the following result is left to the reader.

3.2. Proposition: *If* ${}_RK \in \mathfrak{T}_\tau$, *then the following conditions are equivalent*:

1) K *is an injective object in* $\mathfrak{T}_\tau$.

2) *For every* $M \in R_\tau\text{-LT}$ *and (open) topological submodule* M' *of* M, *every continuous morphism from* M' *into* K *extends to a continuous morphism from* M *into* K.

3) *For every* $I \in \mathcal{F}_\tau$, *every continuous morphism from* I *(endowed with the topology induced by* τ*) into* K *extends to a continuous morphism from* R *into* K. ///

3.3. Let K be a given left R-module. The *K-topology* on a module ${}_RM$ is the topology for which the finite intersections of kernels of morphisms $M \to K$ form a basis of neighbourhoods of zero, that is the weak topology of $\mathrm{Hom}_R(M, K)$. It is easily seen that the K-topology w on ${}_RR$ is a ring topology and M, endowed with the K-topology, is a topological module over the topological ring (R, w). Note that the K-topology on M is Hausdorff if and only if K cogenerates M; in particular (R, w) is Hausdorff if and only if

${}_R K$ is faithful. It is clear that $K \in \mathfrak{C}_w$ and it follows immediately from Proposition 3.2 that ${}_R K$ is quasi-injective if and only if K is an injective object in $\mathfrak{C}_w$.

3.4. A module ${}_R K$ is *strongly quasi-injective* if for every $B <_R K$ and $x \in K \setminus B$, every homomorphism $f: B \to K$ extends to an endomorphism g of ${}_R K$ with $xg \neq 0$. K is a *selfcogenerator* if for every $n \in N$, $B <_R K^n$ and $x \in K^n \setminus B$ there exists $f \in \mathrm{Hom}_R(K^n, K)$ such that $Bf = 0$ and $xf \neq 0$. We recall the following result from [MO$_1$, Corollary 4.5 and Theorem 6.7].

3.5. Proposition: *Given a left R-module K, let w be the K-topology on R. Then the following conditions are equivalent:*

1) *K is strongly quasi-injective.*

2) *K is a quasi-injective selfcogenerator.*

3) *K is an injective cogenerator of the category $\mathfrak{C}_w$.*

4) *K is quasi-injective and contains a copy of every simple left R-module belonging to $\mathfrak{C}_w$.* ///

If τ is any left linear topology on R with $K \in \mathfrak{C}_\tau$, then clearly $T_w \subseteq T_\tau$, that is $w \leq \tau$. Consequently we have the following corollary.

3.6. Corollary: *Let (R, τ) be a left linearly topologized ring and assume that ${}_R K \in \mathfrak{C}_\tau$. Then the following properties hold:*

a) If ${}_R K$ is injective in $\mathfrak{C}_\tau$, then ${}_R K$ is quasi-injective.

b) If ${}_R K$ is an injective cogenerator in $\mathfrak{C}_\tau$, then ${}_R K$ is strongly quasi-injective. ///

We shall need the following two criteria of linear compactness.

3.7. Theorem ([M, Main Theorem]): *Let (R, τ) be a left linearly topologized ring, ${}_R K$ a cogenerator of $\mathfrak{C}_\tau$ and $A = \mathrm{End}({}_R K)$. Then (R, τ) is linearly compact if and only if ${}_R K_A$ is faithfully balanced and K_A is quasi-injective.* ///

3.8. Theorem ([MO$_1$, Theorem 9.4]): *Let ${}_R K$ be a selfcogenerator and set $A = \mathrm{End}({}_R K)$. Then ${}_R K$ is linearly compact discrete if and only if K_A is injective.* ///

3.9. Lemma: *Let (R, τ) be a left linearly topologized ring, let ${}_R K \in \mathfrak{C}_\tau$ and set $A = \mathrm{End}({}_R K)$. Assume that:*

1) *K_A contains a representative of each simple right A-module;*

2) *$\mathrm{Hom}_R(E_\tau(K), K)$ separates points of $E_\tau(K)$.*

Then $K = E_\tau(K)$.

PROOF: Set $E = E_1(K)$, let α: $\mathrm{Hom}_R(E, K) \to \mathrm{Hom}_R(K, K)$ be the restriction morphism and let us write $I = \mathrm{Im}(\alpha)$. We claim that $I = A$. Indeed, if I were a proper right ideal, then there would be a maximal right ideal P of A which contains I. Since A/P maps non-trivially into K_A, there is $x \in K$, $x \neq 0$, such that $xP = 0$, hence $xI = 0$. Thus we would get $xf = x\alpha(f) \in xI = 0$ for all $f \in \mathrm{Hom}_R(E, K)$, in contradiction with 2). Now there is $f \in \mathrm{Hom}_R(E, K)$ such that $\alpha(f) = 1_K$, that is K is a direct summand of E. We conclude $K = E$. ///

3.10. COROLLARY: *If ${}_RK_A$ is faithfully balanced and both K_A and ${}_RK$ are cogenerators, then K_A and ${}_RK$ are injective.* ///

3.11. DEFINITION: Let ${}_RK_A$ be a faithfully balanced bimodule.

a) We say that K is a *Morita bimodule* if both ${}_RK$ and K_A are injective cogenerators.

b) We say that K is a *Generalized Morita bimodule* (GM-bimodule for short) if both ${}_RK$ and K_A are quasi-injective and finitely cogenerated.

3.12. LEMMA ([MO$_1$, Proposition 6.10]): *Let ${}_RK_A$ be a faithfully balanced bimodule with both ${}_RK$ and K_A strongly quasi-injective. Then* $\mathrm{Soc}({}_RK) = \mathrm{Soc}(K_A)$ *and both are essential in ${}_RK$ and K_A.* ///

3.13. PROPOSITION: *Let ${}_RK_A$ be a Morita bimodule. Then:*

a) *The modules K_A, A_A, ${}_RK$, ${}_RR$ are linearly compact discrete.*

b) $\mathrm{Soc}(K_A) = \mathrm{Soc}({}_RK)$ *and both are essential.*

PROOF: It is a consequence of Theorems 3.7, 3.8 and Lemma 3.12. ///

COROLLARY 3.14: *Every Morita bimodule is a* GM-*bimodule.*

PROOF: If ${}_RK_A$ is a Morita bimodule, then ${}_RK$ and K_A are strongly quasi-injective and therefore, by using Lemma 3.12 and Proposition 3.13, we infer that ${}_RK$ and K_A have essential and finitely generated socles and hence they are finitely cogenerated. ///

3.15. PROPOSITION: *Let K be a quasi-injective right A-module with essential socle and let $R = \mathrm{End}(K_A)$. Then K_A is finitely cogenerated if and only if R is semiperfect.*

PROOF: Let σ be the K-topology on A, set $\Sigma = \mathrm{Soc}(K_A)$ and consider in $\mathcal{C}_\sigma$ the exact sequence

$$0 \to \Sigma \to K \to K/\Sigma \to 0 .$$

Since K_A is quasi-injective, then it is injective in $\mathfrak{S}_\tau$ and we get the exact sequence

$$0 \to \mathrm{Hom}_A(K/\Sigma, K) \to R \to \mathrm{Hom}_A(\Sigma, \Sigma) \to 0 \tag{2}$$

of left R-modules. Moreover we have

$$J(R) = \{r \in R : \mathrm{Ker}\,(r) \trianglelefteq K_A\} = \{r \in R : r\Sigma = 0\} \simeq \mathrm{Hom}_A(K/\Sigma, K)$$

and idempotents of $R/J(R)$ lift (see e.g. [F, Theorem 19.27, p. 76]). From the exact sequence (2) we now infer $R/J(R) \simeq \mathrm{Hom}_A(\Sigma, \Sigma)$. This shows that R is semiperfect if and only if $_R\Sigma$ is finitely generated. ///

3.16. Corollary: *Assume that $_RK_A$ is faithfully balanced and both $_RK$ and K_A have essential socle. Then $_RK_A$ is a GM-bimodule if and only if both R and A are semiperfect.* ///

3.17. Proposition: *Let $_RK_A$ be faithfully balanced with both $_RK$ and K_A quasi-injective. If K_A is finitely cogenerated, then $_RK$ contains a copy of each simple left R-module. Consequently $_RK$ is strongly quasi-injective.*

Proof: Our proof is based on the duality D_K (see Theorem 2.7). Let τ be the K-topology on R. According to Proposition 3.15 (see also its proof) R is semiperfect and $J(R) = \mathrm{Ann}_R(\mathrm{Soc}\,(K_A))$ is closed in (R, τ), hence $R/J(R)$ with the quotient topology belongs to R_τ-LT. Since $R/J(R)$ is semisimple, this topology must be the discrete one, therefore $J(R)$ is τ-open. We infer that every simple left R-module belongs to $\mathfrak{S}_\tau$. Let P be a τ-open maximal left ideal and let $i\colon P \to R$ be the inclusion. Since K is an injective object in $\mathfrak{S}_\tau$, according to Proposition 3.2 we have the exact sequence

$$0 \to (R/P)^* \to R^* \xrightarrow{i^*} P^* \to 0$$

which is in $\mathfrak{B}(K_A)$. If $(R/P)^*$ were zero, then i^* would be a continuous A-isomorphism. But since K_A is finitely cogenerated, P^* would be finitely cogenerated as well and hence discrete, therefore i^* would be a topological isomorphism. Inasmuch as D_K is a duality by Theorem 2.7, we would have that i is an isomorphism: a contradiction. This shows that $_RK$ contains a submodule isomorphic to R/P. Finally $_RK$ is strongly quasi-injective by Proposition 3.5. ///

The proposion we have just shown, together with [MO_1, Proposition 6.10], fournishes the following result, which was proven by Kraemer [K, Lemma 2.2] through different arguments.

3.18. Corollary [K, Lemma 2.2]: *Let $_RK_A$ be a GM-bimodule. Then K_A*

(resp. ${}_RK$) contains representatives of all simple right A-modules (resp. left R-modules). Consequently both K_A and ${}_RK$ are strongly quasi-injective and $\mathrm{Soc}(K_A) = \mathrm{Soc}({}_RK)$. ///

3.19. Corollary: *Let ${}_RK_A$ be a GM-bimodule and let τ, σ be the K-topologies of R and A respectively. Then (R, τ) and (A, σ) are linearly compact.*

Proof: Since ${}_RK_A$ is faithfully balanced with ${}_RK$ and K_A both strongly quasi-injective, the thesis is a consequence of Theorem 3.7. ///

4. - Characterization of GM-bimodules by mean of dualities

In this section we show that GM-bimodules are exactly the bimodules associated to dualities between special categories of topological modules, in the sense of 2.3.

4.1. Given a topological module $(M, \varepsilon) \in R_r$-LT, we define the linear topology ε_* on M by taking as basis of neighbourhood of zero the ε-open submodules V of M such that M/V is finitely cogenerated. Then ε_* is Hausdorff and is equivalent to ε, in the sense that a submodule of M is ε-closed if and only if it is ε_*-closed. It can be shown that (M, ε) is complete or linearly compact if and only if (M, ε_*) has the corresponding property (see [B, ch. III, § 2, ex. 18]). The topology ε_* is called the *Leptin topology* of (M, ε).

4.2. Theorem: *Let (A, σ), (R, τ) be respectively right and left linearly topologized rings, let $\mathfrak{B}_A$ and ${}_R\mathfrak{B}$ be subcategories of* LT-A_σ *and* R_τ-LT *respectively. Assume that:*

1) $(A, \sigma) \in \mathfrak{B}_A$ *and* $(R, \tau) \in {}_R\mathfrak{B}$;

2) $\mathfrak{B}_A$ *(resp. ${}_R\mathfrak{B}$) contains all simple right A-modules (resp. left R-modules) and all finitely cogenerated modules in $\mathfrak{T}_\sigma$ (resp. in $\mathfrak{T}_\tau$);*

3) *A duality $H = (H_1\colon \mathfrak{B}_A \to {}_R\mathfrak{B},\ H_2\colon {}_R\mathfrak{B} \to \mathfrak{B}_A)$ is given with associated bimodule $({}_RK_A, {}_R\chi, \chi_A)$.*

Then the following properties hold:

a) ${}_RK_A$ *is a (discrete)* GM-*bimodule.*

b) (A, σ) *and* (R, τ) *are both linearly compact.*

c) For all $M \in \mathfrak{B}_A$ and $N \in {}_R\mathfrak{B}$, $H_1(M)$ and $H_2(N)$ are topologically isomorphic to $\mathrm{Chom}_A(M, K)$ *and* $\mathrm{Chom}_R(N, K)$ *respectively, both endowed with a topology equivalent to the finite topology.*

d) For all $M \in$ LT-A_σ and $N \in R_\tau$-LT the canonical morphisms $\omega_M\colon M \to M^{**}$ and $\omega_N\colon N \to N^{**}$ are continuous isomorphisms.

PROOF: *a*) Let us prove that ${}_RK$, and hence $E_\tau(K)$, is finitely cogenerated. Let us write $\chi = {}_R\chi$ and let $(V_\lambda)_{\lambda\in\Lambda}$ be the family of χ_*-open submodules of ${}_RK$. We have

$$\text{(3)} \qquad \operatorname{Ann}_K \operatorname{Ann}_A(V_\lambda) = V_\lambda \qquad \text{for all } \lambda \in \Lambda .$$

Indeed, assume on the contrary that for some $\lambda \in \Lambda$ there is

$$x \in \operatorname{Ann}_K \operatorname{Ann}_A(V_\lambda) \setminus V_\lambda .$$

By the hypothesis we have $K/V_\lambda \in {}_R\mathfrak{B}$ and, since ${}_RK$ is a cogenerator in ${}_R\mathfrak{B}$, there exists $f \in \operatorname{Hom}_R(K/V_\lambda, K)$ such that $(x + V_\lambda)f \neq 0$. By composing f with the canonical projection $K \to K/V_\lambda$ we get an element $a \in A$ such that $V_\lambda a = 0$ and $xa \neq 0$. This implies $a \in \operatorname{Ann}_A(V_\lambda)$, in contradiction with $x \in \operatorname{Ann}_K \operatorname{Ann}_A(V_\lambda)$.

Let us prove that

$$\text{(4)} \qquad A = \sum_{\lambda\in\Lambda} \operatorname{Ann}_A(V_\lambda) .$$

Let I be the right hand side of (4), assume $I \neq A$ and let P be a maximal right ideal of A containing I. We have $A/P \in \mathfrak{B}_A$ by the assumption and, since K_A is a cogenerator of $\mathfrak{B}_A$, A/P maps non-trivially into K_A. This implies that $xP = 0$ for some non-zero $x \in K$ and hence $xI = 0$. By using (3) and the fact that χ_* is Hausdorff we get $x \in \bigcap_{\lambda\in\Lambda} \operatorname{Ann}_K \operatorname{Ann}_A(V_\lambda) = \bigcap_{\lambda\in\Lambda} V_\lambda = 0$: a contradiction.

Now, since the family $(V_\lambda)_{\lambda\in\Lambda}$ is downward directed, then the family $(\operatorname{Ann}_A(V_\lambda))_{\lambda\in\Lambda}$ is upward directed; thus we infer from (4) that there exists $\mu \in \Lambda$ such that $1 \in \operatorname{Ann}_A(V_\mu)$. We conclude $V_\mu = 0$, that is $K \simeq K/V_\mu$ is finitely cogenerated and so is $E_\tau(K)$. Our assumption implies now that $E_\tau(K) \in {}_R\mathfrak{B}$ and then ${}_RK$ cogenerates $E_\tau(K)$ by Proposition 2.4. Finally $K = E_\tau(K)$ by Lemma 3.9 and then ${}_RK$ is quasi-injective. Since the above arguments apply also to K_A, we conclude that ${}_RK_A$ is a GM-bimodule.

b) Since $E_\tau(K) = K$ and ${}_RK$ contains a representative for each simple left R-module, then ${}_RK$ is a cogenerator of $\mathfrak{C}_\tau$. Moreover ${}_RK_A$ is faithfully balanced and K_A is quasi-injective. Thus (R, τ) is linearly compact by Theorem 3.7. A similar argument applies for (A, σ).

c) Given $M \in \mathfrak{B}_A$, let us prove that the topology ε of $H_1(M)$ is equivalent to the weak topology $\bar{\varepsilon}$ of $\operatorname{Chom}_R(H_1(M), K)$. Since $\bar{\varepsilon} \leq \varepsilon$, we must prove that each ε-closed submodule of $H_1(M)$ is $\bar{\varepsilon}$-closed. Inasmuch as every ε-closed submodule of $H_1(M)$ is an intersection of ε-open submodules, it is sufficient to show that if V is ε-open submodule of $H_1(M)$, then V is an intersection of $\bar{\varepsilon}$-open submodules. Let $x \in H_1(M) \setminus V$ and let W be a submodule of $H_1(M)$ which is maximal with respect to the properties $V \subseteq W$ and $x \notin W$. Then $H_1(M)/W$ belongs to $\mathfrak{C}_\tau$ and contains a simple essential

R-submodule. Since K is a cogenerator of $\mathfrak{C}_r$, $H_1(M)/W$ is isomorphic to a submodule of ${}_RK$ and therefore W is the kernel of an element of $\mathrm{Chom}_R(H_1(M), K)$, that is W is ℓ-open. We conclude that V is an intersection of ℓ-open submodules.

Looking at the commutative diagram (1) in Theorem 2.6 we have

$$(H_1(M), \ell) = \overline{H_1(M)} = {}_RTH_1(M) = D_1T_A(M),$$

where the latter coincides with $\mathrm{Chom}_A(M, K)$ endowed with the topology induced by the product topology of ${}_RK^M$. Since ${}_RK$ is discrete, this topology is just the finite topology of $\mathrm{Chom}_A(M, K)$.

d) Let $(N, \varepsilon) \in R_r$-LT. Since ${}_RK$ is a cogenerator of $\mathfrak{C}_r$, then it is easily seen that $\mathrm{Chom}_R(N, K)$ separates points of N. It follows that $(N, \ell) \in \mathfrak{B}({}_RK)$ and, since $(N, \varepsilon)^* = (N, \ell)^*$, we get $(N, \varepsilon)^{**} = (N, \ell)^{**}$. As D_K is a duality by Theorem 2.7, the canonical morphism $\omega_{(N,\ell)}: (N, \ell) \to (N, \ell)^{**}$ is a topological isomorphism and therefore $\omega_N: (N, \varepsilon) \to (N, \varepsilon)^{**}$ is a continuous isomorphism. ///

The following is, in some sense, the converse of Theorem 4.2.

4.3. THEOREM: *Every GM-bimodule ${}_RK_A$ is the bimodule associated to a duality of the kind considered in Theorem 4.2.*

PROOF: By Theorem 2.7 the pair of functors $D_K = (D_1, D_2)$ gives a duality between $\mathfrak{B}(K_A)$ and $\mathfrak{B}({}_RK)$. Endow R and A with the respective K-topologies τ and σ. Then $(A, \sigma) \in \mathfrak{B}(K_A)$ and $(R, \tau) \in \mathfrak{B}({}_RK)$. Since $K_A \in \mathfrak{B}(K_A)$, it follows from Corollary 3.18 that $\mathfrak{B}(K_A)$ contains all simple right A-modules. Again by Corollary 3.18 K_A is an injective cogenerator of $\mathfrak{C}_\sigma$, therefore every finitely cogenerated module in $\mathfrak{C}_\sigma$ is a submodule of K_A^n for some $n \in N$; thus every finitely cogenerated module in $\mathfrak{C}_\sigma$ belongs to $\mathfrak{B}(K_A)$. A similar argument applies to $\mathfrak{B}({}_RK)$. ///

Theorem 4.2 allows us to improve a result of Müller [Mü$_2$].

4.4. THEOREM: *Let A, R be two discrete rings and let $\mathfrak{B}_A$ and ${}_R\mathfrak{B}$ be subcategories of* LT-A *and* R-LT *respectively. Assume that:*

1) $A_A \in \mathfrak{B}_A$ *and* ${}_RR \in {}_R\mathfrak{B}$;

2) *All finitely cogenerator modules in* Mod-A *and* R-Mod *belongs to* $\mathfrak{B}_A$ *and* ${}_R\mathfrak{B}$ *respectively;*

3) *A duality $H = (H_1, H_2)$ between $\mathfrak{B}_A$ and ${}_R\mathfrak{B}$ is given.*

Then the bimodule ${}_RK_A$ associated to H is a Morita bimodule and therefore the rings A and R have a Morita duality induced by ${}_RK_A$. ///

5. - K-REFLEXIVE MODULES

5.1. Let A, R be given rings and let ${}_RK_A$ be a bimodule. We recall that a right A-module M is *K-reflexive* if the canonical morphism

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_A(M, K_A), {}_RK)$$

is an isomorphism. We denote with $\tilde{M}$ the module M endowed with the K-topology. Let $\mathfrak{D}(K_A)$ be the subcategory of Mod-A of all modules cogenerated by K and let us consider the subcategory $\widetilde{\mathfrak{D}(K_A)} = \{\tilde{M} : M \in \mathfrak{D}(K_A)\}$ of LT-A, which is clearly equivalent to $\mathfrak{D}(K_A)$. Then $\widetilde{\mathfrak{D}(K_A)} \subseteq \mathfrak{B}(K_A)$ and a topological module $M \in \mathfrak{B}(K_A)$ belongs to $\widetilde{\mathfrak{D}(K_A)}$ if and only if

$$\mathrm{Chom}_A(M, K) = \mathrm{Hom}_A(M, K).$$

Let us define $\mathfrak{C}(K_A) = \{M \in \mathfrak{B}(K_A) : M \text{ is complete}\}$. Note that a topological module $M \in \mathfrak{B}(K_A)$ is in $\mathfrak{C}(K_A)$ if and only if M is topologically isomorphic to a closed submodule of the topological product K^X for some set X. The categories $\widetilde{\mathfrak{D}({}_RK)}$ and $\mathfrak{C}({}_RK)$ are defined similarly.

5.2. PROPOSITION: *Let ${}_RK_A$ be a faithfully balanced bimodule and assume that ${}_RK$ and K_A are both strongly quasi-injective. Then we have:*

a) The duality D_K induces a duality between $\widetilde{\mathfrak{D}(K_A)}$ (resp. $\mathfrak{C}(K_A)$) and $\mathfrak{C}({}_RK)$ (resp. $\widetilde{\mathfrak{D}({}_RK)}$). Consequently D_K induces a duality between $\widetilde{\mathfrak{D}(K_A)} \cap \mathfrak{C}(K_A)$ and $\widetilde{\mathfrak{D}({}_RK)} \cap \mathfrak{C}({}_RK)$.

b) A right A-module (resp. left R-module) M is K-reflexive if and only if $\tilde{M}$ is Hausdorff and complete.

PROOF: Since K_A and ${}_RK$ are selfcogenerators by Proposition 3.5, then *a*) is a consequence of [MO$_2$, Theorem 4.12].

Being K_A a selfcogenerator is equivalent to the fact that every factor module of every finite direct sum of copies of K_A is Hausdorff in the K-topology. Thus, according to [V, Corollary 4.5], for every right A-module M the canonical morphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_A(M, K), K)$ is the Hausdorff completion of M and this proves *b*). ///

5.3. COROLLARY: *If ${}_RK_A$ is a GM-bimodule, then the K-reflexive right A-modules (resp. left R-modules) are precisely those which are complete and Hausdorff in the K-topology.*

PROOF: It follows by Corollary 3.18 and Proposition 5.2. ///

We now characterize the K-reflexive modules where ${}_RK_A$ is a GM-bimodule with K_A injective.

5.4. Lemma: *Given a ring A, if K_A is a finitely cogenerated (strongly) quasi-injective cogenerator of* Mod-A, *then K_A is injective.*

Proof: Let σ be the K-topology on A. Then K is an injective object in $\mathfrak{T}_\sigma$ and there is a finite set $\{V_1, \dots, V_n\}$ of representatives for all simple right A-modules. There are positive integers $\alpha_1, \dots, \alpha_n$ such that

$$K \simeq E_\sigma(V_1)^{\alpha_1} \oplus \dots \oplus E_\sigma(V_n)^{\alpha_n}.$$

On the other hand, if we fix $j \in \{1, \dots, n\}$, it follows from the assumption that $E(V_j)$ is isomorphic to a direct summand of K_A and, since $\mathrm{End}\,(E_\sigma(V_i))$ is a local ring for each i, it follows from a well known result of Azumaya that $E(V_j) \simeq E_\sigma(V_j)$. This is enough to conclude that K_A is injective. ///

5.5. Corollary: *If ${}_RK_A$ is a* GM-*bimodule, then K_A is injective if and only if K_A is a cogenerator of* Mod-A.

Proof: It is a consequence of Corollary 3.18 and Lemma 5.4. ///

If (R, τ) is a left linearly topologized ring, we denote by R_τ-LC$_*$ the subcategory of R_τ-LT consisting of all linearly compact modules endowed with their Leptin topologies.

5.6. Lemma: *Assume that (R, τ) is a linearly compact ring, let ${}_RK$ be an injective cogenerator of $\mathfrak{T}_\tau$ with essential socle, set $A = \mathrm{End}({}_RK)$ and let σ be the K-topology on A. Then we have:*

a) *${}_RK_A$ is faithfully balanced, K_A is strongly quasi-injective, (A, σ) is linearly compact, $\mathrm{Soc}\,(K_A)$ is essential in K_A. In particular D_K is a duality between $\mathfrak{B}(K_A)$ and $\mathfrak{B}({}_RK)$.*

b) *The following conditions are equivalent:*

(i) $\mathfrak{C}({}_RK) = R_\tau$-LC$_*$.

(ii) *${}_RK$ is linearly compact discrete.*

(iii) *K_A is an injective cogenerator of* Mod-A.

(iv) *A_A is linearly compact discrete.*

(v) *D_K induces a duality between* $\widetilde{\text{Mod-}A}$ *and* R_τ-LC$_*$.

Proof: *a*) follows from [DO, Corollary 2.12], while *b*) follows from [DO, Corollary 5.12] and Proposition 5.2. ///

5.7. Theorem: *Let ${}_RK_A$ be a* GM-*bimodule and assume that K_A is injective. Then the following properties hold:*

a) *Both ${}_RK$ and A_A are linearly compact discrete.*

b) *A right A-module M is K-reflexive if and only if M is complete in the cofinite topology.*

c) *A left R-module N is K-reflexive if and only if N is linearly compact in the K-topology.*

PROOF: If τ is the K-topology on R, then (R, τ) is linearly compact by Corollary 3.19 and ${}_RK$ is an injective cogenerator of $\mathfrak{C}_\tau$ with essential socle by Proposition 3.5. Therefore both ${}_RK$ and A_A are linearly compact discrete by Lemma 5.6.

According to Corollary 5.5 K_A is a finitely cogenerated injective cogenerator of Mod-A, therefore the K-topology on any right A-module is just the cofinite topology, which is Hausdorff. On the other hand we have from Lemma 5.6 that $\mathfrak{C}({}_RK) = R_\tau$-LC$_*$. Thus both *b*) and *c*) follow from Proposition 5.2. ///

5.8. COROLLARY [Mü$_1$, Theorem 2]: *Let ${}_RK_A$ be a Morita bimodule. Then a right A-module (resp. left R-module) is K-reflexive if and only if it is linearly compact discrete.*

PROOF: According to Theorem 5.7 a module M_A is K-reflexive if and only if M is linearly compact in the cofinite topology, which is equivalent to the discrete topology. ///

6. - RINGS ADMITTING A QUASI-DUALITY

6.1. Following Kraemer [K] we say that a ring R has a *left quasi-duality* if there is a ring A and a GM-bimodule ${}_RK_A$.

6.2. THEOREM: *A ring R has a left quasi-duality if and only if R satisfies the following two conditions:*

1) *R is semiperfect;*
2) *R has a linearly compact left ring topology* τ.

PROOF: If there is a ring A and a GM-bimodule ${}_RK_A$, then we know from Proposition 3.9 that R is semiperfect and it follows from Theorems 4.4 and 4.5 that (R, τ) is linearly compact, where τ is the K-topology.

Conversely, assume that R satisfies 1) and 2). Then $J(R)$ is τ-closed by [L, Satz 8] (see also [M, Corollary 13]), that is the quotient topology δ on $R/J(R)$ is Hausdorff. On the other hand, it follows from 1) that $R/J(R)$ is finitely cogenerated and hence δ must be the discrete topology. We infer that $J(R)$ is τ-open and therefore every simple left R-modules belongs to $\mathfrak{C}_\tau$. Let $V_1, \ldots, V_n$ be representatives of all simple left R-modules, set

$$K = E_\tau(V_1) \oplus \ldots \oplus E_\tau(V_n)$$

and let $A = \mathrm{End}({}_RK)$. Then ${}_RK$ is finitely cogenerated and is an injective cogenerator of $\mathfrak{C}_\tau$, therefore ${}_RK$ is strongly quasi-injective by Corollary 3.6. According to Lemma 5.6 ${}_RK_A$ is faithfully balanced and K_A is strongly quasi-injective with essential socle, thus we infer from Proposition 3.9 that K_A is finitely cogenerated. We conclude that ${}_RK_A$ is a GM-bimodule. ///

6.3. Corollary: *A ring A is right linearly compact discrete if and only if A has a right quasi-duality induced by a* GM-*bimodule ${}_RK_A$ such that ${}_RK$ is linearly compact discrete.*

Proof: If A_A is linearly compact discrete, then A is semiperfect by [S, Corollary, p. 335] and it follows from Theorem 6.2 that A has a right quasi-duality induced by a GM-bimodule ${}_RK_A$. If τ is the K-topology on R, we know that (R, τ) is linearly compact and ${}_RK$ is an injective cogenerator of $\mathfrak{C}_\tau$. Thus ${}_RK$ is linearly compact discrete by Lemma 5.6.

Conversely, assume that there exists a GM-bimodule ${}_RK_A$ with ${}_RK$ linearly compact discrete. Another application of Lemma 5.6 shows that A_A is linearly compact discrete. ///

6.4. Corollary [Mü$_1$]: *A ring R has a left Morita duality if and only if ${}_RR$ and the minimal cogenerator ${}_RU$ are both linearly compact discrete.*

Proof: Assume that R has a left Morita duality and let ${}_RK_A$ be a Morita bimodule. Then ${}_RK$, K_A, ${}_RR$ and A_A are all linearly compact discrete by Proposition 3.13. Since ${}_RU$ is isomorphic to a submodule of ${}_RK$, then ${}_RU$ is linearly compact discrete.

Conversely, assume the both ${}_RR$ and ${}_RU$ are linearly compact discrete and set $A = \mathrm{End}({}_RU)$. Since R is semiperfect, we infer by the same proof of Theorem 6.2 that ${}_RU_A$ is a GM-bimodule. According to Lemma 5.6 U_A is an injective cogenerator of Mod-A and hence ${}_RU_A$ is a Morita bimodule. ///

6.5. Proposition: *There exists a commutative local ring which has a quasi-duality but is not linearly compact discrete.*

Proof: Let F be a commutative field and let X be an infinite set. Consider the F-module $M = F^X$ endowed with the product topology of the discrete topologies. Then M is linearly compact but is not linearly compact discrete, because M has infinite Goldie dimension. Consider the trivial extension ring $R = F \ltimes M$. As an abelian additive group $R = F \times M$, while multiplication is defined by the rule $(\alpha, x)(\beta, y) = (\alpha\beta, \alpha y + \beta x)$. It is clear that R is a commutative local ring with maximal ideal $(0) \times M$, which can be identified with M. In this way every submodule of M becomes an ideal of R. If τ is the product topology in R, then (R, τ) is linearly compact but R is not linearly compact discrete. Finally R has a quasi-duality by Theorem 6.2. ///

The following result generalizes [Mü$_1$, Theorem 3], concerning commutative rings having a Morita duality. Our proof is essentially an adaptation of original Müller's proof to the nondiscrete case.

6.6. Theorem: *If R is a commutative ring having a quasi-duality, then R has a quasi-duality with itself.*

Proof: In view of Theorem 6.2 R is a finite product of local rings, therefore we may assume that R itself is local with Jacobson radical J. Again by Theorem 6.2 there is a ring topology τ on R such that (R, τ) is linearly compact. Let $U = E_\tau(R/J)$ be the minimal cogenerator of $\mathcal{C}_\tau$, let $A = \mathrm{End}(_R U)$ and let σ be the U-topology on A_A. Since R can be identified with a subring of the centre of A, we will achieve the proof if we show that $R = A$.

Given $a \in A$, let us consider the non void set

$$\mathcal{A} = \{X < U_A :\ X(a-r) = 0 \text{ for some } r \in R\}$$

and let us prove that $\mathcal{A}$, ordered by inclusion, is inductive. Let $\mathfrak{C} = \{X_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{A}$. For all $\lambda \in \Lambda$ there is $r_\lambda \in R$ such that $a - r_\lambda \in \mathrm{Ann}_A(X_\lambda)$. Observe now that $\mathrm{Ann}_R(X_\lambda)$ is τ-closed in R and, moreover, the system of congruences

$$r \equiv r_\lambda \ \mathrm{mod.}\ \mathrm{Ann}_R(X_\lambda)\,, \qquad \lambda \in \Lambda\,, \tag{5}$$

is finitely solvable, as it is not difficult to see. Since (R, τ) is linearly compact, then (5) has a solution r and, for each $\lambda \in \Lambda$, there is $t_\lambda \in \mathrm{Ann}_R(X_\lambda)$ such that $r = r_\lambda + t_\lambda$. We infer $a - r \in \mathrm{Ann}_A(X_\lambda)$ and so $\left(\bigcap_{\lambda\in\Lambda} X_\lambda\right)(a-r) = 0$, which implies that $\mathfrak{C}$ has a supremum in $\mathcal{A}$. We conclude that $\mathcal{A}$ is inductive and hence it has a maximal element Y by Zorn's Lemma.

Let us write $I = \mathrm{Ann}_R(Y)$ and let us prove that $\mathrm{Ann}_A(Y) = IA$. To this purpose we first observe that IA is σ-closed in A, because (A, σ) is linearly compact and multiplications by elements of A are continuous. Thus

$$\mathrm{Ann}_A \mathrm{Ann}_U (IA) = IA \tag{6}$$

by [M, Lemma 3]. Taking into account that $_R U$ is a cogenerator of R_τ-LT, we also have

$$Y = \mathrm{Ann}_U \mathrm{Ann}_R(Y) = \mathrm{Ann}_U(I)\,.$$

If $u \in U$ and $uIA = 0$, then $uI = 0$ and so $u \in \mathrm{Ann}_U(I) = Y$; it follows that $\mathrm{Ann}_U(IA) \subseteq Y$. Since the opposite inclusion holds trivially, we get the equality $\mathrm{Ann}_U(IA) = Y$ which, together with (6), yelds the equality $\mathrm{Ann}_A(Y) = IA$.

At this point we can state that for all $a \in A$ there are $r \in R$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in I$ such that

$$a - r = b_1 a_1 + \ldots + b_n a_n\,, \tag{7}$$

and we claim that $b_1 = \ldots = b_n = 0$, from which it will follows $R = A$.

Assume this is not the case and observe first that, since R is local, by the above we may write $A = R + JA$. Thus, given $i \in \{1, \ldots, n\}$, there are $a_{i1}, \ldots, a_{im} \in A$, $c_{i1}, \ldots, c_{im} \in J$, $r_i \in R$ such that

$$a_i = r_i + \sum_{j=1}^{m} c_{ij} a_{ij} . \tag{8}$$

Let us consider the element $r' = r + \sum_{i=1}^{n} r_i b_i \in R$. By using (7) and (8) we see that

$$a - r' \in \left(\sum_{i=1}^{n} Jb_i A \right) . \tag{9}$$

Inasmuch as R is local, our assumption implies that $\sum_{i=1}^{n} Jb_i \subsetneq \sum_{i=1}^{n} Rb_i \subseteq I$. If we set $Y' = \mathrm{Ann}_{E}\left(\sum_{i=1}^{n} Jb_i \right)$, then $Y \subsetneq Y'$ and by (9) we get $Y'(a - r') = 0$. Thus $Y' \in \mathcal{A}$, in contradiction with the maximality of Y. We conclude that $b_1 = \ldots$ $\ldots = b_n = 0$ and so $A = R$. ///

We conclude with an example.

REFERENCES

[AF] F. W. Anderson - K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Springer-Verlag, Heidelberg, 1974.

[B] N. Bourbaki, *Commutative Algebra*, Hermann, Paris, 1971.

[DO] D. Dikranian - A. Orsatti, *On the structure of linearly compact rings and their dualities*, Rend. Accad. Naz. Sc. XL Mem. Mat., 102, Vol. VIII, Fasc. 9 (1984), 143-184.

[F] C. Faith, *Algebra II, Ring Theory*, Grundlehren der Math. Wiss., Bd. 191, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[K] J. Kraemer, *Characterizations of the Existence of (Quasi-) Self-Duality for Complete Tensor Rings*, Algebra Berichte, 56, Verlag Reinhard Fisher, München, 1987.

[L] H. Leptin, *Linear Kompakten Ringe und Moduln*, Math. Z., 62 (1955), 241-267.

[M] C. Menini, *Linearly Compact Rings and Strongly Quasi-Injective Modules*, Rend. Sem. Mat. Univ. Padova, 65 (1980), 252-262. See also *Errata Corrige*, same journal, 69 (1983), 305-306.

[MO_1] C. Menini - A. Orsatti, *Good dualities and strongly quasi-injective modules*, Ann. Mat. Pura Appl., 127 (1981), 182-230.

[MO_2] C. Menini - A. Orsatti, *Dualities between categories of topological modules*, Comm. Algebra, 11 (1), (1983), 21-66.

[$Mü_1$] B. J. Müller, *Linear compactness and Morita duality*, J. Algebra, 16 (1970), 60-66.

[$Mü_2$] B. J. Müller, *All duality theories for linearly topologized modules comes from Morita duality*, Coll. Math. Soc. Janos Bolyai, 6: *Rings Modules and Radicals*, Kesthely (Hungary), 1971.

[O] B. L. Osofsky, *A generalization of QF-rings*, J. Algebra, 4 (1966), 373-387; *Erratum*, 9 (1968), 120.

[S] F. L. Sandomierski, *Linearly compact modules and local Morita duality*, in *Ring Theory*, R. Gordon Ed., New York, 1972.

[V] P. Vamos, *Classical rings*, J. Algebra, 34 (1975), 114-129.

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