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BIAGIO RICCERI (*)

Existence Theorems for Nonlinear Problems (**) (***)

Teoremi di esistenza per problemi non lineari

RIASSUNTO. — In questo lavoro si pongono le basi di una nuova teoria: quella concernente gli operatori D-regolari (Definizione 1.1). In particolare, vengono stabiliti due fondamentali teoremi di esistenza (Teoremi 2.1 e 2.2) e se ne traggono numerose e varie conseguenze, tra le quali figurano: un teorema di suriettività per operatori non lineari definiti in uno spazio di Banach riflessivo e a valori nel duale di esso (Teorema 3.22); un teorema di punto fisso per operatori discontinui (Teorema 3.23); un teorema sulla risolubilità di un sistema omogeneo formato da un'infinità numerabile di equazioni non lineari (Teorema 3.24); una generalizzazione del teorema di Lax-Milgram (Teorema 3.25). Da notare, inoltre, che pure i celeberrimi teoremi di H. Brézis [1], concernenti la risolubilità di equazioni associate ad operatori di tipo (M), sono conseguenze dei Teoremi 2.1 e 2.2 (vedansi i Teoremi 3.4, 3.6 e 3.8).

INTRODUCTION

The main purpose of the present paper is to give a new contribution to the study of a problem that, in its most general form, can be stated as follows:

Given two non-empty sets X, Y and a real function f defined on $X \times Y$, find a point $\hat{x} \in X$ such that $\sup_{y \in Y} f(\hat{x}, y) \leq 0$.

Among the previous results on this problem (for which we refer to the bibliography quoted in [3]), we recall here the following very celebrated and used theorem by Ky Fan (see [2], Theorem 1):

THEOREM A: *Let X be a non-empty convex compact subset of a Hausdorff topological vector space E and let f be a real function on $X \times X$ satisfying the following*

(*) Indirizzo dell'Autore: Dipartimento di Matematica, Città Universitaria, 95125 Catania (Italia).

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conditions:

1) *for every* $x \in X$, *the function* $f(x, \cdot)$ *is concave in* X;

2) *for every* $y \in X$, *the function* $f(\cdot, y)$ *is lower semicontinuous in* X;

3) *for every* $x \in X$, *one has* $f(x, x) \leq 0$.

Then, there exists $\hat{x} \in X$ *such that* $\sup_{y \in X} f(\hat{x}, y) \leq 0$.

Now, to give at once an idea of the nature of our results, we state Theorem B below (a very special case of Theorem 3.2) which has to be compared with Theorem A.

THEOREM B: *Let* X *and* E *be as in Theorem A, with, in addition,* $\theta_E \in X$, *and let* f *be a real function on* $X \times E$ *satisfying the following conditions*:

1) *for every* $x \in X$, *the function* $f(x, \cdot)$ *is concave in* E *and* $f(x, \theta_E) = 0$;

2) *for every* $y \in E$, *the function* $f(\cdot, y)$ *is lower semicontinuous in* X;

3) *for every* $x \in X$ *such that* $X \setminus \bigcup_{\lambda > 0} \lambda(x - X) \neq \emptyset$, *one has* $f(x, x) > 0$.

Then, the conclusion of Theorem A holds.

Thus, we can regard Theorem B as a reasonable substitute of Theorem A in the cases where condition 3) of Theorem A is violated.

Our main results are Theorems 2.1 and 2.2. Their statements, intentionally abstract and detailed, have been formulated in such a way to turn out virtually applicable to a large range of possible different situations. In particular, the celebrated basic existence results by H. Brézis [1], on equations involving operators of type (M), are simple consequences of Theorems 2.1 and 2.2 (see Theorems 3.4, 3.6 and 3.8).

The paper is arranged into three sections. Section 1 contains notations, basic definitions and some preliminary results. In Section 2 we prove Theorems 2.1 and 2.2. Finally, Section 3 is devoted to various consequences of Theorems 2.1 and 2.2, among which, in particular, we point out: a surjectivity theorem for nonlinear operators from a reflexive Banach space into che dual of it (Theorem 3.22); a fixed point theorem for discontinuous operators (Theorem 3.23); a theorem on the solvability of a homogeneous system of countably many nonlinear equations (Theorem 3.24); a generalization of the classical Lax-Milgram theorem (Theorem 3.25).

1. - Notations, basic definitions and preliminary results

Here and (always) in the sequel, E is a real vector space; V is a linear subspace of E; V' is the algebraic dual of V; D is a non-empty subset of V; $\mathcal{F}_D$ is the family of all finite-dimensional linear subspaces of V meeting D;

$\mathfrak{U}_D$ is the collection of all families $\mathcal{F}$ of finite-dimensional linear subspaces of V meeting D such that $\mathcal{F}$ is directed by (set-theoretic) inclusion and $D \subseteq \bigcup_{S \in \mathcal{F}} S$; M_V is the set of all real functions on V, regarded as a vector space in the usual way; $\hat{M}_V$ is the set of all odd real functions on V; C_V is the set of all $\psi \in M_V$ such that $\psi(\Theta_E) < 0$ (Θ_E is the null element of E), the set $\psi^{-1}(]0, +\infty[)$ is convex and finitely open (¹) and

$$\bigcup_{\lambda \in]0,1[} \lambda \psi^{-1}(]0, +\infty[) \subseteq \psi^{-1}(]0, +\infty[) ;$$

$\hat{C}_V$ is the set of all concave real functions ψ on V such that $\psi(\Theta_E) = 0$; $\tilde{C}_V$ is the set of all concave real functions ψ on V such that $\psi(\Theta_E) < 0$ and $\bigcup_{\lambda > 0} \lambda \psi^{-1}(]0, +\infty[) \subseteq \psi^{-1}(]0, +\infty[)$. Observe that $\hat{C}_V \cup \tilde{C} \subseteq C_V$.

Now, we introduce three distinguished families of subsets of M_V. Namely, we denote by A_D the family of all sets $\Gamma \subseteq M_V$ for which there exists $\psi \in \Gamma$ such that $\sup_{x \in D} \psi(x) < 0$. We denote by $\mathcal{G}_D$ the family of all sets $\Gamma \subseteq M_V$ for which there exists $S \in \mathcal{F}_D$ such that $\sup_{x \in D \cap S} \psi(x) > 0$ for all $\psi \in \Gamma$. Moreover, if Γ_* is a non-empty subset of M_V, we put

$$\mathcal{K}_{D,\Gamma_*} = \{\Gamma \subseteq M_V : \Gamma - \psi \in A_D \cup \mathcal{G}_D, \ \forall \psi \in \Gamma_*\} .$$

We denote by τ_D the topology on M_V generated by the family of sets $\{\{\psi \in M_V : \psi(y) < r\}\}_{y \in D, r \in \mathbb{R}}$. It is not hard to check that a net $\{\psi_\alpha\}$ in M_V τ_D-converges to $\psi \in M_V$ if and only if one has $\limsup_\alpha \psi_\alpha(y) \leq \psi(y)$ for all $y \in D$. Further, we denote by $\hat{\tau}_V$ the topology on $\hat{M}_V$ of pointwise convergence. As is known, $(\hat{M}_V, \hat{\tau}_V)$ is a Hausdorff locally convex topological vector space.

Now, we give the most important definition of the paper.

DEFINITION 1.1: Let K be a non-empty subset of E and A be an operator from K into M_V. We say that A is *D-regular in* K if one of the two following conditions is satisfied:

a) $A(K) \in A_D$;

b) there exists $S_0 \in \mathcal{F}_D$ such that, for every $S \in \mathcal{F}_D$, with $S_0 \subseteq S$, one has $\sup_{y \in D \cap S} A(x)(y) > 0$ for all $x \in K \cap S$.

(¹) We say that a set $U \subseteq E$ is finitely open (resp. finitely closed, finitely compact) if, for every finite-dimensional linear subspace S of E, the set $U \cap S$ is open (resp. closed, compact) with respect to the usual Euclidean topology of S. Likewise, if $W \subseteq E$ and (Σ, τ) is a topological space, a function $h : W \to \Sigma$ is said to be finitely τ-continuous if, for every S as above, the function $h|_{W \cap S}$ is τ-continuous with respect to the relative Euclidean topology on $W \cap S$. In any case, for a given S as above, any topological concept relative to a subset of S is referred to the Euclidean topology of S.

Of course, in the above definition (and always in the sequel), $A(x)(y)$ denotes the value of the functional $A(x)(\cdot)$ at y.

The next proposition is an obvious, but important, consequence of Definition 1.1.

PROPOSITION 1.1: *If $A(K) \in A_D \cup \mathfrak{B}_D$, then A is D-regular in K.*

At present, we do not know a full characterization of the family $A_D \cup \mathfrak{B}_D$. Nevertheless, the next six propositions show how it is broad.

PROPOSITION 1.2: *If V is finite-dimensional, then each subset of M_V belongs to $A_D \cup \mathfrak{B}_D$.*

PROOF: Let $\Gamma \subseteq M_V$. Suppose $\Gamma \notin A_D$. Then, for every $\psi \in \Gamma$, one has $\sup_{x \in D} \psi(x) > 0$. Since, by hypothesis, $V \in \mathcal{F}_D$, it follows that $\Gamma \in \mathfrak{B}_D$.

PROPOSITION 1.3: *The family $A_D \cup \mathfrak{B}_D$ is closed under finite union.*

PROOF: Let $\Gamma_1, \Gamma_2 \in A_D \cup \mathfrak{B}_D$: Suppose $\Gamma_1 \cup \Gamma_2 \notin A_D$: Then, of course, $\Gamma_1, \Gamma_2 \notin A_D$, and so $\Gamma_1, \Gamma_2 \in \mathfrak{B}_D$. Let $S_1, S_2 \in \mathcal{F}_D$ be such that $\sup_{x \in D \cap S_1} \psi(x) > 0$ for all $\psi \in \Gamma_1$ and $\sup_{x \in D \cap S_2} \psi(x) > 0$ for all $\psi \in \Gamma_2$. Let S be the linear hull of $S_1 \cup S_2$. Thus, $S \in \mathcal{F}_D$ and $\sup_{x \in D \cap S} \psi(x) > 0$ for all $\psi \in \Gamma_1 \cup \Gamma_2$. Hence, $\Gamma_1 \cup$ $\cup \Gamma_2 \in \mathfrak{B}_D$.

PROPOSITION 1.4: *Let $\Gamma_1 \in A_D \cup \mathfrak{B}_D$, $\Gamma_2 \in \mathfrak{B}_D$ and $\Gamma \subseteq M_V$. If*

$$\Gamma_1 \subseteq \Gamma \subseteq \Big(\bigcup_{\lambda \geq 0} \lambda \Gamma_1 \Big) \cup \Gamma_2, \qquad \textit{then } \Gamma \in A_D \cup \mathfrak{B}_D .$$

PROOF: If $\Gamma_1 \in A_D$, then, as $\Gamma_1 \subseteq \Gamma$, one has $\Gamma \in A_D$. Thus, suppose $\Gamma_1 \notin A_D$. Then, $\Gamma_1 \in \mathfrak{B}_D$ and, therefore, there are $S_1, S_2 \in \mathcal{F}_D$ as in the proof of Proposition 1.3. Let S be the linear hull of $S_1 \cup S_2$. Now, let $\psi \in \Gamma$. If $\psi \in \Gamma_2$, then $0 < \sup_{x \in D \cap S_2} \psi(x) < \sup_{x \in D \cap S} \psi(x)$. Otherwise, there are $\lambda > 0$ and $\varphi \in \Gamma_1$ such that $\psi = \lambda \varphi$. If $\lambda = 0$, then $\Gamma \in A_D$. Thus, suppose $\lambda > 0$. Then, one has $\sup_{x \in D \cap S} \psi(x) = \lambda \sup_{x \in D \cap S} \varphi(x) > \lambda \sup_{x \in D \cap S_1} \varphi(x) > 0$. Hence, $\Gamma \in \mathfrak{B}_D$.

Proceeding as in the proof of Proposition 1.4, one can obtain also the following proposition.

PROPOSITION 1.5: *Let $\Gamma_1, \Gamma_2, \Gamma$ be as in Proposition 1.4. Moreover, suppose that D is symmetric and that $\Gamma_1 \subseteq \tilde{M}_V$. If*

$$\Gamma_1 \subseteq \Gamma \subseteq \Big(\bigcup_{\lambda \in \mathbb{R}} \lambda \Gamma_1 \Big) \cup \Gamma_2, \qquad \textit{then } \Gamma \in A_D \cup \mathfrak{B}_D .$$

PROPOSITION 1.6: *Let $\mathcal{F} \in \mathfrak{U}_D$. For each $S \in \mathcal{F}$, put $\Gamma_S = \bigcup_{y \in D \cap S} \{\psi \in M_V :$ $\psi(y) > 0\}$. Let $\tau_{\mathcal{F}}$ be the topology on M_V generated by the family $\{\Gamma_S\}_{S \in \mathcal{F}}$. Then, each $\tau_{\mathcal{F}}$-compact subset of M_V belongs to the family $A_D \cup \mathfrak{B}_D$.*

Proof: Let Γ be any $\tau_{\mathcal{F}}$-compact subset of M_V. Suppose $\Gamma \notin \mathcal{A}_D$. Then, since $D \subseteq \bigcup_{S\in\mathcal{F}} S$, one has $\Gamma \subseteq \bigcup_{S\in\mathcal{F}} \Gamma_S$. Thus, $\{\Gamma_S\}_{S\in\mathcal{F}}$ is an open cover of Γ, and so there are finitely many $S_1, \ldots, S_n \in \mathcal{F}$ such that $\Gamma \subseteq \bigcup_{i=1}^{n} \Gamma_{S_i}$. As the family $\mathcal{F}$ is directed by inclusion, there is $\hat{S} \in \mathcal{F}$ such that $\bigcup_{i=1}^{n} S_i \subseteq \hat{S}$. Of course, one has $\Gamma \subseteq \Gamma_{\hat{S}}$. Since $\Gamma_{\hat{S}} \in \mathfrak{B}_D$, a fortiori $\Gamma \in \mathfrak{B}_D$.

Remark 1.1: Taking into account that, given $\mathcal{F} \in \mathfrak{U}_D$, the topology $\tau_{\mathcal{F}}$ is weaker than $\hat{\tau}_V$, we infer, from Proposition 1.6, that any $\hat{\tau}_V$-compact subset of M_V belongs to the family $\mathfrak{K}_{D,M_V}$.

Proposition 1.7: *Suppose $V \setminus \{\Theta_E\} \subseteq \bigcup_{\lambda>0} \lambda D$. Let Γ be any non-empty convex subset of V', which is $\hat{\tau}_V$-closed in its linear hull, say L_Γ. Then, $\Gamma \in \mathfrak{K}_{D,L_\Gamma}$.*

Proof: Fix $\psi \in L_\Gamma$ and put $\Gamma_\psi = \Gamma - \psi$. Suppose $\Gamma_\psi \notin \mathcal{A}_D$. Then, $\Theta_{M_V} \notin \Gamma_\psi$. Since Γ_ψ is convex and $\hat{\tau}_V$-closed in L_Γ, thanks to a standard separation theorem, we find a $\hat{\tau}_V$-continuous linear functional T on L_Γ such that $T(\varphi) > 0$ for all $\varphi \in \Gamma_\psi$. Since V and L_Γ are paired in the usual way, the $w(L_\Gamma, V)$-topology being just the relativization of $\hat{\tau}_V$ to L_Γ, by Theorem 16.2 of [6], there exists $\hat{x} \in V \setminus \{\Theta_E\}$ such that $T(\varphi) = \varphi(\hat{x})$ for all $\varphi \in L_\Gamma$. Now, let $S \in \mathcal{F}_D$ be such that $\hat{x} \in S$. By hypothesis, there are $\mu > 0$ and $\hat{y} \in D$ such that $\mu\hat{x} = \hat{y}$. Therefore, one has $\varphi(\hat{y}) > 0$ for all $\varphi \in \Gamma_\psi$. Hence, $\Gamma_\psi \in \mathfrak{B}_D$. This concludes the proof.

Now, come back to Proposition 1.1. Let K and A be as in Definition 1.1. The next propositions provide some sufficient conditions in order that $A(K) \in \mathcal{A}_D \cup \mathfrak{B}_D$.

Proposition 1.8: *Suppose that there exist $\mathcal{F} \in \mathfrak{U}_D$ and a topology on K, with respect to which K is compact, such that, for every $S \in \mathcal{F}$, the set*

$$\bigcup_{y\in D\cap S} \{x \in K:\ A(x)(y) > 0\}$$

is open. Then, $A(K) \in \mathcal{A}_D \cup \mathfrak{B}_D$.

Proof: For each $S \in \mathcal{F}$, let Γ_S be as in Proposition 1.6. Of course, one has $A^{-1}(\Gamma_S) = \bigcup_{y\in D\cap S} \{x \in K:\ A(x)(y) > 0\}$. Since each non-empty, proper and $\tau_{\mathcal{F}}$-open subset of M_V is the union of finite intersections of members of $\{\Gamma_S\}_{S\in\mathcal{F}}$, it follows that the operator A is $\tau_{\mathcal{F}}$-continuous. Hence, $A(K)$ is $\tau_{\mathcal{F}}$-compact. Then, the conclusion follows directly from Proposition 1.6.

Remark 1.2: In the case where V is the linear hull of a countable set, the condition expressed in Proposition 1.8 is also necessary in order that $A(K) \in \mathcal{A}_D \cup \mathfrak{B}_D$. Indeed, in this case, it is possible to find $\mathcal{F} \in \mathfrak{U}_D$ in such a way that each member of $\mathcal{A}_D \cup \mathfrak{B}_D$ is $\tau_{\mathcal{F}}$-compact. Therefore, if $A(K) \in$

$\in \mathcal{A}_D \cup \mathcal{G}_D$, then K turns out to be compact with respect to the topology on K generated by the family $\{A^{-1}(\Gamma_S)\}_{S\in\mathcal{F}}$.

Let us establish now some consequences of Proposition 1.8.

PROPOSITION 1.9: *Let K be a compact topological space such that, for every $y \in D$, the function $A(\cdot)(y)$ is lower semicontinuous. Then, $A(K) \in \mathcal{K}_{D,M_V}$.*

PROOF: Let $\psi \in M_V$. By hypothesis, for every $y \in D$, the set $\{x \in K: (A(x)-\psi)(y) > 0\}$ is open. Therefore, for every $S \in \mathcal{F}_D$, the set $\bigcup_{y\in D\cap S} \{x \in K: (A(x)-\psi)(y) > 0\}$ is open too. Then, thanks to Proposition 1.8, one has $A(K)-\psi \in \mathcal{A}_D \cup \mathcal{G}_D$. Hence, $A(K) \in \mathcal{K}_{D,M_V}$.

PROPOSITION 1.10: *Suppose that D is symmetric and that $A(K) \subseteq \hat{M}_V$. Moreover, let K be a compact topological space such that, for every $y \in D$ (resp. for every $y \in D$ and every $r \in \mathbb{R}$), the set $\{x \in K: A(x)(y) = 0\}$ (resp. $\{x \in K: A(x)(y) = r\}$) is closed. Then, $A(K) \in \mathcal{A}_D \cup \mathcal{G}_D$ (resp. $A(K) \in \mathcal{K}_{D,\hat{M}_V}$).*

PROOF: First, assume that, for every $y \in D$, the set $\{x \in K: A(x)(y) = 0\}$ is closed. For every $S \in \mathcal{F}_D$, thanks to our assumptions, one has

$$\bigcap_{y\in D\cap S}\{x \in K: A(x)(y) < 0\} = \bigcap_{y\in D\cap S}\{x \in K: A(x)(y) = 0\}.$$

Therefore, the set $\bigcup_{y\in D\cap S}\{x \in K: A(x)(y) > 0\}$ is open. Then, by Proposition 1.8, one has $A(K) \in \mathcal{A}_D \cup \mathcal{G}_D$. The respective part of the proposition follows at once by fixing $\psi \in \hat{M}_V$ and then by applying to the operator $A(\cdot)-\psi$ what proved above.

PROPOSITION 1.11: *Suppose that D is the linear hull of a countable set $Y \subseteq V$ and that $A(K) \subseteq V'$. Moreover, let K be a compact topological space such that, for every $y \in Y$ (resp. for every $y \in Y$ and every $r \in \mathbb{R}$) the set $\{x \in K: A(x)(y) = 0\}$ (resp. $\{x \in K: A(x)(y) = r\}$) is closed. Then, $A(K) \in \mathcal{A}_D \cup \mathcal{G}_D$ (resp. $A(K) \in \mathcal{K}_{D,V'}$).*

PROOF: Arrange Y into a sequence $\{y_n\}$. For each $n \in \mathbb{N}$, denote by S_n the linear hull of $\{y_1, \ldots, y_n\}$. Put $\mathcal{F} = \{S_n: n \in \mathbb{N}\}$. Plainly, $\mathcal{F} \in \mathcal{U}_D$. It is easily seen that, for every $n \in \mathbb{N}$, one has

$$\bigcap_{y\in S_n}\{x \in K: A(x)(y) < 0\} = \bigcap_{i=1}^{n}\{x \in K: A(x)(y_i) = 0\}.$$

Now, the proof goes on as that of Proposition 1.10.

Proposition 1.1 provides a first natural way for proving the D-regularity of a given operator. Another way is provided by Proposition 1.12 below, where we use the classical notion of an operator of type (M) [1]. Now, for the reader's convenience, we recall it.

Let G be a real vector space which is in duality with V by means of a bilinear form $\langle \cdot , \cdot \rangle : G \times V \to \mathbf{R}$. Suppose that the pairing (G, V) is separated in the sense of [6], p. 138. Let V be equipped with a vector topology stronger than the $\sigma(V, G)$-topology and let G be equipped with the $\sigma(G, V)$-topology. Then, an operator $\Phi : V \to G$ is said to be *of type* (M) if the two following conditions are satisfied: *a*) Φ is finitely continuous; *b*) for every compact set $K \subset V$, every $\hat{x} \in K$ and every net $\{x_\lambda\}$ in K convergent to $\hat{x}$ and such that the net $\{\Phi(x_\lambda)\}$ converges in G to a g for which

$$\limsup_{\lambda} \langle \Phi(x_\lambda), x_\lambda \rangle \leq \langle g, \hat{x} \rangle , \qquad \text{one has } \Phi(\hat{x}) = g .$$

Now, let V^* be the topological dual of V and let $A_\Phi : V \to V^*$ be the operator defined by putting $A_\Phi(x)(y) = \langle \Phi(x), y \rangle$ for all $x, y \in V$.

PROPOSITION 1.12: *Let* $\Phi : V \to G$ *be an operator of type* (M). *Then, for every compact set* $K \subseteq V$ *and every* $\psi \in V^*$, *the restriction of the operator* $A_\Phi(\cdot) - \psi$ *to* K *is* V*-regular there.*

PROOF: Suppose that condition *b*) of Definition 1.1, applied, taking $D = V$, to the restriction of the operator $A_\Phi(\cdot) - \psi$ to K, does not hold. There are, therefore, a family $\mathcal{F} \in \mathcal{U}_{\mathcal{F}}$ and a net $\{x_S\}_{S \in \mathcal{F}}$ such that $x_S \in K \cap S$ and $A_\Phi(x_S)(y) = \psi(y)$ for all $S \in \mathcal{F}$, $y \in S$. Since K is compact, there is a subnet $\{x_{S_\lambda}\}_{\lambda \in \Lambda}$ convergent to $\hat{x} \in K$. Of course, the net $\{A_\Phi(x_{S_\lambda})\}_{\lambda \in \Lambda}$ f_V-converges to ψ and $\lim_{\lambda} A_\Phi(x_{S_\lambda})(x_{S_\lambda}) = \psi(\hat{x})$. Since the functional ψ is $\sigma(V, G)$-continuous, by Theorem 16.2 of [6], there is $g \in G$ such that $\langle g, y \rangle = \psi(y)$ for all $y \in V$. Thus, as Φ is of type (M), one has $\Phi(\hat{x}) = g$. Hence, one has $A_\Phi(\hat{x}) = \psi$, and so $A_\Phi(K) - \psi \in A_V$. This completes the proof.

REMARK 1.3: Observe that, in the proof of Proposition 1.12, we have used only condition *b*) of the definition of an operator of type (M). However, there are even norm continuous operators Φ acting in V, a Hilbert space, which are not of type (M), with respect to the weak topology, but are such that, for every weakly compact set $K \subseteq V$, the conclusion of Proposition 1.12 holds. Indeed, consider the following example.

EXAMPLE 1.1: Let $(V, (\cdot , \cdot))$ be an infinite-dimensional Hilbert space. Let $G = V$ and $\langle x, y \rangle = (x, y)$ for all $x, y \in V$. Fix any $x_0 \in V \setminus \{\theta_V\}$ and, for every $x \in V$, put $\Phi(x) = (1 - \|x\|)x_0 - \|x\|x$. Let us show that the norm continuous operator Φ is not of type (M) with respect to the weak topology on V. Indeed, let B be the closed unit ball of V. Let $\{x_n\}$ be any sequence in ∂B which converges weakly to θ_V. Since $(\Phi(x_n), y) = -(x_n, y)$ for all $n \in \mathbf{N}$, $y \in V$, it follows that also the sequence $\{\Phi(x_n)\}$ converges weakly to θ_V. On the other hand, one has $(\Phi(x_n), x_n) = -1$ for all $n \in \mathbf{N}$. However, $\Phi(\theta_V) = x_0 \neq \theta_V$. This proves our claim. Let us show, on the contrary, that, for every weakly compact set $K \subseteq V$ and every $\psi \in V^*$, the restriction of $A_\varphi(\cdot)-\psi$ to K is V-regular there. Let $z \in V$ be such that $\psi(y) = (z, y)$ for all $y \in V$. As in the proof of Proposition 1.12, suppose that there are $\mathcal{F} \in \mathfrak{U}_V$ and a net $\{x_S\}_{S\in\mathcal{F}}$ such that $x_S \in K \cap S$ and $A_\varphi(x_S)(y)=\psi(y)$ for all $S \in \mathcal{F}$, $y \in S$. Let $(x_{S_\alpha})_{\alpha\in\Lambda}$ be a subnet which converges weakly to $\hat{x} \in K$. Of course, we can suppose that also the net $\{\|x_{S_\alpha}\|\}_{\alpha\in\Lambda}$ is convergent in $\mathbf{R}$, say to λ. For every $\alpha \in \Lambda$, one has

$$(\Phi(x_{S_\alpha})-z, \hat{x}) = (1-\|x_{S_\alpha}\|)(x_0, \hat{x}) - \|x_{S_\alpha}\|(x_{S_\alpha}, \hat{x}) - (z, \hat{x}) .$$

Thus, passing to the limit and taking into account that $\lim\limits_{\alpha}(\Phi(x_{S_\alpha})-z, \hat{x})=0$, one has

$$(1-\lambda)(x_0, \hat{x}) = \lambda\|\hat{x}\|^2 + (z, \hat{x}) .$$

On the other hand, for all $\alpha \in \Lambda$, one has also

$$0 = (\Phi(x_{S_\alpha})-z, x_{S_\alpha}) = (1-\|x_{S_\alpha}\|)(x_0, x_{S_\alpha}) - \|x_{S_\alpha}\|^2 - (z, x_{S_\alpha}) .$$

Passing to the limit again, we get

$$(1-\lambda)(x_0, \hat{x}) = \lambda^2 + (z, \hat{x}) .$$

Hence, $\lambda\|\hat{x}\|^2 = \lambda^2$. Taking into account that $\|\hat{x}\| < \lambda$, we then infer that $\lambda = \|\hat{x}\|$. Hence, the net $\{x_{S_\alpha}\}_{\alpha\in\Lambda}$ converges strongly to $\hat{x}$. But then the net $\{\Phi(x_{S_\alpha})\}_{\alpha\in\Lambda}$ converges strongly to $\Phi(\hat{x})$. This implies that $\Phi(\hat{x})=z$, and so $A_\varphi(K)-\psi \in A_V$.

Before finishing this section, we introduce two further notations. Namely, if X, Y, Z are three non-empty subsets of E, we put

$$I_{X,Y,Z} = \Big\{x \in X \colon Y \subseteq \bigcup_{\lambda>0} \lambda(x-Z)\Big\} .$$

Moreover, if $\Gamma \subseteq M_V$, we denote by $\overline{(\Gamma)}_{\tau_0}$ the closure of Γ with respect to the topology τ_0. Finally, we point out that in the proof of Theorem 2.1 we will have to consider a suitable multifunction. Thus, for the basic notions on multifunctions we refer, for instance, to [10].

2. - The main results

Our first main result is the following theorem.

Theorem 2.1: *Let $\mathcal{F} \in \mathfrak{U}_V$, with $V = \bigcup_{S\in\mathcal{F}} S$, and let A be an operator from $X \subseteq E$ into C_V. Moreover, for each $S \in \mathcal{F}$, let K_S and X_S be two non-empty subsets of $X \cap S$, with $K_S \subseteq X_S$, satisfying the following conditions:*

1) *K_S is compact in S and X_S is convex and closed in S;*

2) *for every* $y \in X_S - X_S$, *the set* $\{x \in X_S : A(x)(y) \leq 0\}$ *is closed in* S;

3) *for every* $x \in X_S \setminus I_{K_S, D\cap S, X_S}$, *one has* $\sup_{y\in K_S} A(x)(x-y) > 0$.

Under such hypotheses, the following conclusions hold:

i) $\Theta_{M_\varphi} \in \overline{\left(A\left(\bigcup_{S\in\mathcal{F}} K_S\right)\right)}_{\tau_\varphi}$;

ii) *for every set* $K \subseteq X$, *with* $\bigcup_{S\in\mathcal{F}} K_S \subseteq K$, *such that the operator* A *is* D*-regular in* K, *one has* $A(K) \in \mathcal{A}_D$;

iii) *if* $\Gamma \in \mathcal{A}_D \cup \mathcal{G}_D$ *and* $A\left(\bigcup_{S\in\mathcal{F}} K_S\right) \subseteq \Gamma$, *then* $\Gamma \in \mathcal{A}_D$.

PROOF: We proceed as in the proof of Theorem 1 of [8]. Fix $S \in \mathcal{F}$. Denote by L_S the affine hull of X_S. Let T be a convex and compact subset of L_S such that $\mathrm{int}_{L_S}(T) \neq \emptyset$ (of course, $\mathrm{int}_{L_S}(T)$ denotes the interior of T in L_S) and $K_S \subseteq T \subseteq X_S$. Such a set T exists by virtue condition 1), taking into account that $\mathrm{int}_{L_S}(X_S) \neq \emptyset$. For every $x \in T$, put

$$F(x) = \{y \in T : A(x)(x-y) > 0\}.$$

We claim that there is some $x \in T$ for which $F(x) = \emptyset$. Assume the contrary, that is $F(x) \neq \emptyset$ for all $x \in T$. Let us show that the multifunction F is lower semicontinuous. Therefore, let Ω be any open set in S and $x' \in T$, $y' \in T \cap \Omega$ be such that $A(x')(x'-y') > 0$. Since $A(x') \in C_\Gamma$, one has $\mathrm{int}_{L_S}(F(x')) \neq \emptyset$. Therefore, we can suppose that $y' \in \mathrm{int}_{L_S}(T)$. By condition 2), the set

$$U = \{x \in X_S : A(x)(x'-y') > 0\}$$

is an open neighbourhood of x' in X_S. Put

$$W = U \cap [(x'-y') + (\mathrm{int}_{L_S}(T) \cap \Omega)] \cap T.$$

It is possible to show that W is a neighbourhood of x' in T. Let $x \in W$. If we put $y = x - x' + y'$, we have $y \in \mathrm{int}_{L_S}(T) \cap \Omega$ as well as $A(x)(x-y) = A(x)(x'-y') > 0$. Hence, $y \in F(x) \cap \Omega$. This proves the lower semicontinuity of F. But then, taking into account that F is convex-valued, by Proposition 3 of [8], there exists $\bar{x} \in T$ such that $\bar{x} \in F(\bar{x})$, that is $A(\bar{x})(\Theta_X) > 0$, against the fact that $A(\bar{x}) \in C_\Gamma$. Now, let $\{T_n\}$ be a non-decreasing sequence of convex and compact subsets of L_S such that $\mathrm{int}_{L_S}(T_n) \neq \emptyset$ for all $n \in \mathbb{N}$, $K_S \subseteq \bigcap_{n\in\mathbb{N}} T_n$ and $X_S = \bigcup_{n\in\mathbb{N}} T_n$. On the basis of what seen above, for each $n \in \mathbb{N}$, there exists $x_n \in T_n$ such that

$$\sup_{y\in T_n} A(x_n)(x_n - y) < 0.$$

Thanks to condition 3), one has $x_n \in I_{K_S, D\cap S, X_S}$. Consider now the sequence $\{x_n\}_{n\in\mathbf{N}}$. Since K_S is compact, $\{x_n\}_{n\in\mathbf{N}}$ admits a cluster point $\hat{x}_S$ in K_S. We claim that $\sup\limits_{y\in X_S} A(\hat{x}_S)(\hat{x}_S - y) < 0$. Assume the contrary. Hence, there is $y_0 \in X_S$ such that $A(\hat{x}_S)(\hat{x}_S - y_0) > 0$. Then, it is possible to find $\nu \in \mathbf{N}$ and $\bar{y} \in \operatorname{int}_{L_S}(T_\nu)$ is such a way that $A(\hat{x}_S)(\hat{x}_S - \bar{y}) > 0$. Thanks to condition 2) again, the set $\hat{U} = \{x \in X_S\colon A(x)(\hat{x}_S - \bar{y}) > 0\}$ is an open neighbourhood of $\hat{x}_S$ in X_S. Put $\hat{W} = \hat{U} \cap [(\hat{x}_S - \bar{y}) + \operatorname{int}_{L_S}(T_\nu)]$. Since $\hat{W}$ is a neighbourhood of $\hat{x}_S$ in X_S, there is $n > \nu$ such that $x_n \in \hat{W}$. If we put $y_n = x_n - \hat{x}_S + \bar{y}$, we have $y_n \in \operatorname{int}_{L_S}(T_\nu) \subseteq T_n$, and so $A(x_n)(x_n - y_n) = A(x_n)(\hat{x}_S - \bar{y}) < 0$. But, since $x_n \in \hat{U}$, we have also $A(x_n)(\hat{x}_S - \bar{y}) > 0$, a contradiction. Then, by condition 3) again, the point $\hat{x}_S$ must belong to $I_{K_S, D\cap S, X_S}$. Hence, if $y \in D \cap S$, there is $\lambda > 0$ such that $\hat{x}_S - \lambda y \in X_S$, and so $A(\hat{x}_S)(\lambda y) < 0$. Since $A(\hat{x}_S) \in C_V$, it follows that

$$\bigcup_{\varrho\in]0,1[} \varrho(A(\hat{x}_S))^{-1}(]0, +\infty[) \subseteq (A(\hat{x}_S))^{-1}(]0, +\infty[)\,.$$

Plainly, as X_S is convex, we can suppose that $\lambda < 1$. Hence, we have $A(\hat{x}_S)(y) < 0$. Summarizing, we have proved that, for each $S \in \mathcal{F}$, there is $\hat{x}_S \in K_S$ such that $\sup\limits_{y\in D\cap S} A(\hat{x}_S)(y) < 0$. Then, as $D \subseteq \bigcup\limits_{S\in\mathcal{F}} S$, it follows that $\limsup\limits_{S} A(\hat{x}_S)(y) < 0$ for all $y \in D$. Of course, from this, conclusion i) follows at once. Now, let $K \subseteq X$, with $\bigcup\limits_{S\in\mathcal{F}} K_S \subseteq K$, be such that the operator A is D-regular in K. Thus, to prove that $A(K) \in \mathcal{A}_D$, it suffices to show that condition b) of Definition 1.1 does not hold. To this end, observe that if $S_0 \in \mathcal{F}_D$, then, since the family $\mathcal{F}$ is directed by inclusion and $V = \bigcup\limits_{S\in\mathcal{F}} S$, it is possible to find $\hat{S} \in \mathcal{F}$ such that $S_0 \subseteq \hat{S}$. We know that there is $\hat{x}_{\hat{S}} \in K_{\hat{S}} \subseteq K \cap \hat{S}$ such that $\sup\limits_{y\in D\cap\hat{S}} A(\hat{x}_{\hat{S}})(y) < 0$. Thus, also conclusion ii) is proved. Finally, suppose that $\Gamma \in \mathcal{A}_D \cup \mathcal{B}_D$ and $A\Big(\bigcup\limits_{S\in\mathcal{F}} K_S\Big) \subseteq \Gamma$. If $\Gamma \notin \mathcal{A}_D$, then $\Gamma \in \mathcal{B}_D$. Hence, a fortiori, $A\Big(\bigcup\limits_{S\in\mathcal{F}} K_S\Big) \in \mathcal{B}_D$. But then, thanks to conclusion ii) (take into account Proposition 1.1), one has $A\Big(\bigcup\limits_{S\in\mathcal{F}} K_S\Big) \in \mathcal{A}_D$, and so, a fortiori, $\Gamma \in \mathcal{A}_D$. This contradiction proves conclusion iii).

The other main result is the following theorem.

THEOREM 2.2: *Let* $\mathcal{F} \in \mathfrak{U}_D$, *with* $V = \bigcup\limits_{S\in\mathcal{F}} S$, *and let* A *be an operator from* $X \subseteq E$ *into* $\hat{C}_V$. *Moreover, for each* $S \in \mathcal{F}$, *let* K_S *and* X_S *be two non-empty subsets of* $X \cap S$, *with* $K_S \subseteq X_S$, *satisfying the following conditions*:

1) K_S *is compact in* S *and* X_S *is convex and compact in* S;
2) *for every* $y \in X_S - X_S$, *the set* $\{x \in X_S\colon A(x)(y) < 0\}$ *is closed in* S;
3) *for every* $x \in X_S \setminus K_S$, *one has* $\sup\limits_{y\in X_S} A(x)(x - y) > 0$;
4) $\theta_E \in I_{K_S, D\cap S, X_S}$ *and, for every* $x \in K_S \setminus I_{K_S, D\cap S, X_S}$, *one has* $A(x)(x) > 0$.

Under such hypotheses, the conclusions of Theorem 2.1 *hold.*

PROOF: From the proof of Theorem 2.1, we know that conditions 1) and 2) are enough to show that, for each $S \in \mathcal{F}$, there exists $x_s \in X_s$ such that $\sup_{y \in X_s} A(x_s)(x_s - y) < 0$. Thanks to condition 3), one has $x_s \in K_s$. Thus, always by the proof of Theorem 2.1, to get our conclusions it suffices to show that $\sup_{y \in D \cap S} A(x_s)(y) < 0$ for all $S \in \mathcal{F}$. Fix, therefore, $S \in \mathcal{F}$ and $y \in D \cap S$. First, suppose $x_s \in I_{K_s, D \cap S, X_s}$. Thus, there is $\lambda > 0$ such that $x_s - \lambda y \in X_s$. Hence, $A(x_s)(\lambda y) < 0$. Since $A(x_s) \in \bar{C}_v$, one has $A(x_s)(y) < 0$. Now, suppose $x_s \notin I_{K_s, D \cap S, X_s}$. Then, by condition 4), one has $A(x_s)(x_s) > 0$. Since $\Theta_E \in I_{K_s, D \cap S, X_s}$ and X_s is convex, there is $\mu \in]0, 1[$ such that $-(1-\mu)y \in X_s$. Hence, one has

$$A(x_s)\left(\frac{\mu x_s + (1-\mu)\mu y}{\mu}\right) < 0 .$$

From this inequality, always because $A(x_s) \in \bar{C}_v$, it follows that

$$A(x_s)(\mu x_s + (1-\mu)\mu y) < 0 .$$

Further, one has

$$(1-\mu) A(x_s)(\mu y) < \mu A(x_s)(x_s) + (1-\mu) A(x_s)(\mu y) < A(x_s)(\mu x_s + (1-\mu)\mu y) < 0 .$$

Therefore, $A(x_s)(\mu y) < 0$, and so $A(x_s)(y) < 0$. This completes the proof.

3. - SOME CONSEQUENCES OF THEOREMS 2.1 AND 2.2

First, we want explicitly to point out that Theorems 2.1 and 2.2 are mutually independent. The reader can realize it by means of very simple examples.

Moreover, it is worth noticing that in the proof of Theorem 2.1 the following result is implicitly proved.

THEOREM 3.1: *Let X be a non-empty, convex and compact subset of $\mathbf{R}^n$ and let f be a real function on $X \times \mathbf{R}^n$ satisfying the following conditions:*

1) *for every $x \in X$, the function $f(x, \cdot)$ is concave;*
2) *for every $y \in X - X$, the function $f(\cdot, y)$ (resp. $x \to f(x, x-y)$) is lower semicontinuous.*

Then, for any convex real function ψ on $\mathbf{R}^n$, with $\sup_{x \in X} f(x, \Theta_{\mathbf{R}^n}) < \psi(\Theta_{\mathbf{R}^n})$ (resp. $f(x, x) < \psi(x)$ for all $x \in X$), there exists $\hat{x} \in X$ such that $f(\hat{x}, \hat{x} - y) < \psi(\hat{x} - y)$ (resp. $f(\hat{x}, y) < \psi(y)$) for all $y \in X$.

As already announced in the Introduction, Theorem 3.2 below is a much more sophisticated version of Theorem B.

THEOREM 3.2: *Let X be a finitely closed and convex subset of E, K a finitely compact subset of X, with $\Theta_E \in K$, τ a topology on K, with respect to which K is compact, f a real function on $X \times V$ satisfying the following conditions:*

1) *for every $x \in X$, the function $f(x, \cdot)$ is concave;*
2) *the function $f(\cdot, y)$ is finitely lower semicontinuous in X for every $y \in (X-X) \cap V$, is τ-lower semicontinuous in K for every $y \in D$, is finitely continuous in X and τ-continuous in K for $y = \Theta_E$.*

Then, for any convex real function ψ on V, with $\psi(\Theta_E) = 0$ and

$$f(x, x) > f(x, \Theta_E) + \psi(x) \qquad \text{for all } x \in (X \cap V) \setminus I_{K,D,X},$$

there exists $\hat{x} \in K$ such that

$$f(\hat{x}, y) < f(\hat{x}, \Theta_E) + \psi(y) \qquad \text{for all } y \in D.$$

PROOF: For each $S \in \mathcal{F}_D$, put $X_S = X \cap S$ and $K_S = K \cap S$. Of course, $X_S \setminus I_{K_S, D\cap S, X_S} \subseteq X \setminus I_{K,D,X}$. Now, define $A: X \to M_V$ by putting $A(x)(\cdot) = f(x, \cdot) - f(x, \Theta_E) - \psi(\cdot)$ for all $x \in X$. Thus, $A(X) \subseteq \hat{C}_V$, and so, a fortiori, $A(X) \subseteq C_V$. Thanks to condition 2), the function $A(\cdot)(y)$ is finitely lower semicontinuous in X for every $y \in (X-X) \cap V$ and τ-lower semicontinuous in K for every $y \in D$. Taking into accout that $\Theta_E \in K$ and that $A(x)(x) > 0$ for all $x \in (X \cap V) \setminus I_{K,D,X}$, we realize that each hypothesis of Theorem 2.1 is satisfied. On the other hand, thanks to Propositions 1.1 and 1.9, the operator A is D-regular in K. Therefore, by conclusion ii) of Theorem 2.1, there exists $\hat{x} \in K$ such that $\sup_{y \in D} A(\hat{x})(y) < 0$. This completes the proof.

REMARK 3.1: To deduce Theorem B from Theorem 3.2 take $V = E$, $X = K = D$, τ being the relativization to K of the given Hausdorff vector topology on E, $\psi = \Theta_{M_E}$ and then observe that a point $x \in X$ does not belong to $I_{X,X,X}$ if and only if $X \setminus \bigcup_{\lambda > 0} \lambda(x - X) \neq \emptyset$.

Now, we show how one can obtain, from Theorems 2.1 and 2.2, some improved versions of the celebrated results by H. Brézis [1] on equations involving operators of type (M). Thus, in the next six theorems, V, G and Φ are as in Proposition 1.12.

THEOREM 3.3: *Let $\mathcal{F} \in \mathfrak{U}_V$. For each $S \in \mathcal{F}$, let X_S be a closed convex of S and K_S be a non-empty compact subset of X_S such that, for every $x \in X_S \setminus I_{K_S, S, X_S}$, one has $\sup_{y \in K_S} \langle \Phi(x), x - y \rangle > 0$. Moreover, assume that the set $\overline{\bigcup_{S \in \mathcal{F}} K_S}$ is compact. Then, the set $\left\{x \in \overline{\bigcup_{S \in \mathcal{F}} K_S} : \Phi(x) = \Theta_G\right\}$ is non-empty and compact.*

Proof: Let $A_\Phi: V \to V^*$ be as in Proposition 1.12. It is easily seen that each assumption of Theorem 2.1 is satisfied, provided we take $D = V$ and $A = A_\Phi$. In particular, condition (2) follows from the fact that the operator Φ is finitely $w(G, V)$-continuous. On the other hand, by Proposition 1.12, the operator A_Φ is V-regular in $\overline{\bigcup_{S \in \mathcal{F}} K_S}$. Therefore, thanks to conclusion ii) of Theorem 2.1, one has $A_\Phi\left(\overline{\bigcup_{S \in \mathcal{F}} K_S}\right) \in A_V$, that is the set $\left\{x \in \overline{\bigcup_{S \in \mathcal{F}} K_S}: \Phi(x) = \Theta_a\right\}$ is non-empty. The compactness of this set follows at once from the fact that Φ is of type (M).

The following result is a remarkable particular case of Theorem 3.3.

Theorem 3.4: *Let K be a compact subset of V containing Θ_v. Suppose $\langle \Phi(x), x \rangle > 0$ for all $x \in V \setminus K$. Then, the set $\{x \in K: \Phi(x) = \Theta_a\}$ is non-empty and compact.*

Proof: Take $\mathcal{F} = \mathcal{F}_V$ and, for each $S \in \mathcal{F}_V$, $X_S = S$ and $K_S = K \cap S$. Then, apply Theorem 3.3 observing, in particular, that $I_{K \cap S, S, S} = K \cap S$.

Remark 3.2: Theorem 3.4 is a slight improvement of Theorem 11 of [1], since K is not assumed to be convex.

Theorem 3.5: *Let $\mathcal{F} \in \mathfrak{U}_V$. For each $S \in \mathcal{F}$, let X_S be a compact convex subset of S and K_S be a compact subset of X_S, with $\Theta_V \in I_{K_S, S, X_S}$, such that, for every $x \in X_S \setminus K_S$, one has $\sup_{y \in X_S} \langle \Phi(x), x - y \rangle > 0$, and, for every $x \in K_S \setminus I_{K_S, S, X_S}$, one has $\langle \Phi(x), x \rangle > 0$. Moreover, assume that $\overline{\bigcup_{S \in \mathcal{F}} K_S}$ is compact. Then, the set $\left\{x \in \overline{\bigcup_{S \in \mathcal{F}} K_S}: \Phi(x) = \Theta_a\right\}$ is non-empty and compact.*

Proof: The proof goes on exactly as that of Theorem 3.3, except that one must apply Theorem 2.2 instead of Theorem 2.1.

In particular, from Theorem 3.5, we get the following result.

Theorem 3.6: *Let X be a finitely compact convex subset of V and K be a compact subset of X, with $\Theta_V \in I_{K, V, X}$. Suppose $\langle \Phi(x), x \rangle > 0$ for all $x \in X \setminus K$ and $\langle \Phi(x), x \rangle > 0$ for all $x \in X \setminus I_{K, V, X}$. Then, the set $\{x \in K: \Phi(x) = \Theta_a\}$ is non-empty and compact.*

Proof: Apply Theorem 3.5 by taking $\mathcal{F} = \mathcal{F}_V$ and, for each $S \in \mathcal{F}$, $X_S = X \cap S$ and $K_S = K \cap S$.

Remark 3.3: When $X = K$, Theorem 3.6 reduces to Theorem 10 of [1].

Theorem 3.7: *Let $\mathcal{F} \in \mathfrak{U}_V$. For each $S \in \mathcal{F}$, let K_S, X_S be two non-empty subsets of S and Ψ_S be a continuous function from X_S into K_S satisfying the following conditions:*

1) *K_s is convex and compact, X_s is convex and closed,* $\dim(X_s) > 2$ *and* $K_s \subseteq \mathrm{int}_s(X_s)$;

2) $\{x \in X_s : \langle \Phi(x), x - \Psi_s(x) \rangle = 0\} \subseteq K_s$.

Moreover, assume that $\overline{\bigcup_{s \in \mathcal{F}} K_s}$ *is compact. Then, the set* $\left\{x \in \overline{\bigcup_{s \in \mathcal{F}} K_s} : \Phi(x) = \Theta_0\right\}$ *is non-empty and compact.*

Proof: Fix $S \in \mathcal{F}$. Condition 1) implies that the set $X_s \setminus K_s$ is connected. On the other hand, it is easily seen that the real function $x \to \langle \Phi(x), x - \Psi_s(x) \rangle$ is continuous in X_s. Therefore, thanks to condition 2), such function has a constant sign in $X_s \setminus K_s$. Taking into account Propositions 1.1 and 1.2, we can apply Theorem 2.1 to $(A_\Phi)|_{X_s}$ or to $-(A_\Phi)|_{X_s}$ according to whether the sign in $X_s \setminus K_s$ of the above function is $+$ or $-$. Thus, we get a point $\hat{x}_s \in K_s$ such that $\langle \Phi(\hat{x}_s), y \rangle = 0$ for all $y \in S$. Since this holds for every $S \in \mathcal{F}$, our conclusion follows at once from the fact that the set $\overline{\bigcup_{s \in \mathcal{F}} K_s}$ is compact and the operator Φ is of type (M).

As a particular case of Theorem 3.7, we obtain the following result.

Theorem 3.8: *Let* $\dim(V) > 2$ *and let* $\Psi : V \to V$ *be a finitely continuous function, with a finite-dimentional range, such that the closed convex hull of the set* $\Psi(V) \cup \{x \in V : \langle \Phi(x), x - \Psi(x) \rangle = 0\}$, *say* K, *is compact. Then, the set* $\{x \in K : \Phi(x) = \Theta_e\}$ *is non-empty and compact.*

Proof: Let $\mathcal{F}$ be the family of all $S \in \mathcal{F}_V$ such that $\dim(S) > 2$ and $\Psi(V) \subseteq S$. Of course, $\mathcal{F} \in \mathcal{U}_V$. For each $S \in \mathcal{F}$, take $X_s = S$, $K_s = K \cap S$ and $\Psi_s = \Psi|_s$. Now, apply Theorem 3.7.

Remark 3.4: When $\Theta_V \in K$ and the function Ψ is identically null, Theorem 3.8 reduces to Theorem 12 of [1].

Remark 3.5: For reasons of heuristic nature, we have limited ourselves to do a thorough comparison between some of the simplest particular cases of Theorems 2.1 and 2.2 and the original existence results by H. Brézis on operators of type (M), contained in his seminal paper [1]. Later, other mathematicians have proposed some variants of the notion of an operator of type (M) given by Brézis. For instance, P. Hess [4], in the setting of separable and reflexive Banach spaces, gave the notion of an operator of type (M) with respect to two spaces. Also the theory of Hess is a particular case of ours. But we do not want to insist on this, here. The reader himself, in any case, can check easily our assertion.

Now, we establish some further existence results (partially similar to the last six ones) for equations involving operators not necessarily of type (M).

Thus, in the next five theorems, X is a non-empty subset of E, K is a non-empty subset of X, τ is a topology on K, with respect to which K is compact, A is an operator from X into C_V, with $A(K) \subseteq \hat{M}_V$, the set D is symmetric.

THEOREM 3.9: *Let the hypotheses of Theorem 2.1 (resp. Theorem 2.2) be satisfied, with* $\bigcup_{S\in\mathcal{F}} K_S \subseteq K$. *Moreover, assume that, for every* $y \in D$, *the set* $\{x \in K: A(x)(y) = 0\}$ *is* $\mathfrak{k}$*-closed. Then, there exists* $\hat{x} \in K$ *such that* $A(\hat{x})(y) = 0$ *for all* $y \in D$.

PROOF: Thanks to Propositions 1.1 and 1.10, the operator A is D-regular in K. Thus our assertion is a simple consequence of conclusion ii) of Theorem 2.1.

We point out the two following particular cases of Theorem 3.9, whose proofs are similar to that of Theorems 3.4 and 3.6, respectively.

THEOREM 3.10: *Let* $V \subseteq X$ *and let* K *be finitely compact, with* $\Theta_X \in K$. *Moreover, assume that, for every* $y \in V$, *the set* $\{x \in V: A(x)(y) < 0\}$ *is finitely closed, that, for every* $y \in D$, *the set* $\{x \in K: A(x)(y) = 0\}$ *is* $\mathfrak{k}$*-closed and that* $A(x)(x) > 0$ *for all* $x \in V \setminus K$. *Then, the conclusion of Theorem 3.9 holds.*

THEOREM 3.11: *Let* X *be finitely compact and convex, and* K *be finitely compact, with* $\Theta_X \in I_{K,D,X}$. *Moreover, assume that:* $A(X) \subseteq \bar{C}_V$; *for every* $y \in (X - X) \cap V$, *the set* $\{x \in X \cap V: A(x)(y) < 0\}$ *is finitely closed; for every* $y \in D$, *the set* $\{x \in K: A(x)(y) = 0\}$ *is* $\mathfrak{k}$*-closed;* $A(x)(x) > 0$ *for all* $x \in X \setminus K$; $A(x)(x) > 0$ *for all* $x \in X \setminus I_{K,D,X}$. *Then, the conclusion of Theorem 3.9 holds.*

For the sake of completeness, now we state explicitly two theorems whose proofs are quite similar to that of Theorems 3.7 and 3.8, respectively.

THEOREM 3.12: *Let* $\mathcal{F} \in \mathfrak{U}_D$, *with* $V = \bigcup_{S\in\mathcal{F}} S$. *For each* $S \in \mathcal{F}$, *let* K_S, X_S *be two non-empty subsets of* $X \cap S$ *and* Ψ_S *be a continuous function from* X_S *into* K_S *satisfying the following conditions:*

1) K_S *is convex, compact in* S *and contained in* K, X_S *is convex and closed in* S, $\dim(X_S) > 2$ *and* $K_S \subseteq \mathrm{int}_S(X_S)$;
2) *the operator* A *is* $\mathfrak{t}_V$*-continuous in* X_S;
3) $\{x \in X_S: A(x)(x - \Psi_S(x)) = 0\} \subseteq K_S$.

Moreover, assume that $A(X \cap V) \subseteq V'$ *and that, for every* $y \in D$, *the set* $\{x \in K: A(x)(y) = 0\}$ *is* $\mathfrak{k}$*-closed. Then, the conclusion of Theorem 3.9 holds.*

REMARK 3.6: We want to point out that, in Theorem 3.12, the assumption « $A(X \cap V) \subseteq V'$ » serves only to prove that, for each $S \in \mathcal{F}$, the real function $x \to A(x)(x - \Psi_S(x))$ is continuous in X_S.

THEOREM 3.13: *Let* $\dim(V) > 2$, $V \subseteq X$ *and* K *be convex and finitely compact. Moreover, let* $\Psi: V \to V$ *be a function, with* $\dim(\Psi(V)) < +\infty$, *finitely continuous with respect to the Euclidean topology on* $\Psi(V)$, *such that* $\Psi(V) \cup \cup \{x \in V: A(x)(x - \Psi(x)) = 0\} \subseteq K$. *Finally, assume that the operator* A *is* $\mathfrak{t}_V$*-continuous in* V, *that* $A(V) \subseteq V'$ *and that, for every* $y \in D$, *the set* $\{x \in K: A(x)(y) = 0\}$ *is* $\mathfrak{k}$*-closed. Then, the conclusion of Theorem 3.9 holds.*

REMARK 3.7: Of course, it is possible to give the versions of Theorems 3.9-3.13 in terms of sets D and operators A satisfying the assumptions of Proposition 1.11. We leave to the reader the task of formulating the appropriate statements.

In the consequences of Theorems 2.1 and 2.2 we have presented up to now, only conclusion ii) of them has been utilized. In some of the next results, we shall utilize also conclusion i).

THEOREM 3.14: *Let X be a convex and finitely closed subset of E, K a finitely compact subset of X, with $\Theta_E \in K$, $A\colon X \to C_V$ an operator which is finitely τ_V-continuous in $X \cap V$. Suppose that there exists $\alpha > 0$ such that $A(x)(x) > \alpha$ for all $x \in (X \cap V) \setminus I_{K,D,X}$. Put*

$$P_{X\cap V} = \left\{\psi \in M_V\colon -\psi \in C_V \text{ and } \sup_{x\in X\cap V} \psi(x) \leq 1\right\}$$

and

$$Q_{X\cap V} = \left\{\psi \in M_V\colon -\psi \in C_V \text{ and } \sup_{x\in X\cap V} \psi(x) < +\infty\right\}.$$

Then, the following conclusions hold:

i) $\alpha P_{X\cap V} \subseteq \overline{(A(K\cap V))}_{\tau_D}$;

ii) *for every sequence $\{\beta_n\}$ in R^+, with $\sup\limits_{n\in\mathrm{N}} \beta_n = +\infty$, one has*

$$Q_{X\cap V} \subseteq \bigcup_{n\in\mathrm{N}} \overline{(\beta_n A(K\cap V))}_{\tau_D}.$$

PROOF: Let $\varepsilon \in]0, \alpha[$ and $\psi \in P_{X\cap V}$. For every $x \in (X\cap V)\setminus I_{K,D,X}$, one has $A(x)(x) - (\alpha-\varepsilon)\psi(x) > \alpha - (\alpha-\varepsilon) = \varepsilon$. Thus, arguing in a way by now evident, we realize that it is possible to apply Theorem 2.1 to the operator $x \to A(x) - (\alpha-\varepsilon)\psi$. Then, by conclusion i) of that theorem, one has

$$\Theta_{M_V} \in \overline{(A(K\cap V) - (\alpha-\varepsilon)\psi)}_{\tau_D}.$$

On the other hand, it is easily seen that

$$\overline{(A(K\cap V) - (\alpha-\varepsilon)\psi)}_{\tau_D} = \overline{(A(K\cap V))}_{\tau_D} - (\alpha-\varepsilon)\psi.$$

Hence, $(\alpha-\varepsilon)\psi \in \overline{(A(K\cap V))}_{\tau_D}$. Plainly, the function $\sigma \to \sigma\psi$, from R into M_V, is τ_D-continuous. Hence, $\alpha\psi \in \overline{(A(K\cap V))}_{\tau_D}$. This proves conclusion i). Now, let us prove conclusion ii). Let $\varphi \in Q_{X\cap V}$. Choose $n' \in \mathrm{N}$ such that $\sup\limits_{x\in X\cap V} \varphi(x) \leq \alpha\beta_{n'}$. Thus, $\varphi/\alpha\beta_{n'} \in P_{X\cap V}$. But then, thanks to conclusion i), one has $\varphi/\beta_{n'} \in \overline{(A(K\cap V))}_{\tau_D}$, and so $\varphi \in \overline{(\beta_{n'} A(K\cap V))}_{\tau_D}$. This completes the proof.

Proceeding as in the proof of Theorem 3.14, taking into account Proposition 1.1 and, respectively, Propositions 1.9, 1.10, 1.11, and using conclusion ii) of Theorem 2.1 instead of conclusion i), we obtain the three following results.

THEOREM 3.15: *Let the hypotheses of Theorem 3.14 be satisfied. In addition, let K be a compact topological space such that, for every $y \in D$, the function $A(\cdot)(y)$ is lower semicontinuous in K. Then, for every $\psi \in P_{X \cap V}$, there exists $\hat{x} \in K$ such that $A(\hat{x})(y) < \alpha\psi(y)$ for all $y \in D$. Moreover, for every sequence $\{\beta_n\}$ in $\mathbf{R}^+$, with $\sup_{n \in \mathbf{N}} \beta_n = +\infty$, and every $\varphi \in Q_{X \cap V}$ there exist $\bar{x} \in K$ and $\bar{n} \in \mathbf{N}$ such that $\beta_{\bar{n}} A(\bar{x})(y) < \varphi(y)$ for all $y \in D$.*

THEOREM 3.16: *Let the hypotheses of Theorem 3.14 be satisfied. In addition, let D be symmetric, $A(K) \subset \hat{M}_V$ and let K be a compact topological space such that, for every $y \in D$ and every $r \in \mathbf{R}$, the set $\{x \in K\colon A(x)(y) = r\}$ is closed. Then, for every $\psi \in P_{X \cap V} \cap \hat{M}_V$ there exists $\hat{x} \in K$ such that $A(\hat{x})(y) = \alpha\psi(y)$ for all $y \in D$. Moreover, for every sequence $\{\beta_n\}$ in $\mathbf{R}^+$, with $\sup_{n \in \mathbf{N}} \beta_n = +\infty$, and every $\varphi \in Q_{X \cap V} \cap \hat{M}_V$, there exist $\bar{x} \in K$ and $\bar{n} \in \mathbf{N}$ such that $\beta_{\bar{n}} A(\bar{x})(y) = \varphi(y)$ for all $y \in D$.*

REMARK 3.8: From Theorem 3.16 one can deduce at once Theorem I of [5].

THEOREM 3.17: *Let the hypotheses of Theorem 3.14 be satisfied. In addition, let D be the linear hull of a countable set $Y \subseteq V$, let $A(K) \subseteq V'$ and let K be a compact topological space such that, for every $y \in Y$ and every $r \in \mathbf{R}$, the set $\{x \in K\colon A(x)(y) = r\}$ is closed. Then, the conclusions of Theorem 3.16 hold with V' instead of $\hat{M}_V$.*

Unless the operator A has some homogeneity property, Theorems 3.14-3.17 cannot provide proper density or surjectivity results concerning A. To get them we need a more stringent coercivity condition. Now, we present the versions of the above quoted theorems where this new coercivity condition appears.

THEOREM 3.18: *Let X, K be as in Theorem 3.14, $W \subseteq E$ a cone containing X, $A\colon W \to V'$ an operator which is finitely $\mathfrak{t}_V$-continuous in $W \cap V$. Further, let $\{\gamma_n\}$ and $\{\delta_n\}$ be two sequences in $\mathbf{R}^+$, with $\sup_{n \in \mathbf{N}} \delta_n/\gamma_n = +\infty$, such that, for every $n \in \mathbf{N}$ and every $x \in (X \cap V) \setminus I_{K,D,X}$, one has $A(\gamma_n x)(\gamma_n x) > \delta_n$. Then, one has*

$$Q_{X \cap V} \cap V' \subseteq \bigcup_{n \in \mathbf{N}} \overline{(A(\gamma_n K \cap V))}_{\tau_D}.$$

PROOF: Let $\psi \in Q_{X \cap V} \cap V'$. Fix $\bar{n} \in \mathbf{N}$ such that $\sup_{x \in X \cap V} \psi(x) < \delta_{\bar{n}}/\gamma_{\bar{n}}$. Since $(\gamma_{\bar{n}} X \cap V) \setminus I_{\gamma_{\bar{n}} K, D, \gamma_{\bar{n}} X} = \gamma_{\bar{n}}((X \cap V) \setminus I_{K,D,X})$, by hypothesis, one has $A(y)(y) > \delta_{\bar{n}}$ for all $y \in (\gamma_{\bar{n}} X \cap V) \setminus I_{\gamma_{\bar{n}} K, D, \gamma_{\bar{n}} X}$. Therefore, thanks to conclusion i) of Theorem 3.14, one has $\delta_{\bar{n}} P_{\gamma_{\bar{n}} X \cap V} \subseteq \overline{(A(\gamma_{\bar{n}} K \cap V))}_{\tau_D}$. Now, observe that

$\sup_{x \in X \cap Y} \psi(\gamma_{\bar{n}} x) < \delta_{\bar{n}}$, because $\psi \in V'$. Hence, $\psi \in \delta_{\bar{n}} P_{\gamma_{\bar{n}} X \cap Y}$. This completes the proof.

Proceeding as in the proof of Theorem 3.18 and using, respectively, Theorems 3.15, 3.16, 3.17, we obtain the three following results.

THEOREM 3.19: *Let the hypotheses of Theorem 3.18 be satisfied. In addition, for each $n \in \mathbf{N}$, let τ_n be a topology on $\gamma_n K$, with respect to which $\gamma_n K$ is compact, such that, for every $y \in D$, the function $A(\cdot)(y)$ is τ_n-lower semicontinuous in $\gamma_n K$. Then, for every $\psi \in Q_{X \cap Y} \cap V'$, there exist $\bar{x} \in K$ and $\bar{n} \in \mathbf{N}$ such that $A(\gamma_{\bar{n}} \bar{x})(y) < \psi(y)$ for all $y \in D$.*

THEOREM 3.20: *Let the hypotheses of Theorem 3.18 be satisfied. In addition, let D be symmetric and, for each $n \in \mathbf{N}$, let τ_n be a topology on $\gamma_n K$, with respect to which $\gamma_n K$ is compact, such that, for every $y \in D$ and every $r \in \mathbf{R}$, the set $\{x \in \gamma_n K: A(x)(y) = r\}$ is τ_n-closed. Then, for every $\psi \in Q_{X \cap Y} \cap V'$, there exist $\bar{x} \in K$ and $\bar{n} \in \mathbf{N}$ such that $A(\gamma_{\bar{n}} \bar{x})(y) = \psi(y)$ for all $y \in D$.*

THEOREM 3.21: *Let the hypotheses of Theorem 3.18 be satisfied. In addition, let D be the linear hull of a countable set $Y \subseteq V$ and, for each $n \in \mathbf{N}$, let τ_n be a topology on $\gamma_n K$, with respect to which $\gamma_n K$ is compact, such that, for every $y \in Y$ and every $r \in \mathbf{R}$, the set $\{x \in \gamma_n K: A(x)(y) = r\}$ is τ_n-closed. Then, the conclusion of Theorem 3.20 holds.*

Now, we present a remarkable application of Theorem 3.21.

THEOREM 3.22: *Let $(E, \|\cdot\|)$ be a reflexive Banach space; W a closed linear subspace of E such that $W \cap V$ is the linear hull of a countable set $Y \subseteq E$; G a linear subspace of E, containing V, such that $W \cap V$ is dense in G; $T: W \to G^*$ an operator such that, for every $y \in V$, the functional $T(\cdot)(y)$ is finitely continuous in $W \cap V$. Moreover, let $\{\gamma_n\}$ and $\{\delta_n\}$ be two sequences in $\mathbf{R}^+$, with $\sup_{n \in \mathbf{N}} \delta_n / \gamma_n = +\infty$, such that, for every $n \in \mathbf{N}$ and every $x \in W \cap V$ with $\|x\| = \gamma_n$, one has $T(x)(x) > \delta_n$. Finally, suppose that, for every $y \in Y$ and every $r \in \mathbf{R}$, the set $\{x \in W: T(x)(y) = r\}$ is sequentially weakly closed.*

Under such hypotheses, one has $T(W) = G^*$.

PROOF: Take $X = K = \{x \in W: \|x\| < 1\}$ and $D = W \cap V$. So that

$$(X \cap V) \setminus I_{K,D,X} = \{x \in W \cap V: \|x\| = 1\}.$$

Moreover, for each $n \in \mathbf{N}$, choose as τ_n the relative weak topology on $\gamma_n X$. Since W is reflexive, $\gamma_n X$ is weakly compact. For every $y \in Y$ and every $r \in \mathbf{R}$, the set $\{x \in \gamma_n X: T(x)(y) = r\}$ is sequentially weakly closed and bounded, and so, by Theorem 7 on p. 313 of [7], it is weakly closed. Therefore, we can apply Theorem 3.21 by taking as A the operator $x \to T(x)(\cdot)|_Y$ $(x \in W)$. First, observe that $V^* \subseteq Q_{X \cap Y} \cap V'$. Then, given $\psi \in G^*$, by Theorem 3.21, there exists $\hat{x} \in W$ such that $T(\hat{x})(y) = \varphi(y)$ for all $y \in W \cap V$. Since $W \cap V$ is dense in G, it follows that $T(\hat{x})(y) = \varphi(y)$ for all $y \in G$. This proves that $T(W) = G^*$.

REMARK 3.9: A natural way of finding two sequences $\{\gamma_n\}$, $\{\delta_n\}$ as in the statement of Theorem 3.22 is to assume that the following classical coercivity condition holds:

$$\lim_{\substack{\|x\| \to \infty \\ x \in W \cap V}} \frac{T(x)(x)}{\|x\|} = +\infty .$$

Indeed, if this condition is satisfied, then for every $n \in \mathbf{N}$, there is $\varrho_n > 0$ such that $T(x)(x) > n\|x\|$ for all $x \in W \cap V$ with $\|x\| > \varrho_n$. Thus, it suffices to take $\gamma_n = \varrho_n$ and $\delta_n = n\varrho_n$.

The final part of the paper, starting now, is devoted to some consequences of Theorems 2.1 and 2.2 in the setting of Hilbert spaces. Thus, from now on, $(E, (\cdot, \cdot))$ is a Hilbert space.

THEOREM 3.23: *Let X be a closed, convex, bounded subset of E, with $\Theta_E \in \operatorname{int}(X)$. Let Φ: $X \to E$ be an operator such that, for every $y \in V$, the set $\{x \in X \cap V$: $(x - \Phi(x), y) < 0\}$ is finitely closed. Suppose that V is dense in E and that, for every $y \in V$, the set $\{x \in X$: $(x - \Phi(x), y) = 0\}$ is weakly closed. Finally, suppose that $(\Phi(x), x) < \|x\|^2$ for all $x \in \partial X \cap V$.*

Under such hypotheses, there exists $\hat{x} \in X$ such that $\hat{x} = \Phi(\hat{x})$.

PROOF: Let A be the operator, from X into V^*, defined by $A(x)(\cdot) = (x - \Phi(x), \cdot)|_V$ $(x \in X)$. Moreover, take $D = V$ and, for each $S \in \mathcal{F}_V$, $X_S = K_S = X \cap S$. It is easily seen that, with these choices, the assumptions of Theorem 2.2 are satisfied. Since X is weakly compact, Propositions 1.1 and 1.10 enable us to use conclusion ii) of Theorem 2.2, with $K = X$. Therefore, there exists $\hat{x} \in X$ such that $(\hat{x} - \Phi(\hat{x}), y) = 0$ for all $y \in V$. Our conclusion follows then from the density of V in E.

REMARK 3.10: Theorem 3.23 improves Theorem 6 of [8]. Moreover, if V is the linear hull of a countable set $Y \subseteq E$, then, thanks to Proposition 1.11, it suffices to suppose that the set $\{x \in X$: $(x - \Phi(x), y) = 0\}$ is weakly closed only for each $y \in Y$.

The following proposition will be useful in the proof of Theorem 3.24.

PROPOSITION 3.1: *Let W be a closed linear subspace of E, $r > 0$, $X = \{v \in W$: $\|v\| < r\}$, $x \in W$, $\|x\| = r$, $z \in E$. Then, the following conditions are equivalent:*

a) $\sup_{y \in X} (z, x - y) > 0$;

b) $z \notin W^\perp$ *and* $x \notin \bigcup_{\lambda < 0} \lambda \Pi_W(z)$, *where $W^\perp$ is the orthogonal complement of W and $\Pi_W(z)$ is the orthogonal projection of z on W.*

Proof: Let a) hold. Then, $(z, x) > \inf_{y \in X}(z, y)$. So, in particular, $z \notin W^{\perp}$. Observe that, given any $\psi \in W^* \setminus \{\Theta_{W^*}\}$, the equation $\psi(v) = \inf_{y \in X} \psi(y)$ admits a unique solution in X given by $-rv_\psi/\|v_\psi\|$, where v_ψ denotes the point of W which, by Riesz's theorem, represents ψ. Now, apply this observation to the functional $\psi = (z, \cdot)|_W$. In this case, one has $v_\psi = \Pi_W(z)$. Hence, $x \neq -(r\Pi_W(z))/\|\Pi_W(z)\|$. Of course, this implies the second assertion of b), since if $x = \lambda \Pi_W(z)$, for some $\lambda < 0$, then, recalling that $\|x\| = r$, we would have $r = |\lambda| \, \|\Pi_W(z)\|$, and so $\lambda = -r/\|\Pi_W(z)\|$, a contradiction. Now, let b) hold. Suppose that a) does not hold. Then, on the basis of the above discussion, one has necessarily $x = -(r\Pi_W(z))/\|\Pi_W(z)\|$, since $x \in X$. This contradiction completes the proof.

Theorem 3.24: *Let E be infinite-dimensional and separable and V be the linear hull of an orthonormal basis, $\{e_n\}$, of E. For each $n \in \mathbf{N}$, let S_n be the linear hull of $\{e_1, \ldots, e_n\}$. Further, let $r > 0$, $X = \{x \in E : \|x\| < r\}$ and let $\{\varphi_n\}$ be a sequence of real functions on X such that $\sum_{n=1}^{\infty} |\varphi_n(x)|^2 < +\infty$ for all $x \in X$. Suppose that, for each $n \in \mathbf{N}$, the function φ_n is finitely continuous in $X \cap V$ and that the set $\{x \in X : \varphi_n(x) = 0\}$ is weakly closed. Finally, let $\hat{\mathbf{N}}$ be an infinite subset of $\mathbf{N}$ and $\{r_n\}$ be a sequence in $]0, r]$ such that, for every $n \in \hat{\mathbf{N}}$ and every $x \in \partial((r_n/r)X) \cap S_n$, there exist $v \in S_n$ and $k \in \mathbf{N}$, with $k \leq n$, such that $\sum_{i=1}^{n} \varphi_i(x)(v, e_i) \neq 0$ and $\varphi_k(x)(x, e_k) > 0$.*

Under such hypotheses, there exists $\hat{x} \in X$ such that $\varphi_n(\hat{x}) = 0$ for all $n \in \mathbf{N}$.

Proof: For each $x \in X$, put $\Phi(x) = \sum_{n=1}^{\infty} \varphi_n(x) e_n$. Thanks to the Riesz-Fischer theorem, the operator $\Phi : X \to E$ is well defined. Let A be the operator, from X into V^*, defined by $A(x)(\cdot) = (\Phi(x), \cdot)|_V$ $(x \in X)$. Fix $n' \in \hat{\mathbf{N}}$ and $x \in \partial((r_{n'}/r)X) \cap S_{n'}$. In correspondence to them, consider v and k as in the statement. Then, since $\sum_{i=1}^{n'} \varphi_i(x)(v, e_i) = (\Phi(x), v)$, the inequality $\sum_{i=1}^{n'} \varphi_i(x)(v, e_i) \neq 0$ implies directly that $\Phi(x) \notin S_{n'}^{\perp}$. On the other hand, since $\Pi_{S_{n'}}(\Phi(x)) = \sum_{i=1}^{n'} \varphi_i(x) e_i$, the inequality $\varphi_k(x)(x, e_k) > 0$ implies easily that

$$x \notin \bigcup_{\lambda < 0} \lambda \Pi_{S_{n'}}(\Phi(x)) .$$

Therefore, thanks to Proposition 3.1, one has

$$\sup_{y \in (r_{n'}/r)X \cap S_{n'}} (\Phi(x), x - y) > 0 .$$

Now, if we take $D = V$, $\mathcal{F} = \{S_n : n \in \hat{\mathbf{N}}\}$ and, for each $n \in \hat{\mathbf{N}}$, $X_{S_n} = K_{S_n} = (r_n/r)X \cap S_n$, we realize that the hypotheses of Theorem 2.1 are satisfied. In particular, condition (3) follows from the above discussion, while condition (2) follows from the fact that, for every $y \in V$, the function $A(\cdot)(y)$ turns out to be the sum of a finite number of functions which are finitely continuous in $X \cap V$. Finally, taking into account that, for each $n \in \mathbf{N}$, the set $\{x \in X\colon A(x)(e_n) = 0\}$ is weakly closed, being equal to $\{x \in X\colon \varphi_n(x) = 0\}$, we see that Propositions 1.1 and 1.11 allow us to utilize conclusion ii) of Theorem 2.1, with $K = X$. Therefore, there exists $\hat{x} \in X$ such that $(\Phi(\hat{x}), y) = 0$ for all $y \in V$. But V is dense in E, and so $\Phi(\hat{x}) = \theta_E$. This means that $\varphi_n(\hat{x}) = 0$ for all $n \in \mathbf{N}$.

Remark 3.11: The conclusion of Theorem 3.24 is still true if, instead of the last assumption of it, one assumes that, for every $n \in \hat{\mathbf{N}}$ and every $x \in \partial((r_n/r)X) \cap S_n$, one has $\sum_{i=1}^{n} \varphi_i(x)(x, e_i) > 0$. To see this, one must apply Theorem 2.2 instead of Theorem 2.1.

Our next result is a generalization of the classical Lax-Milgram theorem.

Theorem 3.25: *Let $\Phi\colon E \to E$ be a continuous linear operator and $\{y_1, \dots, y_k\}$ be a finite orthonormal subset of E such that*

$$\inf_{\|x\|=1}\left[(\Phi(x), x) + \sqrt{\sum_{i=1}^{k} |(\Phi(x), y_i)|^2}\right] > 0 .$$

Then, one has $\Phi(E) = E$.

Proof: Put

$$\alpha = \inf_{\|x\|=1}\left[(\Phi(x), x) + \sqrt{\sum_{i=1}^{k} |(\Phi(x), y_i)|^2}\right].$$

Denote by $\hat{S}$ the linear hull of $\{y_1, \dots, y_k\}$ and by X the closed unit ball of E. Fix $x \in \partial X$ and $z \in (\alpha/3)X$. Observe that

$$\sqrt{\sum_{i=1}^{k} |(\Phi(x), y_i)|^2} = \|\Pi_{\hat{S}}(\Phi(x))\| = \sup_{y \in X \cap \hat{S}} (\Phi(x), y) .$$

Thus, there is $\hat{y} \in X \cap \hat{S}$ such that

$$(\Phi(x), \hat{y}) = -\sqrt{\sum_{i=1}^{k} |(\Phi(x), y_i)|^2} .$$

Then, one has

$$(\Phi(x) - z, x - \hat{y}) > (\Phi(x), x) + \sqrt{\sum_{i=1}^{k} |(\Phi(x), y_i)|^2} - \|z\|\,\|x - \hat{y}\| > \alpha - \frac{2\alpha}{3} > 0 .$$

Now, if we take

$$D = V = E , \quad \mathcal{F} = \{S \in \mathcal{F}_E\colon \hat{S} \subseteq S\} , \quad X_S = K_S = X \cap S \qquad (S \in \mathcal{F}) ,$$

$$A(x)(\cdot) = (\Phi(x) - z, \cdot) \qquad (x \in X) ,$$

we see that the hypotheses of Theorem 2.1 are satisfied. Obviously, for every $y \in E$, the set $\{x \in X: A(x)(y) = 0\}$ is weakly closed. Thus, thanks to Propositions 1.1 and 1.10, conclusion ii) of Theorem 2.1 holds, with $K = X$. Therefore, there exists $\hat{x} \in X$ such that $\Phi(\hat{x}) = z$. Hence, $(\alpha/3)X \subseteq \Phi(X)$. Then, as $\Phi(E)$ is a linear subspace of E, one has $\Phi(E) = E$.

In particular, from Theorem 3.25, we obtain directly the following result.

Theorem 3.26: *Let* $\dim(E) < +\infty$ *and let* $\Phi: E \to E$ *be a linear operator such that* $\inf_{\|x\|=1} [(\Phi(x), x) + \|\Phi(x)\|] > 0$.

Then, one has $\Phi(E) = E$.

Theorem 3.26 is no longer true if $\dim(E) = +\infty$. Indeed, we have the following result.

Theorem 3.27: *Let* E *be infinite-dimensional and separable. Then, there exists a linear isometry* $\Phi: E \to E$, *with* $\Phi(E) \neq E$, *such that*

$$\inf_{\|x\|=1} [(\Phi(x), x) + \|\Phi(x)\|] > 0 .$$

Proof: Fix an orthonormal basis $\{e_n\}$ of E. Next, define an infinite matrix $[a_{kn}]$ $(k, n \in \mathbb{N})$ in the following way:

$$a_{kn} = \begin{cases} 1 & \text{if } k = n = 1 , \\ 0 & \text{if } n > 1 \text{ and } (k - 2n + 2)(k - 2n + 1) \neq 0 , \\ \dfrac{1}{\sqrt{2}} & \text{if } n > 1 \text{ and } k = 2n - 2 , \\ -\dfrac{1}{\sqrt{2}} & \text{if } n > 1 \text{ and } k = 2n - 1 . \end{cases}$$

It is easy to see that, for every $\{\xi_n\} \in l^2$, one has $\sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} a_{kn} \xi_n \right|^2 = \sum_{k=1}^{\infty} |\xi_k|^2$. This implies that the matrix $[a_{kn}]$ represents a continuous linear operator $\Phi: E \to E$, by means of the equality $\Phi(x) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{kn}(x, e_n) \right) e_k$ $(x \in E)$. Thanks to Parseval's identity and to the previous relation again, one has

$$\|\Phi(x)\|^2 = \sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} a_{kn}(x, e_n) \right|^2 = \sum_{k=1}^{\infty} |(x, e_k)|^2 = \|x\|^2 \qquad \text{for all } x \in E .$$

Hence, Φ is a linear isometry. Moreover, it is easily seen that, for every $x \in E$, one has also $(\Phi(x), x) > -\|x\|^2/\sqrt{2}$. Therefore,

$$\inf_{\|x\|=1} [(\Phi(x), x) + \|\Phi(x)\|] > 1 - \frac{1}{\sqrt{2}} > 0 .$$

Let us show, finally, that Φ is not surjective. To this end, observe that if β is any positive real number and p is any odd integer such that $\sum_{k=1}^{p} 1/k > \beta$, then one has

$$\sum_{k=1}^{p} \frac{1}{k} > \beta \sum_{n=1}^{\infty} \left| \sum_{k=1}^{p} a_{kn} \right|^2 .$$

This fact, thanks to a well-known result (see [9], p. 49), implies that the point $\sum_{n=1}^{\infty} e_n/n$ does not belong to $\Phi(E)$.

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