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GIOVANNI PROUSE - ANNA ZARETTI (*)

On the inequalities associated to a model of Graffi for the motion of a mixture of two viscous incompressible fluids (**) (***)

SUMMARY. — A global existence and uniqueness theorem is proved for the solution of a system of differential inequalities associated to a model of Graffi relative to the motion of a mixture of two viscous incompressible fluids.

Sulle disuguaglianze associate a un modello di Graffi per il moto di una miscela di due fluidi viscosi ed incomprimibili

RIASSUNTO. — Si dimostra un teorema di esistenza ed unicità in grande della soluzione di un sistema di disequazioni differenziali associato ad un modello di Graffi per lo studio del moto di un miscuglio di due fluidi viscosi incomprimibili.

1. - INTRODUCTION

The study of the motion of a mixture of two viscous, incompressible fluids in a closed basin is of particular interest, for example, in the analysis of problems connected with pollution. The equations can be deduced, under more or less stringent hypotheses, from the general equations governing the motion of a mixture, these in turn being obtained from the principles of conservation of mass and momentum.

(*) Indirizzo degli Autori: Dipartimento di Matematica del Politecnico di Milano.
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Let Ω be the part of the $x=(x_1, x_2, x_3)$-space $\mathbb{R}^3$ in which the motion takes place and let α, β be the two components of the mixture. We introduce the following notations:

— ϱ_α, ϱ_β densities of the fluids (which we assume are constants);
— $c_{v\alpha}(x, t)$ local volume concentration of the fluid α;
— $c_{m\alpha}(x, t)$ local mass concentration of the fluid α;
— $\varrho(x, t) = c_{v\alpha}\varrho_\alpha + (1 - c_{v\alpha})\varrho_\beta$ local density of the mixture;
— μ viscosity of the mixture (which we assume is constant);
— $\boldsymbol{f}(x, t)$ external force;
— $p(x, t)$ pressure;
— $\boldsymbol{u}_\alpha(x, t)$, $\boldsymbol{u}_\beta(x, t)$ velocities of the fluids;
— $\boldsymbol{u}_m(x, t) = c_{m\alpha}\boldsymbol{u}_\alpha + (1 - c_{m\alpha})\boldsymbol{u}_\beta$ mass velocity of the mixture;
— $\boldsymbol{u}_v(x, t) = c_{v\alpha}\boldsymbol{u}_\alpha + (1 - c_{v\alpha})\boldsymbol{u}_\beta$ volume velocity of the mixture.

The fundamental equations of motion are given in $Q = \Omega \times [0, T]$ by ([1])

$$
(1.1) \qquad \left\{
\begin{aligned}
&\varrho\left(\frac{\partial \boldsymbol{u}_m}{\partial t} + (\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m - \boldsymbol{f}\right) = -\nabla p - \mu\,\Delta \boldsymbol{u}_m + \frac{\mu}{3}\nabla(\nabla \cdot \boldsymbol{u}_m),\\
&\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{u}_m) = 0,\\
&\nabla \cdot \boldsymbol{u}_v = 0,
\end{aligned}
\right.
$$

with suitable initial and boundary conditions.

From Fick's experimental law (see for instance [1]) which connects the local velocities of the fluids with the mass velocity, the following relationship can easily be deduced

$$
(1.2) \qquad \boldsymbol{u}_m = \boldsymbol{u}_v - \frac{\lambda}{\varrho}\nabla\varrho,
$$

where $\lambda > 0$ is the *molecular diffusion coefficient*, assumed constant ([2]).

Substituting (1.2) into (1.1) we obtain equations expressed entirely in terms of the mass or the volume velocity. Since the condition $\nabla \cdot \boldsymbol{u}_v = 0$ is of considerable importance in the theoretical study of the system of equations, in the sequel we shall consider the equations written in terms of the volume velocity (which, for the sake of simplicity, will from now on be denoted

([1]) In what follows we shall always assume that the equations are written in dimensionless form.
([2]) In particular, our model assumes that the quantities λ and μ do not vary with the concentration.

by $\boldsymbol{u}$). This gives

$$
(1.3)\qquad \begin{cases} \varrho\left(\dfrac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\boldsymbol{f}\right)-\lambda(\boldsymbol{u}\cdot\nabla)\nabla\varrho-\lambda(\nabla\varrho\cdot\nabla)\boldsymbol{u}+ \\ \qquad +\dfrac{\lambda^2}{\varrho}(\nabla\varrho\cdot\nabla)\nabla\varrho-\dfrac{\lambda^2}{\varrho^2}(\nabla\varrho\cdot\nabla\varrho)\nabla\varrho+\dfrac{\lambda^2}{\varrho}\nabla\varrho\cdot\nabla\varrho=-\nabla\tilde{p}+\mu\,\Delta\boldsymbol{u}, \\ \dfrac{\partial\varrho}{\partial t}+\boldsymbol{u}\cdot\nabla\varrho=\lambda\,\Delta\varrho, \\ \nabla\cdot\boldsymbol{u}=0, \end{cases}
$$

where

$$
\tilde{p}=p-\lambda^2\Delta\varrho+\lambda(\boldsymbol{u}\cdot\nabla)\varrho+\frac{\mu}{3}\nabla\frac{\lambda}{\varrho}\Delta\varrho+\mu\lambda\,\Delta\log\varrho\,.
$$

We shall always assume in what follows that the open, bounded set Ω is either of class C^3 or is convex, with boundary Γ constituted by a finite number of surfaces of class C^2. Thus, in addition to the usual initial conditions

$$
(1.4)\qquad \boldsymbol{u}(x,0)=\bar{\boldsymbol{u}}(x)\,,\quad \varrho(x,0)=\bar{\varrho}\qquad x\in\Omega\,,
$$

we have the classical boundary conditions

$$
(1.5)\qquad \boldsymbol{u}(x,t)=0\,,\qquad \frac{\partial\varrho(x,t)}{\partial\boldsymbol{\nu}}=0\qquad x\in\Omega\,,\ t\in(0,T)\,,
$$

which interpret the fact that the velocity and the density flux vanish on Γ, where $\boldsymbol{\nu}$ debotes the normal to Γ. In the sequel *we shall always assume that* $0<\varrho_1<\bar{\varrho}(x)<\varrho_2<+\infty$.

This model has been studied by Beirão da Veiga [2] and, in the case $\Omega=\mathbf{R}^3$, by Secchi [3]; in the inviscid case ($\mu=0$) it has also been studied by Beirão da Veiga, Serapioni and Valli [4]. All these authors have proved *local* existence and uniqueness theorems, under the assumption that Γ is « sufficiently smooth » (for example of class C^4).

Previously, other simplified models had been considered by various authors. Kazhikov and Smagulov [5] have studied a model obtained from (1.3) by eliminating the terms containing λ^2 (which, being in general λ small, are assumed to be negligeable); under the additional assumption that μ is « large », Kazhikov and Smagulov prove a *local* existence and uniqueness theorem for a « strong » solution and a *global* existence theorem for a (not necessarily unique) « turbulent » solution.

If all the terms containing λ and λ^2 are omitted, one obtains from (1.3) a model corresponding to the equations

$$
(1.6) \qquad \left|\begin{array}{l}
\varrho\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f}\right) = -\nabla p + \mu\,\Delta\boldsymbol{u}\,, \\[2ex]
\dfrac{\partial \varrho}{\partial t} + (\boldsymbol{u}\cdot\nabla)\varrho = 0\,, \\[2ex]
\nabla\cdot\boldsymbol{u} = 0\,.
\end{array}\right.
$$

These have been extensively studied by many authors; in particular, Antonov and Kazhikov [6] and Ladyzenskaja and Solonnikov [7] (see also Lions [8]) have extended existence and uniqueness results for the incompressible Navier-Stokes equations to the system (1.6).

Such a system coincides with the equations of motion of a viscous, incompressible, inhomogeneous fluid and corresponds to the case when, in the motion of the mixture, the molecular diffusion is negligeable; it is, therefore, in some cases, an oversimplified model.

Another model associated to the motion of a mixture of two incompressible fluids has been introduced by Graffi [9].

Starting from equations (1.6) and observing that, if the molecular diffusion had to be taken into account, the term $\lambda\,\Delta\varrho$ would have to be added to the right hand side of the second of (1.6), Graffi proposed the system

$$
(1.7) \qquad \left|\begin{array}{l}
\varrho\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f}\right) = -\nabla p + \mu\,\nabla\boldsymbol{u}\,, \\[2ex]
\dfrac{\partial \varrho}{\partial t} + \boldsymbol{u}\cdot\nabla\varrho = \lambda\,\Delta\varrho\,, \\[2ex]
\nabla\cdot\boldsymbol{u} = 0\,.
\end{array}\right.
$$

It is worth noting that (1.7) can be obtained from (1.3) by neglecting only in the first equation all terms involving λ. This procedure, which at first sight may appear incoherent, can in fact be justified by observing that the elimination of the term in λ in the second of (1.7) would critically modify the system by transforming the parabolic second equation into a first order one; on the other hand, elimination of the terms in λ^2 and λ from the first of (1.3) simplifies the system in a practically useful way without changing its mathematical and physical features.

We shall therefore adopt this approach to the problem and consider in the sequel system (1.7) which, by what has been said above, will be called the *Graffi model*. To equations (1.7) we associate the initial and boundary conditions (1.4), (1.5); it is obvious that the results obtained by Beirão da Veiga and by Kazhikov and Smagulov recalled above hold also, in particular, in our case. On the other hand, the problem of the *global existence and uniqueness* of a solution of (1.7), (1.4), (1.5) is still open.

Although not usually noted in practice, it is important for our particular purposes to observe that *any mathematical model should include a set of appropriate consistency conditions which define analytically the physical conditions under which the model maintains its validity or, in other words, is physically consistent.*

For the general model characterized by (1.1), it is evident that the consistency conditions must correspond to the following physical assumptions:

a) the velocities $|\boldsymbol{u}_\alpha|$, $|\boldsymbol{u}_\beta|$ must not approach the velocity of light, since the model is not relativistic; hence $|\boldsymbol{u}_m|$, $|\boldsymbol{u}_v|$ must be bounded;

b) the density ϱ of the mixture must be strictly positive and bounded;

c) the pressure p must be bounded;

d) the internal stress must be bounded; hence, $|\nabla\cdot\boldsymbol{u}_m|$ must be bounded. The same conditions must obviously hold also for the Graffi model (1.8).

Since we shall be concerned with « strong » solutions (in appropriate function spaces) of (1.7), condition *c*) follows from the other conditions (see the remark at the end of section 3). Moreover it follows from the second equation of (1.7), from the initial and boundary conditions (1.4) and (1.5) and from the maximum principle (see, for instance, [10]) that if $0<\varrho_1<\bar{\varrho}<\varrho_2$, then $0<\varrho_1<\varrho(x,t)<\varrho_2$.

On the other hand, condition *a*) can be expressed by

$$|\boldsymbol{u}|<M_1\,, \tag{1.8}$$

while, by (1.2) and conditions *a*), *b*),

$$|\nabla\varrho|<M_2\,; \tag{1.9}$$

moreover, applying to (1.2) the operator $\nabla\cdot$ and bearing in mind *d*), we have

$$|\Delta\varrho|<M_3\,, \tag{1.10}$$

from which also follows, by (1.8) and the second of (1.7) that

$$\left|\frac{\partial\varrho}{\partial t}\right|<M_4\,. \tag{1.11}$$

Thus (1.8), (1.9), (1.10), (1.11) are the *consistency conditions* for Graffi's model.

In order to take the consistency conditions into account when studying the model, we shall require the solutions to belong to appropriate convex sets; this, in turn, leads us to replace (1.7) by a suitable system of *differential* *inequalities.* We shall replace (1.7) by the system

$$
(1.12)\qquad
\left|\begin{aligned}
&\int_0^t\!\!\int_\Omega \left[\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\boldsymbol{f}\right)+\nabla p-\mu\,\Delta\boldsymbol{u}\right](\boldsymbol{u}-\boldsymbol{\varphi})\,d\Omega\,d\eta\leqslant 0\,,\\
&\int_0^t\!\!\int_\Omega \left(\frac{\partial\varrho}{\partial t}+\boldsymbol{u}\cdot\nabla\varrho-\lambda\,\Delta\varrho\right)\left(\frac{\partial\varrho}{\partial t}-\frac{\partial\psi}{\partial t}\right)d\Omega\,d\eta\leqslant 0\,,\\
&\nabla\cdot\boldsymbol{u}=0\,,
\end{aligned}\right.
$$

where $\boldsymbol{\varphi}$, ψ are test functions belonging to appropriate convex sets which, together with the exact definition of solution, will be introduced in section 2.

While however in (1.7) the condition $0<\varrho_1<\varrho<\varrho_2$ was imposed by the maximum principle related to the second equation, in system (1.12) the maximum principle no longer holds, having replaced the equations by inequalities. Hence, the coefficient ϱ in the first inequality of (1.12) could vanish, or even become negative or infinite, creating serious difficulties in the study of the system itself. We shall therefore in the sequel substitute (1.12) by the system

$$
(1.13)\qquad
\left|\begin{aligned}
&\int_0^t\!\!\int_\Omega \left[\bar{\varrho}\left(\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\boldsymbol{f}\right)+\nabla p-\mu\,\Delta\boldsymbol{u}\right](\boldsymbol{u}-\boldsymbol{\varphi})\,d\Omega\,d\eta\leqslant 0\,,\\
&\int_0^t\!\!\int_\Omega \left(\frac{\partial\varrho}{\partial t}+\boldsymbol{u}\cdot\nabla\varrho-\lambda\,\Delta\varrho\right)\left(\frac{\partial\varrho}{\partial t}-\frac{\partial\psi}{\partial t}\right)d\Omega\,d\eta\leqslant 0\,,\\
&\nabla\cdot\boldsymbol{u}=0
\end{aligned}\right.
$$

where we have set

$$
G\varrho=\bar{\varrho}=\left|\begin{array}{ll}
\varrho & \text{where } \varrho_1\leqslant\varrho\leqslant\varrho_2\,,\\
\varrho_1 & \text{where } \varrho<\varrho_1\,,\\
\varrho_2 & \text{where } \varrho>\varrho_2\,.
\end{array}\right.
$$

As we shall see in section 3, systems (1.12) and (1.13) are, for the purposes of our study, *perfectly equivalent*, since both reduce to (1.7) if the consistency conditions hold; the substitution of (1.12) by (1.13) is therefore justified. In section 3 we shall also illustrate the basic ideas underlying the substitution of the Graffi model by inequalities, together with a physical interpretation of the existence and uniqueness theorem for (1.13), (1.4), (1.5) which will be proved in sections 4 and 5.

REMARK 1: A simple interpretation of the meaning of a differential inequality can be obtained in the following way. Consider the inequality

$$\int_0^t \int_\Omega (Av - f)(v - \varphi)\, d\Omega\, d\eta$$

with v, f sufficiently smooth, v, φ belonging to a closed, convex set. Assume that, at the point $(\bar{x}, \bar{t})$, $v \in \mathring{K}$; then we can take $\varphi = u + \varepsilon\psi$ (ψ arbitrary, $|\varepsilon|$ sufficiently small), from which follows that it must necessarily be $Av(\bar{x}, \bar{t}) - f(\bar{x}, \bar{t}) = 0$. If, on the other hand, $v(\bar{x}, \bar{t}) \in \partial K$, the from the condition

$$(Av(\bar{x}, \bar{t}) - f(\bar{x}, \bar{t}), v(\bar{x}, \bar{t}) - \varphi(\bar{x}, \bar{t})) \leq 0\,, \qquad \forall \varphi \in K$$

it follows that the vector $Av(\bar{x}, \bar{t}) - f(\bar{x}, \bar{t})$ must be orthogonal to ∂K. At these points the external force f is therefore modified so as to satisfy such a condition.

REMARK 2: The substitution of inequalities for equations in the study of hydrodynamical models is not new to this paper. For a bibliography on this subject we refer to [11].

2. - Basic definitions and notations

Let Ω be an open, bounded set of $\mathbb{R}^3$, with boundary Γ and $\boldsymbol{v}(x) = \{v_1(x), v_2(x), v_3(x)\}$ a vector defined on Ω. Denoting by $L^2 = L^2(\Omega)$, $H^s = H^s(\Omega)$, $H_0^s = H_0^s(\Omega)$ the usual Sobolev spaces and by $\mathfrak{D}$ the space of functions which are indefinitely differentiable and with compact support in Ω, let us introduce the following notations:

— $\mathcal{N} = \{\boldsymbol{v}(x) : v_j \in \mathfrak{D}, \nabla \cdot \boldsymbol{v} = 0\}$,
— N^s = closure of $\mathcal{N}$ in H^s, with $(\boldsymbol{u}, \boldsymbol{v})_{N^s} = (\boldsymbol{u}, \boldsymbol{v})_{H_0^s}$ $(s > 0)$,
— $(N^s)'$ dual space of N^s, with $(N^0)' = N^0$,
— $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w})_{L^2} = \sum_{j,k}^{1,2,3} \int_\Omega u_j \frac{\partial v_k}{\partial x_j} w_k\, d\Omega$.

As already pointed out in section 1, our aim is to study system (1.13), which, with the notations introduced, can now be written as follows

$$(2.1) \quad \left| \begin{aligned} & \int_0^t (\bar{\varrho}\boldsymbol{u}' + \bar{\varrho}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \mu\, \Delta\boldsymbol{u} - \bar{\varrho}\boldsymbol{f} + \nabla p, \boldsymbol{u} - \boldsymbol{\varphi})_{L^2}\, d\eta \leq 0\,, \\ & \int_0^t (\varrho' + \boldsymbol{u} \cdot \nabla\varrho - \lambda\, \Delta\varrho, \varrho' - \psi')_{L^2}\, d\eta \leq 0\,, \\ & \nabla \cdot \boldsymbol{u} = 0\,, \end{aligned} \right.$$

where we have set

$$\boldsymbol{u}'(t)=\left\{\frac{\partial \boldsymbol{u}(x,t)}{\partial t};\, x\in\Omega\right\},\qquad \varrho'(t)=\left\{\frac{\partial \varrho(x,t)}{\partial t};\, x\in\Omega\right\},$$

$$\psi'(t)=\left\{\frac{\partial \psi(x,t)}{\partial t};\, x\in\Omega\right\}.$$

To the system (2.1) are associated the initial and boundary conditions (corresponding to (1.4), (1.5))

$$\boldsymbol{u}(x,0)=\bar{\boldsymbol{u}}(x)\,,\qquad \varrho(x,0)=\bar{\varrho}(x)\qquad (x\in\Omega)\,, \tag{2.2}$$

$$\boldsymbol{u}|_{\Gamma\times(0,T)}=0\,,\qquad \frac{\partial\varrho}{\partial\nu}\Big|_{\Gamma\times(0,T)}=0\,, \tag{2.3}$$

with $0<\varrho_1<\bar{\varrho}<\varrho_2$.

In order to give a precise formulation of the problem, let us introduce the following closed, convex sets:

$$K_1=\{\boldsymbol{v}\in L^2\colon |\boldsymbol{v}|<M_1 \text{ a.e.}\}\,,$$

$$K_2=\{g\in L^2\colon |\nabla g|<M_2,\ |\Delta g|<M_3 \text{ a.e.}\}\,,$$

$$K_3=\{g\in L^2\colon |g|<M_4 \text{ a.e.}\}\,.$$

The consistency conditions (1.8), (1.9), (1.10), (1.11) will then be satisfied in $(0,t')$ if and only if

$$\boldsymbol{u}(t)\in\mathring{K}_1\,,\qquad \varrho(t)\in\mathring{K}_2\,,\qquad \varrho'(t)\in\mathring{K}_3\,,\qquad \text{a.e. in } (0,T)\,. \tag{2.4}$$

Observe now that, if $\boldsymbol{\varphi}(t)\in L^2(0,T;N^1)$, $\psi(t)\in H^1(0,T;H^1)$, we have, by the third equation of (2.1) and the initial and boundary conditions (2.2), (2.3),

$$\left|\begin{aligned}
&\int_0^t(\nabla p,\boldsymbol{u}-\boldsymbol{\varphi})_{L^2}\,d\eta=-\int_0^t(p,\nabla\cdot(\boldsymbol{u}-\boldsymbol{\varphi}))_{L^2}\,d\eta=0\,,\\
&-\int_0^t(\nabla\varrho,\varrho'-\psi')_{L^2}\,d\eta=\int_0^t(\varrho,\varrho'-\psi')_{H^1}\,d\eta=\\
&\qquad\qquad=\tfrac12\|\varrho(t)\|^2_{H^1}-\tfrac12\|\bar{\varrho}\|^2_{H^1}-\int_0^t(\varrho,\psi)_{H^1}\,.
\end{aligned}\right. \tag{2.5}$$

Hence, assuming that $\boldsymbol{u}(t)$ takes its values in N^0, the system (2.1) can be written in the form

$$
(2.6)\quad \begin{cases} \displaystyle\int_0^t \{(\varrho\boldsymbol{u}' - \mu\Delta\boldsymbol{u} - \varrho\boldsymbol{f}, \boldsymbol{u} - \boldsymbol{\varphi})_{L^2} + b(\varrho\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{\varphi})\}\, d\eta \leq 0\,, \\ \displaystyle\int_0^t \{(\varrho' + \boldsymbol{u}\cdot\nabla\varrho, \varrho' - \psi')_{L^2} - \lambda(\varrho, \psi')_{H^1}\}\, d\eta + \frac{\lambda}{2}\|\varrho(t)\|^2_{H^1} - \frac{\lambda}{2}\|\bar\varrho\|^2_{H^1} \leq 0\,, \\ \nabla\cdot\boldsymbol{u} = 0\,. \end{cases}
$$

We shall then say that $\{\boldsymbol{u}, \varrho\}$ *is a solution of* (2.1) *in* $(0, T)$ *satisfying the initial and boundary conditions* (2.2), (2.3) *if*:

i) $\boldsymbol{u}(t) \in L^\infty(0, T; N^1 \cap K_1) \cap L^2(0, T; H^2)$, $\boldsymbol{u}'(t) \in L^\infty(0, T; N_0) \cap L^2(0, T; N^1)$, $\varrho(t) \in L^\infty(0, T; K_2)$, $\varrho'(t) \in L^\infty(0, T; K_2)$, $\boldsymbol{u}(0) = \bar{\boldsymbol{u}}$;

ii) $\boldsymbol{u}$, ϱ *satisfy* (2.6) *a.e. in* $(0, T)$, $\forall\boldsymbol{\varphi}(t) \in L^2(0, T; N^1 \cap K_1)$, $\psi(t) \in L^\infty(0, T; K_2)$, *with* $\psi'(t) \in L^\infty(0, T; H^1 \cap K_2)$.

3. - Physical interpretation of system (2.1)

The physical interpretation of (2.1) is based on the following property, which is a direct consequence of a general statement concerning the solutions of differential inequalities (see, for instance, [14]). The proof will be given in Appendix 1.

Let $\{\boldsymbol{u}, \varrho\}$ *be a solution* (in the sense indicated in section 2) *of* (2.1) *and assume that the consistency conditions* (2.4) *are satisfied a.e. in* $(0, t')$ $(0 < t' \leq T)$. *Then* $\{\boldsymbol{u}, \varrho\}$ *is a solution of* (1.8) *in the sense of distribution on* $\Omega \times (0, t')$. Hence, *if the solutions of inequalities* (2.1) *satisfy in* $(0, t')$ *the consistency conditions, they are also solutions of Graffi's model.*

As already mentioned, *we shall prove in the next sections an existence and uniqueness theorem for a global solution* $\{\boldsymbol{u}, \varrho\}$ *of* (2.1), (2.2), (2.3). Let us consider the two following possibilities:

a) there exists an interval $(0, t')$ in which the solution $\{\boldsymbol{u}, \varrho\}$ satisfies the consistency conditions (2.4);

b) no such interval exists, i.e. in every neighbourhood of $t = 0$ there exists a set of positive measure in which one at least of the consistency conditions is not satisfied.

If condition *b*) holds, we must conclude that the Graffi model is not suitable for the description of our physical problem, i.e. is not physically consistent. Indeed, even if the Graffi model (1.7) had a solution, this solution could not satisfy in any neighbourhood of $t = 0$ the consistency conditions and would therefore be physically meaningless.

Assume now that there exists $t' > 0$ such that (2.4) hold a.e. in $(0, t')$; then the solution $\{\boldsymbol{u}, \varrho\}$ is also, by the property recalled above, the only solution of the Graffi model in $(0, t')$ satisfying (2.4) and we obtain therefore a local existence and uniqueness theorem for (1.7), (2.4). The introduction and study of the inequalities (2.1) enable us then to *obtain a unique solution of the Graffi model wherever it can be expected that this solution is physically significant*, i.e. to state that *the Graffi model is well posed wherever it is physically consistent.*

W would like to emphasize a difference between the «local» theorem we thus obtain and, for instance, the «local» results recalled in section 1. In the latter, existence and uniqueness of the solution are proved in an interval $(0, t')$, where t' does not have special physical significance, since in its expression there appear embedding constants and other quantities which do not have any direct physical interpretation. In our case, on the other hand, *the interval* $(0, t')$ *represents the largest time interval in which the solution of* (1.7), (1.4), (1.5) *interprets the physical problem*; since, in fact, the consistency conditions are no longer verified for $t > t'$, the solution of (1.7), (1.4), (1.5), even if it existed, would have no physical meaning for $t > t'$.

Remark: In (2.6) the pressure p no longer appears explicitely, having been eliminated through the first of equations (2.5). Assume now that $\{\boldsymbol{u}, \varrho\}$ satisfies (2.6) and that the consistency conditions (2.4) are satisfied in $(0, t')$; by what has been said above, system (2.6) is then equivalent to (1.7) in $(0, t')$ and we calculate ∇p by means of the first od equations (1.7):

$$\nabla p = \mu \nabla \boldsymbol{u} - \varrho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f}\right).$$

Bearing in mind conditions i), ii) of section 2 and that, by the maximum principle, $0 < \varrho_1 < \varrho < \varrho_2$, we have

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f}\right) - \mu\,\Delta\boldsymbol{u} \in L^2(0, t'; H^{-1,\infty})$$

and, consequently,

$$p(t) \in L^2(0, t'; L^\infty).$$

Hence, the consistency condition c) of section 1 is satisfied in the function class chosen for the solution of our problem.

4. - An auxiliary theorem

In the present section we shall prove the following existence theorem for the solution of an «approximate» proboem.

Theorem 1: *Assume that* $\bar{\boldsymbol{u}} \in N^1 \cap H^2 \cap K_1$, $\bar{\varrho} \in K_2$, $\boldsymbol{f}(t) \in H^1(0, T; N^0)$ *and that* Ω *satisfies the conditions set in section* 1. *There exists then,* $\forall \delta > 0$, $\{\boldsymbol{u}, \varrho\}$ *such that* i$_0$) $\boldsymbol{u}(t) \in L^\infty(0, T; N^1 \cap K_1) \cap L^2(0, T; H^2)$, $\boldsymbol{u}'(t) \in L^\infty(0, T; N_0) \cap L^2(0, T; N^1)$, $\varrho(t) \in L^\infty(0, T; K_2)$, $\varrho'(t) \in L^\infty(0, T; K_2)$, $\boldsymbol{u}(0) = \bar{\boldsymbol{u}}$;

ii$_0$) $\boldsymbol{u}$, ϱ *satisfy a.e. in* $(0, T)$ *the inequalities*

$$\int_0^t \{(\bar{\varrho}\boldsymbol{u}' - \mu\,\Delta\boldsymbol{u} - \bar{\varrho}\boldsymbol{f}, \boldsymbol{u} - \boldsymbol{\varphi})_{L^2} + b(\bar{\varrho}\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{\varphi})\}\, d\eta < 0\,, \tag{4.1}$$

$$\begin{aligned}\int_0^t \{(\varrho' + \boldsymbol{u}\cdot\nabla\varrho, \varrho' - \psi')_{L^2} + \delta(\varrho', \psi')_{L^2} - \lambda(\varrho, \psi')_{H^1}\}\, d\eta\, + \\ + \frac{\delta}{2}\|\varrho'(t)\|^2_{L^2} + \frac{\lambda}{2}\|\varrho(t)\|^2_{H^1} - \frac{\lambda}{2}\|\bar{\varrho}\|^2_{H^1} - \delta(\varrho'(t), \psi'(t))_{L^2} < 0\end{aligned} \tag{4.2}$$

$\forall \boldsymbol{\varphi}(t) \in L^\infty(0, T; K_1)$, $\psi(t) \in L^\infty(0, T; K_2)$, $\psi'(t) \in L^\infty(0, T; H^1 \cap K_2)$, $\psi'(t) \in L^2(0, T; L^2)$.

Given $\boldsymbol{v} \in L(Q)$, let P be the projection operator defined by

$$P\boldsymbol{v}(x, t) = \begin{cases} \boldsymbol{v}(x, t) & \text{where } |\boldsymbol{v}(x, t)| < M_1\,, \\ M_1 \dfrac{\boldsymbol{v}(x, t)}{|\boldsymbol{v}(x, t)|} & \text{where } |\boldsymbol{v}(x, t)| > M_1\,; \end{cases} \tag{4.3}$$

it is obvious that $\|P\boldsymbol{v}\|_{L^\infty(Q)} < M_1$. Let, moreover, $\{\boldsymbol{g}_i\}$ be the basis in N^1 constituted by the eigenfunctions of the operator $-\Delta$ and denote by $\{\lambda_i\}$ the corresponding eigenvalues. Hence

$$(\boldsymbol{g}_j, \boldsymbol{\varphi})_{N^1} = \lambda_j(\boldsymbol{g}_j, \boldsymbol{\varphi})_{N^0} \qquad \forall \boldsymbol{\varphi} \in N^1,\ (\boldsymbol{g}_j, \boldsymbol{g}_k) = \delta_{jk}\,, \tag{4.4}$$

while, by the assumptions made on Ω, $\boldsymbol{g}_j \in H^2$ (cfr. [12]).

Setting $\boldsymbol{u}_m(t) = \sum_{j=1}^m \alpha_{jm}(t)\boldsymbol{g}_j$, we consider the following « approximate » system of ordinary differential equations

$$\begin{aligned}(\bar{\varrho}_m\boldsymbol{u}_m', \boldsymbol{g}_j)_{L^2} + \mu(\boldsymbol{u}_m, \boldsymbol{g}_j)_{N^1} + b(\bar{\varrho}_m P\boldsymbol{u}_m, P\boldsymbol{u}_m, \boldsymbol{g}_j)\, + \\ + m(\boldsymbol{u}_m - P\boldsymbol{u}_m, \boldsymbol{g}_j)_{L^2} - (\bar{\varrho}_m\boldsymbol{f}, \boldsymbol{g}_j)_{L^2} = 0 \qquad (j = 1, \dots, m)\end{aligned} \tag{4.5}$$

coupled with the inequality

$$\begin{aligned}\int_0^t \{(\varrho_m' + \boldsymbol{u}_m\cdot\nabla\varrho_m, \varrho_m' - \psi')_{L^2} + \delta(\varrho_m', \psi')_{L^2} - \lambda(\varrho_m, \psi')_{H^1}\}\, d\eta\, + \\ + \frac{\delta}{2}\|\varrho_m'(t)\|^2_{L^2} + \frac{\lambda}{2}\|\varrho_m(t)\|^2_{H^1} - \frac{\lambda}{2}\|\bar{\varrho}\|^2_{H^1} - \delta(\varrho_m'(t), \psi'(t))_{L^2} < 0\,.\end{aligned} \tag{4.6}$$

Denoting by Π_m the operator « projection on the subspace spanned by

$\boldsymbol{g}_1, \dots, \boldsymbol{g}'_m$, we associate to (4.5), (4.6) the initial conditions

$$(4.7) \qquad \boldsymbol{u}_m(0) = \Pi_m \bar{\boldsymbol{u}} = \boldsymbol{u}_m \,, \qquad \varrho_m(0) = \bar{\varrho} \,.$$

Let us give, to begin with, some a priori estimated which will enable us prove the existence of a solution of (4.5), (4.6), (4.7). Setting $\beta(\boldsymbol{u}) = \boldsymbol{u} - P\boldsymbol{u}$, we have, by (4.1),

$$(4.8) \qquad \beta(\boldsymbol{u}) = 0 \Leftrightarrow \|\boldsymbol{u}\|_{L^\infty(\Omega)} \leq M_1 \,,$$

$$(4.9) \qquad (\beta(\boldsymbol{u}), \boldsymbol{u})_{L^2} \geq 0 \,,$$

$$(4.10) \qquad (\beta(\boldsymbol{u}), \boldsymbol{u})_{L^2} = (|\beta(\boldsymbol{u})|, |\boldsymbol{u}|)_{L^2} \,.$$

Moreover, setting

$$(4.11) \quad R = \left\{ v \in C^2(\overline{Q}) : \left| \int_0^t \nabla v(x, \eta)\, d\eta + \nabla \bar{\varrho} \right| \leq M_2 \,, \left| \int_0^t \Delta v(x, \eta)\, d\eta + \Delta \bar{\varrho} \right| \leq M_3 \right\} ,$$

$$(4.12) \quad R_0 = \text{closure of } R \text{ in } L^2(Q) \,,$$

it can be proved in a straightforward way (see, for instance, [13], note 1) that

$$(4.13) \qquad \varrho_m(t) \in L^\infty(0, T; K_2) \Leftrightarrow \varrho'_m(t) \in R_0 \,,$$

$\forall \varrho_m(t) \in H^1(0, T; L^2)$ with $\varrho_m(0) = \bar{\varrho}$. Hence, conditions $\varrho_m(t) \in L^\infty(0, T; K_2)$, $\varrho'(t) \in L^\infty(0, T; K_2)$ can be substituted by

$$(4.14) \qquad \varrho'_m(t) \in L^\infty(0, T; K_2) \cap R_0 \,.$$

Since, by definition, $\boldsymbol{u}_m \in H^{2,m}(Q)$ and K_2, R_0 are closed, convex sets, it can be proved that (4.6) admits, $\forall$ fixed $\boldsymbol{u}_m$, a unique solution $\varrho_m(t) \in L^\infty(0, T; H^1)$, with $\varrho'_m(t) \in L^\infty(0, T; K_2) \cap R_0$, $\forall \psi(t) \in L^\infty(0, T; H^1)$, with $\psi'(t) \in L^\infty(0, T; K_2) \cap R_0$, $\psi'(t) \in L^2(0, T; L^2)$. The proof is given in Appendix 2. Bearing in mind the definitions of K_2 and R_0, we have then

$$(4.15) \qquad \|\nabla \varrho_m(t)\|_{L^\infty(0,T;L^\infty)} \leq M_2 \,, \qquad \|\Delta \varrho_m(t)\|_{L^\infty(0,T;L^\infty)} \leq M_3 \,,$$

$$(4.16) \qquad \|\varrho'_m(t)\|_{L^\infty(0,T;L^\infty)} \leq M_4 \,.$$

In order to obtain corresponding a priori estimates for $\boldsymbol{u}_m(t)$, let us multiply (4.5) by α_{jm} and add; observing that

$$(4.17) \qquad (\bar{\varrho}_m \boldsymbol{u}'_m, \boldsymbol{u}_m)_{L^2} = \frac{1}{2} \frac{d}{dt} \|\sqrt{\bar{\varrho}_m}\, \boldsymbol{u}_m\|^2_{L^2} - \frac{1}{2} (\bar{\varrho}'_m \boldsymbol{u}_m, \boldsymbol{u}_m)_{L^2} \,,$$

we obtain

$$(4.18)\qquad \frac{1}{2}\frac{d}{dt}\|\sqrt{\bar\varrho_m}\,u_m\|^2_{L^2}+\mu\|u_m\|^2_{H^1}+b(\bar\varrho_m Pu_m, Pu_m, u_m)-$$
$$-\frac{1}{2}(\bar\varrho'_m u_m, u_m)_{L^2}-(\bar\varrho_m f, u_m)_{L^2}+m(\beta(u_m), u_m)_{L^2}=0\,.$$

Hence, integrating between 0 and $t\in(0, T]$, and denoting by c_i quantities which do not depend on m and δ, we have, by (4.9), (4.3), (4.15), (4.16),

$$(4.19)\qquad \frac{1}{2}\|\sqrt{\bar\varrho_m(t)}\,u_m(t)\|^2_{L^2}+\mu\|u_m\|^2_{L^2(0,t;H^1)}\le\frac{1}{2}\|\sqrt{\bar\varrho}\,u_m\|^2_{L^2}+$$
$$+c_1\|\bar\varrho_m Pu_m\|_{L^\infty(0,t;L^\infty)}\|Pu_m\|_{L^2(0,t;H^1)}\|u_m\|_{L^2(0,t;H^1)}+$$
$$+c_2\|\bar\varrho'_m\|_{L^\infty(0,t;L^\infty)}\|u_m\|^2_{L^2(0,t;H^1)}+c_3\|Pu_m\|_{L^\infty(0,t;L^\infty)}\|u_m\|_{L^2(0,t;H^1)}\le$$
$$\le c_4\|u_m\|_{L^2(0,t;H^1)}\|u_m\|_{L^2(0,t;H^1)}+c_5\|u_m\|_{L^2(0,t;H^1)}\,.$$

Consequently, by Gronwall's lemma,

$$(4.20)\qquad \|u_m(t)\|_{L^\infty(0,T;H^1)\cap L^2(0,T;H^1)}\le M_6\,.$$

Under the assumptions we have made, the system (4.5) admits, by (4.20), $\forall$ fixed $\varrho_m(t)\in L^\infty(0, T; K_2)$ with $\varrho'_m(t)\in L^\infty(0, T; K_2)$, a global solution satisfying the first of conditions (4.7). From the a priori estimates (4.15), (4.16), (4.20) and the existence theorem given for (4.5) and (4.6) it follows then, by Schauder's principle, that system (4.5), (4.6) admits, $\forall\delta$ and m, a solution $\{\varrho_m, u_m\}$ satisfying (4.7), with $\varrho_m(t)\in L^\infty(0, T; H^1\cap K_2)$, $\varrho'_m(t)\in L^\infty(0, T; K_2)$, $\psi(t)\in L^\infty(0, T; K_2)$, $\psi'(t)\in L^\infty(0, T; H^1\cap K_2)$, $\psi'(t)\in L^2(0, T; L^2)$.

In order to be able to pass to the limit in (4.5), (4.6) when $m\to\infty$, we need some estimates on the derivatives of u_m. Multiplying (4.5) by $-\lambda_j\alpha_{jm}$, adding and integrating between 0 and $t\in(0, T]$ we have then, bearing in mind (4.17), (4.4) and setting $D_i=\partial/\partial x_i$,

$$(4.21)\qquad \frac{1}{2}\sum_{i=1}^{3}\|\sqrt{\bar\varrho_m(t)}\,D_i u_m(t)\|^2_{L^2}-\frac{1}{2}\sum_{i=1}^{3}\|\sqrt{\bar\varrho}\,D_i\bar u_m\|^2_{L^2}+$$
$$+\int_0^t\Big\{\sum_{i=1}^{3}\Big[(u'_m D_i\bar\varrho_m, D_i u_m)_{L^2}-\frac{1}{2}(\bar\varrho' D_i u_m, D_i u_m)_{L^2}\Big]+$$
$$+\mu\|\Delta u_m\|^2_{L^2}-b(\bar\varrho_m Pu_m, Pu_m, \Delta u_m)-m(\beta(u_m), \Delta u_m)_{L^2}+(\bar\varrho_m f, \Delta u_m)_{L^2}\Big\}d\eta=0\,.$$

Hence, by (4.15), (4.16), (4.20),

$$(4.22)\qquad \frac{1}{2}\sum_{i=1}^{3}\|\sqrt{\bar\varrho_m(t)}\,D_i u_m(t)\|^2_{L^2}+\mu\|\Delta u_m\|^2_{L^2(0,t;H^1)}\le\frac{1}{2}\sum_{i=1}^{3}\|\sqrt{\bar\varrho}\,D_i\bar u_m\|^2_{L^2}+$$
$$+c_6\|\bar\varrho_m\|_{L^\infty(0,t;H^1)}\|u'_m\|_{L^2(0,t;H^1)}\|u_m\|_{L^2(0,t;H^1)}+c_7\|u_m\|^2_{L^2(0,t;H^1)}+$$
$$+c_8\|Pu_m\|_{L^2(0,t;H^1)}\|\Delta u_m\|_{L^2(0,t;H^1)}+c_9\|\Delta u_m\|_{L^2(0,t;H^1)}\le$$
$$\le c_{10}\|u'_m\|_{L^2(0,t;H^1)}+c_{11}\|\Delta u_m\|_{L^2(0,t;H^1)}+c_{12}\,.$$

In fact, by the monotonicity of β,

$$(4.23)\qquad -\int_0^t (\beta(\mathbf{u}_m), \Delta\mathbf{u}_m)_{L^2}\,d\eta = \sum_{i=1}^{3}\int_0^t (D_i\beta(\mathbf{u}_m), D_i\mathbf{u}_m)_{L^2}\,d\eta \geq 0\,.$$

Differentiating (4.5), multiplying by α'_{jm}, adding and integrating between 0 and $t\in(0, T]$, we obtain, on the other hand,

$$(4.24)\qquad \frac{1}{2}\|\sqrt{\bar{\varrho}_m(t)}\,\mathbf{u}'_m(t)\|^2_{L^2} - \frac{1}{2}\|\sqrt{\bar{\varrho}}\,\mathbf{u}'_m(0)\|^2_{L^2} + \int_0^t\Big\{-\frac{1}{2}(\bar{\varrho}'_m\mathbf{u}'_m, \mathbf{u}'_m)_{L^2} +$$
$$+\mu\|\mathbf{u}'_m\|^2_{H^1} + b((\bar{\varrho}_m P\mathbf{u}_m)', P\mathbf{u}_m, \mathbf{u}'_m) + b(\bar{\varrho}_m P\mathbf{u}_m, (P\mathbf{u}_m)', \mathbf{u}'_m) +$$
$$+ m(\beta'(\mathbf{u}_m)\mathbf{u}'_m, \mathbf{u}'_m)_{L^2} - (\bar{\varrho}'_m\mathbf{f}, \mathbf{u}'_m)_{L^2} - (\bar{\varrho}_m\mathbf{f}', \mathbf{u}'_m)_{L^2}\Big\}\,d\eta = 0\,.$$

Observe now that, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w}\in H^1_0$,

$$(4.25)\qquad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) - ((\nabla\cdot\mathbf{u})\mathbf{v}, \mathbf{w})_{L^2}\,;$$

hence, by (4.15), (4.16), (4.20), (4.25),

$$(4.26)\qquad \int_0^t |b((\bar{\varrho}_m P\mathbf{u}_m)', P\mathbf{u}_m, \mathbf{u}'_m)|\,d\eta \leq \int_0^t\{|b(\bar{\varrho}'_m P\mathbf{u}_m, P\mathbf{u}_m, \mathbf{u}'_m)| +$$
$$+ |b(\bar{\varrho}_m(P\mathbf{u}_m)', P\mathbf{u}_m, \mathbf{u}'_m)|\}\,d\eta \leq c_{12}\|\bar{\varrho}'_m\|_{L^\infty(0,t;L^\infty)}\cdot$$
$$\cdot\|P\mathbf{u}_m\|_{L^\infty(0,t;L^\infty)}\cdot\|P\mathbf{u}_m\|_{L^2(0,t;H^1)}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)} +$$
$$+ c_{13}\|\bar{\varrho}_m\|_{L^\infty(0,t;L^\infty)}\|(P\mathbf{u}_m)'\|_{L^2(0,t;H^1)}\|P\mathbf{u}_m\|_{L^\infty(0,t;L^\infty)}\cdot$$
$$\cdot\|\mathbf{u}'_m\|_{L^2(0,t;H^1)} + c_{14}\|\bar{\varrho}_m\|_{L^\infty(0,t;L^\infty)}\|(P\mathbf{u})_m'\|_{L^2(0,t;H^1)}\cdot$$
$$\cdot\|P\mathbf{u}_m\|_{L^\infty(0,t;L^\infty)}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)} + c_{15}\|\nabla\bar{\varrho}_m\|_{L^\infty(0,t;L^\infty)}\cdot$$
$$\cdot\|(P\mathbf{u}_m)'\|_{L^2(0,t;H^1)}\|P\mathbf{u}_m\|_{L^\infty(0,t;L^\infty)}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)} \leq$$
$$\leq c_{16}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)} + c_{17}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)}\,.$$

Substituting (4.25), (4.26) into (4.24) and bearing in mind that $(\beta'(\mathbf{u}_m)\cdot\mathbf{u}'_m, \mathbf{u}'_m)_{L^2}\geq 0$, we obtain then

$$(4.27)\qquad \frac{1}{2}\|\sqrt{\bar{\varrho}_m(t)}\,\mathbf{u}'_m(t)\|^2_{L^2} + \mu\|\mathbf{u}'_m\|^2_{L^2(0,t;H^1)} \leq \frac{1}{2}\|\sqrt{\bar{\varrho}}\,\mathbf{u}'_m(0)\|^2_{L^2} +$$
$$+ c_{18}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)} + c_{19}\|\mathbf{u}'_m\|_{L^2(0,t;H^1)}\,.$$

Multiplying (4.5) written for $t = 0$ by $\alpha'_{jm}(0)$ and adding, we have, on the other hand,

$$\|\sqrt{\bar{\varrho}}\,\boldsymbol{u}'_m(0)\|^2_{L^2} - \mu(\Delta\boldsymbol{u}_m(0), \boldsymbol{u}'_m(0))_{L^2} + \\ + b(\bar{\varrho}\bar{\boldsymbol{u}}_m, \bar{\boldsymbol{u}}_m, \boldsymbol{u}'_m(0)) - (\bar{\varrho}f(0), \boldsymbol{u}'_m(0))_{L^2} = 0 , \tag{4.28}$$

from which follows, by the assumptions made on the initial data, that

$$\|\sqrt{\bar{\varrho}}\,\boldsymbol{u}'_m(0)\|_{L^2} < c_{18} . \tag{4.29}$$

Hence, adding (4.22) and (4.27), we obtain

$$\frac{1}{2}\sum_{i=1}^{3} \|\sqrt{\varrho_m(t)}\,D_i\boldsymbol{u}_m(t)\|^2_{L^2} + \frac{1}{2}\|\sqrt{\varrho_m(t)}\,\boldsymbol{u}'_m(t)\|^2_{L^2} + \\ + \mu\|\Delta\boldsymbol{u}_m\|^2_{L^2(0,t;N^0)} + \mu\|\boldsymbol{u}'_m\|^2_{L^2(0,t;N^1)} < c_{19}\|\Delta\boldsymbol{u}_m\|_{L^2(0,t;N^0)} + \\ + c_{20}\|\boldsymbol{u}'_m\|_{L^2(0,t;N^0)}\|\boldsymbol{u}'_m\|_{L^2(0,t;N^1)} + c_{21}\|\boldsymbol{u}'_m\|_{L^2(0,t;N^0)} + c_{22} ; \tag{4.30}$$

consequently, by Gronwall's lemma, bearing in mind that $\varrho_m > \varrho_1 > 0$ and the smootheness assumptions on Ω,

$$\|\boldsymbol{u}_m\|_{L^\infty(0,T;N^1)\cap L^2(0,T;H^2)} < M_7 , \tag{4.31}$$

$$\|\boldsymbol{u}'_m\|_{L^\infty(0,T;N^0)\cap L^2(0,T;N^1)} < M_8 . \tag{4.32}$$

Denoting again by $\{\boldsymbol{u}_m\}$, $\{\varrho_m\}$ appropriate subsequences selected from $\{\boldsymbol{u}_m\}$, $\{\varrho_m\}$, we have then, by (4.15), (4.16), (4.20), (4.31), (4.32) and by well known embedding and compactness theorems,

$$\lim_{m\to\infty} \boldsymbol{u}_m(t) = \boldsymbol{u}(t) \tag{4.33}$$

in the strong topology of $L^2(0, T; N^1)$, the weak topology of $L^2(0, T; H^2) \cap$ $\cap H^1(0, T; N^1)$, the weak-star topology of $L^\infty(0, T; N^1) \cap H^{1,\infty}(0, T; N^0)$ and

$$\lim_{m\to\infty} \varrho_m(t) = \varrho(t) \tag{4.34}$$

in the weak-star topology of $L^\infty(0, T; H^{1,\infty}) \cap H^{1,\infty}(0, T; L^\infty)$ and, consequently, in the strong topology of $L^\infty(0, T; H^{1,\infty})$. Moreover, by (4.15), (4.16),

$$\|\nabla\bar{\varrho}_m\|_{L^\infty(0,T;L^\infty)} < M_3 , \qquad \|\bar{\varrho}'_m\|_{L^\infty(0,T;L^\infty)} < M_4 \tag{4.35}$$

and, consequently,

$$\lim_{m\to\infty} \bar{\varrho}_m = \bar{\varrho} \tag{4.36}$$

in the strong topology of $L^2(Q)$ and the weak topology of $H^1(Q)$. On the other hand, by (4.33) and the definition of the operator P,

$$\lim_{m\to\infty} P\boldsymbol{u}_m = P\boldsymbol{u} = \boldsymbol{u} \tag{4.37}$$

in the strong topology of $L^2(Q)$ and the weak topology of $H^1(Q)$, while, by (4.18), (4.33),

$$\lim_{m\to\infty} (\beta(\boldsymbol{u}_m), \boldsymbol{u}_m)_{L^2} = 0 \Rightarrow \boldsymbol{u}(t) \in L^\infty(0, T; K_1)\,. \tag{4.38}$$

Moreover, since $|P\boldsymbol{u}_m|$ and $|\bar{\varrho}_m|$ are uniformly bounded a.e. in Q, (4.36), (4.37) hold also in the strong topology of $L^p(Q)$, $\forall p$.

Let $\boldsymbol{\varphi}(t)$ be an arbitrary function $\in H^1(0, T; \mathcal{N})$, with $|\boldsymbol{\varphi}(x, t)| < M_1$; setting

$$\boldsymbol{\varphi}(t) = \sum_{j=1}^{\infty} \gamma_j(t)\boldsymbol{g}_j\,, \qquad \boldsymbol{\varphi}_p(t) = \sum_{j=1}^{p} \gamma_j(t)\boldsymbol{g}_j\,, \tag{4.39}$$

it is obvious that, since the embedding of $H^1(0, T; \mathcal{N})$ in $C^0(\bar{Q})$ is completely continuous, $|\boldsymbol{\varphi}_p| < M_1$ when $p > \bar{p}$ sufficiently large. Assuming that $p > \bar{p}$ and setting $\sigma_j = \gamma_j$ when $j < p$, $\sigma_j = 0$ when $j > p$, let us multiply (4.5) by $x_{jm} - \sigma_j$; taking $m > p$ and adding, we obtain

$$(\bar{\varrho}_m \boldsymbol{u}_m' - \mu\,\Delta\boldsymbol{u}_m + m\beta(\boldsymbol{u}_m) - \bar{\varrho}_m \boldsymbol{f}, \boldsymbol{u}_m - \boldsymbol{\varphi}_p)_{L^2} + \\ + b(\bar{\varrho} P\boldsymbol{u}_m, P\boldsymbol{u}_m, \boldsymbol{u}_m - \boldsymbol{\varphi}_p) = 0\,. \tag{4.40}$$

Consequently, bearing in mind that, since $|\boldsymbol{\varphi}_p| < M_1$,

$$(\beta(\boldsymbol{u}_m), \boldsymbol{u}_m - \boldsymbol{\varphi}_p)_{L^2} = (\beta(\boldsymbol{u}_m) - \beta(\boldsymbol{\varphi}_p), \boldsymbol{u}_m - \boldsymbol{\varphi}_p)_{L^2} > 0\,,$$

we have

$$\int_0^t \{(\bar{\varrho}_m \boldsymbol{u}_m' - \mu\,\Delta\boldsymbol{u}_m - \bar{\varrho}_m \boldsymbol{f}, \boldsymbol{u}_m - \boldsymbol{\varphi}_p)_{L^2} + \\ + b(\bar{\varrho}_m P\boldsymbol{u}_m, P\boldsymbol{u}_m, \boldsymbol{u}_m - \boldsymbol{\varphi}_p)\}\, d\eta < 0\,. \tag{4.41}$$

Letting $m \to \infty$, it follows from (4.41), by (4.33), (4.34), (4.36), (4.37), (4.38) that the limit functions ϱ, $\bar{\varrho}$, $\boldsymbol{u}$ satisfy condition i_g) and relation (4.2) $\forall \boldsymbol{\varphi}_p$ defined by (4.39). Since the space of these functions is dense in that of test functions considered in ii_g), (4.2) holds.

By (4.6), (4.33), (4.34), letting $m \to \infty$ and observing that, by a well known property of the weak limit,

$$\lim_{m\to\infty} \|\varrho_m'(t)\|_{L^2}^2 > \|\varrho'(t)\|_{L^2}^2\,, \tag{4.42}$$

we may conclude that also (4.3) is verified. Since the initial conditions for $\boldsymbol{u}$ and ϱ are obviously fulfilled, the theorem is proved.

5. - Proof of the main theorem

We now prove the following existence and uniqueness theorem for the global solution of system (2.1) with the initial and boundary conditions (2.2), (2.3).

Theorem 2: Assume that

$$\bar{u} \in N^1 \cap H^2 \cap K_1, \qquad \bar{\varrho} \in H^1 \cap K_2, \qquad f(t) \in H^1(0, T; N^0)$$

and that Ω satisfies the conditions set in section 1. There exists then one, and only one, couple $\{u, \varrho\}$ which satisfies conditions i), ii) *of section* 2.

We begin by proving the existence of a solution. Let $\{u_\delta, \varrho_\delta\}$ be the solution given in Theorem 1 corresponding to a given value of δ. By what has been proved in the preceding section, we have, a.e. in $(0, T)$,

$$\int_0^t \{(\bar{\varrho}_\delta u'_\delta - \mu\, \Delta u_\delta - \bar{\varrho}_\delta f, u_\delta - \varphi)_{L^2} + b(\bar{\varrho}_\delta u_\delta, u_\delta, u_\delta - \varphi)\}\, d\eta \leq 0\,, \tag{5.1}$$

$$\begin{aligned}\int_0^t \{(\varrho'_\delta + u_\delta \cdot \nabla \varrho_\delta, \varrho'_\delta - \psi')_{L^2} + \delta(\varrho'_\delta, \psi')_{L^2} - \lambda(\varrho_\delta, \psi')_{H^1}\}\, d\eta\, + \\ + \frac{\delta}{2}\|\varrho'_\delta(t)\|^2_{L^2} + \frac{\lambda}{2}\|\varrho_\delta(t)\|^2_{H^1} - \frac{\lambda}{2}\|\bar{\varrho}\|^2_{H^1} - \delta(\varrho'_\delta(t), \psi'(t))_{L^2} \leq 0\end{aligned} \tag{5.2}$$

$\forall \varphi(t) \in L^\infty(0, T; K_1)$, $\psi(t) \in L^\infty(0, T; K_2)$, $\psi'(t) \in L^\infty(0, T; H^1 \cap K_2)$, $\psi'(t) \in L^2(0, T; L^2)$ and

$$u_\delta(0) = \bar{u}\,; \tag{5.3}$$

moreover, bearing in mind (4.15), (4.16), (4.31), (4.32),

$$\|\nabla \varrho_\delta\|_{L^\infty(Q)} \leq M_2\,, \qquad \|\varrho'_\delta\|_{L^\infty(Q)} \leq M_3\,, \tag{5.4}$$

$$\|u_\delta\|_{L^\infty(0,T;N^1) \cap L^2(0,T;H^2)} \leq M_7\,, \qquad \|u'_\delta\|_{L^\infty(0,T;N^0) \cap L^2(0,T;N^1)} \leq M_8\,, \tag{5.5}$$

where the quantities M_i do not depend on δ. Hence, we can select from $\{u_\delta\}$, $\{\varrho_\delta\}$ two subsequences, again denoted by $\{u_\delta\}$, $\{\varrho_\delta\}$ such that

$$\lim_{\delta \to 0} u_\delta(t) = u(t) \tag{5.6}$$

in the strong topology of $L^2(0, T; N^1)$, the weak topology of $L^2(0, T; H^2) \cap H^1(0, T; N^1)$, the weak-star topology of $L^\infty(0, T; N^1) \cap H^{1,\infty}(0, T; N^0)$ and

$$\lim_{\delta \to 0} \varrho_\delta = \varrho\,, \qquad \lim_{\delta \to 0} \bar{\varrho}_\delta = \bar{\varrho} \tag{5.7}$$

in the strong topology of $L^2(Q)$ and the weak-star topology of $H^{1,\infty}(Q)$. Since $|\boldsymbol{u}_\delta|$, $|\varrho_\delta|$ are uniformly bounded a.e. in Q, the limits (5.6), (5.7) hold also in the strong topology of $L^p(Q)$, $\forall p$.

By (5.6), (5.7), (5.4), (5.5), (5.3) it is obvious that the limit functions $\boldsymbol{u}$, ϱ satisfy condition i); moreover,

$$\lim_{\delta\to 0}\left(\frac{\delta}{2}\|\varrho_\delta'(t)\|^2_{L^2}-\delta(\varrho_\delta'(t),\psi'(t))_{L^2}+\delta\int_0^t(\varrho_\delta',\psi')_{L^2}\,d\eta\right)=0\,. \tag{5.8}$$

Hence, passing to the limit in (5.1), (5.2) and bearing in mind (5.4), (5.5), we can conclude that $\boldsymbol{u}$, ϱ satisfy also condition ii), provided the test function ψ is such that $\psi'(t)\in L^2(0,T;L^2)$. Since however the space of such test functions is dense in the one considered in ii), the existence of a solution is proved.

Let us now show that the solution in unique. Assume that $\{\boldsymbol{u}_1,\varrho_1\}$, $\{\boldsymbol{u}_2,\varrho_2\}$ satisfy conditions i), ii); setting

$$\boldsymbol{w}=\boldsymbol{u}_1-\boldsymbol{u}_2\,,\qquad \sigma=\tfrac{1}{2}(\varrho_1+\varrho_2)$$

and deoting by σ_ε the solution of the differential equation

$$-\varepsilon\sigma_\varepsilon'+\sigma_\varepsilon=\sigma\qquad\text{with } \sigma_\varepsilon(0)=\bar\varrho\,,\ \varepsilon>0\,, \tag{5.9}$$

we set in ii) respectively $\boldsymbol{u}=\boldsymbol{u}_1$, $\varrho=\varrho_1$, $\boldsymbol{\varphi}=\boldsymbol{u}_2$, $\psi=\sigma_\varepsilon$ and $\boldsymbol{u}=\boldsymbol{u}_2$, $\varrho=\varrho_2$, $\boldsymbol{\varphi}=\boldsymbol{u}_1$, $\psi=\sigma_\varepsilon$; adding the equations thus obtained, we have then

$$\int_0^t\{(\tilde\varrho_1\boldsymbol{u}_1'-\mu\,\Delta\boldsymbol{u}_1-\tilde\varrho_1\boldsymbol{f},\boldsymbol{w})_{L^2}-(\tilde\varrho_2\boldsymbol{u}_2'-\mu\,\Delta\boldsymbol{u}_2-\tilde\varrho_2\boldsymbol{f},\boldsymbol{w})_{L^2}+ \\ +b(\tilde\varrho_1\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{w})-b(\tilde\varrho_2\boldsymbol{u}_2,\boldsymbol{u}_2,\boldsymbol{w})\}\,d\eta\leq 0\,, \tag{5.10}$$

$$\int_0^t\{(\varrho_1'+\boldsymbol{u}_1\cdot\nabla\varrho_1,\varrho_1'-\sigma_\varepsilon')_{L^2}+(\varrho_2'+\boldsymbol{u}_2\cdot\nabla\varrho_2,\varrho_2'-\sigma_\varepsilon')_{L^2}- \\ -\lambda(\varrho_1,\sigma_\varepsilon')_{H^1}-\lambda(\varrho_2,\sigma_\varepsilon')_{H^1}\}\,d\eta+\frac{\lambda}{2}\|\varrho_1(t)\|^2_{H^1}+\frac{\lambda}{2}\|\varrho_2(t)\|^2_{H^1}-\lambda\|\bar\varrho\|^2_{H^1}\leq 0\,, \tag{5.11}$$

from the first of which follows, bearing in mind (4.17),

$$\frac{1}{2}\|\sqrt{\tilde\varrho_1(t)}\,\boldsymbol{w}(t)\|^2_{L^2}+\int_0^t\Big\{-\frac{1}{2}(\tilde\varrho_1',\boldsymbol{w},\boldsymbol{w})_{L^2}+ \\ +((\tilde\varrho_1-\tilde\varrho_2)\boldsymbol{u}_2',\boldsymbol{w})_{L^2}+\mu\|\boldsymbol{w}\|^2_{H^1}+b(\tilde\varrho_1\boldsymbol{w},\boldsymbol{u}_1,\boldsymbol{w})+ \\ +b((\tilde\varrho_1-\tilde\varrho_2)\boldsymbol{u}_2,\boldsymbol{u}_1,\boldsymbol{w})+b(\tilde\varrho_2\boldsymbol{u}_2,\boldsymbol{w},\boldsymbol{w})-((\tilde\varrho_1-\tilde\varrho_2)\boldsymbol{f},\boldsymbol{w})_{L^2}\Big\}\,d\eta\leq 0\,. \tag{5.12}$$

Observing that, by (5.9),

$$(5.13)\qquad (\varrho_1+\varrho_2, \sigma_\varepsilon')_{H^1} = 2(\sigma, \sigma_\varepsilon')_{H^1} = 2(\sigma_\varepsilon, \sigma_\varepsilon')_{H^1} + 2(\sigma-\sigma_\varepsilon, \sigma_\varepsilon')_{H^1} < \frac{d}{dt}\|\sigma_\varepsilon(t)\|^2_{H^1}$$

and that

$$(5.14)\qquad \|\varrho_1\|^2_{H^1} + \|\varrho_2\|^2_{H^1} - \tfrac{1}{2}\|\varrho_1+\varrho_2\|^2_{H^1} > \tfrac{1}{2}\|\varrho_1-\varrho_2\|^2_{H^1},$$

we obtain, on the other hand, from (5.11)

$$(5.15)\qquad -\lambda\|\sigma_\varepsilon(t)\|^2_{H^1} + \frac{\lambda}{4}\|\varrho_1(t)+\varrho_2(t)\|^2_{H^1} + \frac{\lambda}{4}\|\varrho_1(t)-\varrho_2(t)\|^2_{H^1} +$$
$$+\int_0^t \{(\varrho_1'+\boldsymbol{u}_1\cdot\nabla\varrho_1, \varrho_1'-\sigma_\varepsilon')_{L^2} + (\varrho_2'+\boldsymbol{u}_2\cdot\nabla\varrho_2, \varrho_2'-\sigma_\varepsilon')_{L^2}\}\,d\eta < 0.$$

Letting $\varepsilon\to 0$ and setting $\chi = \varrho_1-\varrho_2$, it follows from (5.12), (5.15) that

$$(5.16)\qquad \frac{1}{2}\|\sqrt{\bar\varrho_1(t)}\,\boldsymbol{w}(t)\|^2_{L^2} + \int_0^t\Big\{\mu\|\boldsymbol{w}\|^2_{L^2} - \frac{1}{2}(\bar\varrho_1'\boldsymbol{w}, \boldsymbol{w})_{L^2} +$$
$$+(\chi\boldsymbol{u}_2', \boldsymbol{w})_{L^2} + b(\bar\varrho_1\boldsymbol{w}, \boldsymbol{u}_1, \boldsymbol{w}) + b(\chi\boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{w}) +$$
$$+ b(\bar\varrho_2\boldsymbol{u}_2, \boldsymbol{w}, \boldsymbol{w}) - (\chi f, \boldsymbol{w})_{L^2}\Big\}\,d\eta < 0,$$

$$(5.17)\qquad \frac{\lambda}{4}\|\chi(t)\|^2_{H^1} + \frac{1}{2}\int_0^t\{\|\chi'\|^2_{L^2} + (\boldsymbol{u}_1\cdot\nabla\chi, \chi')_{L^2} + (\boldsymbol{w}\cdot\nabla\varrho_2, \chi')_{L^2}\}\,d\eta < 0.$$

Since $\boldsymbol{w}(0) = \chi(0) = 0$, we have then, by Gronwall's lemma and condition i), $\boldsymbol{w}(t) = \chi(t) = 0$. This proves the uniqueness of the solution.

Appendix 1

Let us prove the following proposition, stated at the beginning of section 3.

Let $\{\boldsymbol{u}, \varrho\}$ *satisfy conditions* i), ii) *of section* 2 *and assume that* (2.4)*hold a.e. in* $(0, t')$. *Then* $\{\boldsymbol{u}, \varrho\}$ *is a solution of the Graffi model* (1.7), *in the sense of distributions in* $\Omega\times(0, t')$.

Let $\boldsymbol{\zeta}(t)$, $\theta(t)$ be two arbitrary functions, belonging respectively to $L^2(0, T; N^1)$ and to $H^1(0, T; \mathfrak{D})$ and let $\varrho_j(t)$ be the solution of the differential equation

$$(\mathrm{A1.1})\qquad -\frac{1}{j}\varrho_j'(t) + \varrho_j(t) = \varrho(t), \qquad \text{with } \varrho_j(0) = \bar\varrho \ (j = 1, 2, \ldots).$$

Since $\{\boldsymbol{u}, \varrho\}$ satisfy (2.4), it is possible, assuming that $t < t'$, to choose in (2.6)

$$\text{(A1.2)} \qquad \boldsymbol{\varphi}(t) = \boldsymbol{u}(t) - \varepsilon\boldsymbol{\zeta}(t)\,, \qquad \psi(t) = \varrho_j(t) - \varepsilon\theta(t)$$

with $|\varepsilon|$ sufficiently small.

With this choice of the test functions, the first two equations of (2.6) become

$$\text{(A1.3)} \qquad \int_0^t \{(\bar{\varrho}\boldsymbol{u}' - \mu\,\Delta\boldsymbol{u} - \bar{\varrho}\boldsymbol{f}, \varepsilon\boldsymbol{\zeta})_{L^2} + b(\bar{\varrho}\boldsymbol{u}, \boldsymbol{u}, \varepsilon\boldsymbol{\zeta})\}\,d\eta < 0\,,$$

$$\text{(A1.4)} \qquad \int_0^t \{(\varrho' + \boldsymbol{u}\cdot\nabla\varrho, \varrho' - \varrho_j' + \varepsilon\theta')_{L^2} - \lambda(\varrho, \varrho_j' - \varepsilon\theta')_{H^1}\}\,d\eta + {} + \frac{\lambda}{2}\|\varrho(t)\|^2_{H^1} - \frac{\lambda}{2}\|\bar{\varrho}\|^2_{H^1} < 0\,.$$

Since we can change ε in $-\varepsilon$, it follows from (A1.3) that

$$\text{(A1.5)} \qquad \int_0^t \{(\bar{\varrho}\boldsymbol{u}' - \mu\,\Delta\boldsymbol{u} - \bar{\varrho}\boldsymbol{f}, \boldsymbol{\zeta})_{L^2} + b(\bar{\varrho}\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\zeta})\}\,d\eta = 0$$

$\forall\boldsymbol{\zeta}(t) \in L^2(0, T; N^0)$. Hence $\bar{\varrho}\boldsymbol{u}' - \mu\,\Delta\boldsymbol{u} + (\bar{\varrho}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \bar{\varrho}\boldsymbol{f}$ is orthogonal to $L^2(0, T; N^0)$; consequently, by a well known property of the space N^0, there exists a function p such that

$$\bar{\varrho}\boldsymbol{u}' - \mu\,\Delta\boldsymbol{u} + (\bar{\varrho}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \bar{\varrho}\boldsymbol{f} = -\nabla p$$

in the sense of distributions.

Let us now consider (A1.4); observe, to begin with, that, by the definition of ϱ_j,

$$\text{(A1.7)} \qquad \int_0^t (\varrho, \varrho_j')_{H^1}\,d\eta = \int_0^t (\varrho_j, \varrho_j')_{H^1}\,d\eta + \int_0^t (\varrho - \varrho_j, \varrho_j')_{H^1}\,d\eta < \frac{1}{2}\|\varrho_j(t)\|^2_{H^1} - \frac{1}{2}\|\bar{\varrho}\|^2_{H^1}\,.$$

Hence, it follows from (A1.4) that

$$\int_0^t \{(\varrho' + \boldsymbol{u}\cdot\nabla\varrho, \varrho' - \varrho_j' + \varepsilon\theta')_{L^2} + \lambda(\varrho, \varepsilon\theta')_{H^1}\}\,d\eta - \frac{\lambda}{2}\|\varrho_j(t)\|^2_{H^1} + \frac{\lambda}{2}\|\varrho(t)\|^2_{H^1} < 0$$

and, letting $j \to \infty$,

$$\text{(A1.8)} \qquad \int_0^t \{(\varrho' + \boldsymbol{u}\cdot\nabla\varrho, \varepsilon\theta')_{L^2} + \lambda(\varrho, \varepsilon\theta')_{H^1}\} < 0\,.$$

Changing ε in $-\varepsilon$ we obtain then

$$\varrho' + \mathbf{u}\cdot\nabla\varrho - \lambda\,\Delta\varrho = 0 \tag{A1.9}$$

in the sense of distributions.

By the maximum principle $\varrho_1 < \varrho < \varrho_2$ ($\Rightarrow \hat{\varrho} = \varrho$) and the system formed by (A1.6), (A1.9) coincides therefore with the first two equations of (1.7).

Appendix 2

Let us prove that, *if Ω satisfies the conditions set in section* 1 *and* $\mathbf{v}\in H^{1,\infty}(Q)$, *there exists*, $\forall\delta>0$, *a unique function* $\varrho(t)\in L^\infty(0,T;H^1)$, *with* $\varrho'(t)\in L^\infty(0,T;K_\lambda)\cap R_0$, $\varrho'(0)=0$ *satisfying a.e. in* $[0,T]$ *the inequality*

$$\int_0^t \{(\varrho' + \mathbf{v}\cdot\nabla\varrho, \varrho' - \psi')_{L^2} + \delta(\varrho', \psi')_{L^2} - \lambda(\varrho, \psi')_{H^1}\}\,d\eta + \tag{A2.1}$$

$$+ \frac{\delta}{2}\|\varrho'(t)\|^2_{L^2} + \frac{\lambda}{2}\|\varrho(t)\|^2_{H^1} - \frac{\lambda}{2}\|\bar{\varrho}\|^2_{H^1} - \delta(\varrho'(t), \psi'(t))_{L^2} < 0$$

$\forall\psi(t)\in L^\infty(0,T;H^1)$, *with* $\psi'(t)\in L^\infty(0,T;K_\lambda)\cap R_0$, $\psi'(t)\in L^2(0,T;L^2)$. Let us denote by $\{b_i\}$ the basis in H^1 constituted by the eigenfunctions of the operator $-\Delta$ and by ν_i the corresponding eigenvalues:

$$(b_j, \chi)_{H^1} = \nu_j(b_j, \chi)_{L^2} \qquad \forall\chi\in H^1,\ (b_j, b_k)_{L^2} = \delta_{jk}\,.$$

By the assumptions made on Ω, $b_j\in H^2$.

Setting $\varrho_k(t) = \sum_{j=1}^{k}\gamma_{jk}(t)b_j$ and denoting by β a penalization operator associated to the closed, convex set $L^\infty(0,T;K_\lambda)\cap R_0$ (*), we consider the system

$$\big(\varrho_k'(t) + \delta\varrho_k''(t) - \lambda\,\Delta\varrho_k(t) + \mathbf{v}(t)\cdot\nabla\varrho_k(t) + k\beta(\varrho_k'(t)), b_j\big)_{L^2} = 0 \tag{A2.2}$$

$$(j = 1, \dots, k)\,,$$

with the initial conditions

$$\varrho_k(0) = \Pi_k\bar{\varrho}\,, \qquad \varrho_k'(0) = 0 \tag{A2.3}$$

having denoted by Π_k the operator « projection on the subspace spanned by $b_1, \dots, b_k$ ».

Multiplying (A2.2) first by $\gamma_{jk}'(t)$, then by $\nu_j\gamma_{jk}'(t)$ and adding we obtain

(*) Apart from some small differences due to the particular structure of (A2.1), our proof coincides with that of a general theorem for hyperbolic inequalities (Lions, [14], ch. 3, th. 7).

directly the a priori estimates

$$\text{(A2.4)} \qquad \|\varrho_k(t)\|_{L^\infty(0,T;H^2)} \leq c_1\,, \qquad \|\varrho_k'(t)\|^2_{L^\infty(0,T;H^1)} \leq c_2\,,$$

with c_1, c_2 independent of k. Hence, we can select from $\{\varrho_k\}$ a subsequence (again denoted by $\{\varrho_k\}$) such that

$$\text{(A2.5)} \qquad \lim_{k\to\infty} \varrho_k(t) = \varrho(t)$$

in weak-star topology of $L^\infty(0, T; H^2) \cap H^{1,\infty}(0, T; H^1)$ (and, consequently, in the strong topology of $L^2(0, T; L^2)$).

Since (A2.2) is linear, it can be proved by means of standard techniques (see, for instance, [14], ch. 3) that the function $\varrho(t)$ defined by (A2.5) is a solution of the problem considered. The uniqueness of the solution can be obtained by a classical procedure. Let ϱ_1, ϱ_2 be two solutions and set $\sigma = \frac{1}{2}(\varrho_1 + \varrho_2)$; denote, moreover, by σ_ε the solution of the differential equation

$$\text{(A2.6)} \qquad -\varepsilon\sigma_\varepsilon''(t) + \sigma_\varepsilon'(t) = \sigma'(t)\,, \qquad \text{with } \sigma_\varepsilon(0) = \bar{\varrho}\,,\ \sigma_\varepsilon'(0) = 0\,,\ \varepsilon > 0\,.$$

Setting in (A2.1), written for ϱ_1 and ϱ_2, $\psi = \sigma_\varepsilon$, we obtain

$$\begin{aligned}\text{(A2.7)} \qquad \int_0^t &\{(\varrho_1' + v\cdot\nabla\varrho_1, \varrho_1' - \sigma_\varepsilon')_{L^2} + (\varrho_2' + v\cdot\nabla\varrho_2, \varrho_2' - \sigma_\varepsilon')_{L^2} - \\ &- \delta(\varrho_1' + \varrho_2', \sigma_\varepsilon'')_{L^2} - \lambda(\varrho_1 + \varrho_2, \sigma_\varepsilon')_{H^1}\}\, d\eta + \frac{\delta}{2}\|\varrho_1'(t)\|^2_{L^2} + \frac{\delta}{2}\|\varrho_2'(t)\|^2_{L^2} + \\ &+ \frac{\lambda}{2}\|\varrho_1(t)\|^2_{H^1} + \frac{\lambda}{2}\|\varrho_2(t)\|^2_{H^1} - \lambda\|\bar{\varrho}\|^2_{H^1} - \delta(\varrho_1'(t) + \varrho_2'(t), \sigma_\varepsilon'(t))_{L^2} \leq 0\,.\end{aligned}$$

From (A2.7) follows, by a straightforward calculation (see, for instance, the uniqueness theorem proved in section 5 and [14], ch. 3) that $\varrho_1 = \varrho_2$.

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