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On an Inequality Associated with Stationary Flows of Viscous, Incompressible Fluids (**)

SUMMARY. — In this paper we study a variational inequality associated with a boundary-value problem for the stationary motion of viscous incompressible fluids. This inequality replaces the description of the flow given in terms of the boundary-value problem in the case when some additional constraints are imposed on the flow.

We prove the existence of solutions of the inequality, study their regularity, uniqueness, dependence on data and relations to solutions of the equations of motion.

A proposito di una disuguaglianza relativa alle correnti stazionarie dei fluidi viscosi ed incompressibili

SUNTO. — Nella presente nota si studia una disequazione variazionale associata ad un problema al contorno relativo al moto stazionario di un fluido viscoso incomprimibile.

Tale disequazione sostituisce la corrispondente equazione quando si impongano ulteriori vincoli alla soluzione.

Si dimostra l'esistenza di una soluzione della disequazione e se ne studia la regolarità, l'unicità, la dipendenza dai dati e la relazione con le soluzioni dell'equazione.

0. - Introduction and Main Results

In this paper we consider a variational inequality related to the following boundary-value problem

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } D, \tag{0.1}$$

$$\operatorname{div} u = 0 \quad \text{in } D, \tag{0.2}$$

$$u = 0 \quad \text{on } S, \tag{0.3}$$

where D is a bounded domain in R^3 with a smooth boundary S, $\nu = \text{cost} > 0$.

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Equations (0.1), (0.2) describe the stationary motion of viscous, incompressible fluids. The functions $u(x)=(u_1(x), u_2(x), u_3(x))$ and $p(x)$ denote the velocity vector and pressure of the fluid, $f(x)=(f_1(x), f_2(x), f_3(x))$ denotes the exterior mass forces. By ∇, Δ and div we mean the usual gradient, Laplacian and divergence operators, so that Δu, $(u\cdot\nabla)u$ and ∇p are vectors with components Δu_i, $u_j(\partial/\partial x_j)u_i$, $(\partial/\partial x_i)p$, $i=1,2,3$, respectively (repeated indices are summed), $\operatorname{div} u=(\partial/\partial x_j)u_j$.

Through the paper we are interested in the situation when the velocity vector u is subjected to the constraint

$$u\in K=\{v\in L^2(D)\colon |v(x)|<C_1 \text{ a.e. in } D\} \tag{0.4}$$

where C_1 is an arbitrarily fixed positive number. In this case, to describe the fluid motion, we replace the above boundary-value problem by a variational inequality (inequality (0.6) below).

Similar variational problems related to the non-stationary Navier-Stokes equations were studied in [1], [4], [8] (see also [5], [6], [7], [9], [10] for more examples of variational problems of hydrodynamics). The main aims of all these investigations were to prove the existence and regularity of solutions of given variational problems as well as to characterize those convex and closed sets K for which the related variational problems *hav* solutions with postulated regularity.

Considerations of variational problems for the stationary Navier-Stokes equations as in this paper sem to be new. We pay special attention to formulating our results in terms of both functions u *and* p.

Before stating the results we introduce the basic notation and definitions.

— $L^q(D)=$ the set of classes of functions $f\colon D\to R^k$, L^q integrable in D, with the norm

$$|f|_q=\Big(\int_D |f|^q\Big)^{1/q} \qquad (k=1 \text{ or } 3,\ q>1),$$

— $W_q^m(D)=$ closure of $C^\infty(D;R^k)$, $k=1$ or 3, in the norm

$$\|f\|_{m,q}=\Big(\sum_{|\alpha|\le m}|D^\alpha f|_q^q\Big)^{1/q} \qquad (q>1,\ m\text{-nonnegative integer}),$$

— $\mathring{W}_q^m(D)=$ closure of $C_0^\infty(D;R^3)$ in $W_q^m(D)$,

— $W_{q'}^{-m}(D)=$ dual space to $\mathring{W}_q^m(D)$, $1/q+1/q'=1$,

— $H_0^1(D)=$ closure of $C_0^\infty(D;R^3)$ in the norm

$$\|u\|_1=\Big(\int_D|\nabla u|^2\Big)^{\frac12},$$

— $\mathcal{V}=\{u\in C_0^\infty(D;R^3)\colon \operatorname{div} u=0\}$,

— V = closure of $\mathcal{V}$ in $H_0^1(D)$,

— V' = dual space to V, with the usual norm.

By $(\,,\,)$ and $((\,,\,))$ we denote the scalar products in $L^2(D)$ and $H_0^1(D)$ respectively, $b(u, v, w) = ((u\cdot\nabla)v, w)$.

WEAK SOLUTIONS OF PROBLEM (0.1)-(0.3). We say that a pair of functions (u, p) is a weak solution of the boundary-value problem (0.1)-(0.3) if $u \in V$, $p \in L^q(D)$ for some $q > 1$, $\int_D p(x)\,dx = 0$ and if the following integral identity holds

$$\nu((u, a)) + b(u, u, a) - (p, \operatorname{div} a) = (f, a) \tag{0.5}$$

for all functions $a \in H_0^1(D)$.

VARIATIONAL INEQUALITY ASSOCIATED WITH PROBLEM (0.1)-(0.3). We say that a pair of functions (u, p) satisfies variational inequality (0.6) below if $u \in V \cap K$, $p \in L^q(D)$ for same $q > 1$, $\int_D p(x)\,dx = 0$ and if the following integral inequality holds

$$\nu((u, u-\varphi)) + b(u, u, u-\varphi) - (p, \operatorname{div}(u-\varphi)) < (f, u-\varphi) \tag{0.6}$$

for all functions $\varphi \in H_0^1(D) \cap K$.

The aim of this paper is to prove the following theorems.

THEOREM 0.1: If $f \in L^2(D)$ then there exists a pair of functions (u, p) satisfying variational inequality (0.6).

THEOREM 0.2: Suppose that (u, p) satisfies variational inequality (0.6) and that $u \in \operatorname{Int} K$. Then (u, p) is a weak solution of the boundary-value problem (0.1)-(0.3). Moreover, $p \in W_2^1(D)$, $u \in W_2^2(D)$.

Conversely, if (u, p) is a weak solution of problem (0.1)-(0.3) and if $u \in K$ then (u, p) satisfies variational inequality (0.6).

THEOREM 0.3: If $f \in L^q(D)$, $q > 3$ and if the L^q norm of f is sufficiently small then there exists a solution (u, p) of problem (0.1)-(0.3) such that $u \in \operatorname{Int} K$. In this case there exists a constant C_2 such that

$$|p(x)| < C_2, \qquad |\nabla u(x)| < C_2 \qquad \text{for almost all } x \text{ in } D.$$

THEOREM 0.4: Suppose that the constant C_1 in (0.4) is sufficiently small and that (u_1, p_1) and (u_2, p_2) satisfy variational inequality (0.6) with $f = f_1$ and $f = f_2$ respectively. Then

(i) $\|u_1 - u_2\|_1 < C\|f_1 - f_2\|_1$;

(ii) if, in addition, $u_1, u_2 \in \operatorname{Int} K$, then also

$$\|p_1 - p_2\|_{1,2} < C\|f_1 - f_2\|_1 . \tag{0.7}$$

In particular, if $f_1 = f_2$ and $u_1, u_2 \in \operatorname{Int} K$, then $u_1 = u_2$ and $p_1 = p_2$.

To prove the existence of solutions of variational inequality (0.6) we use the penalty method.

The plan of the paper is as follows. In Section 1 we consider the penalty equation for problem (0.1)-(0.3) and then prove Theorem 0.1. In Section 2 we prove Theorem 0.2 and 0.3. To prove Theorem 0.3 we linearize problem (0.1)-(0.3), iterate Cattabriga's estimates for the Stokes problem several times and then make use of Schauder's principle to show the existence of a solution of the nonlinear problem. Section 3 presents the proof of Theorem 0.4.

For convenience, we denote different positive numeric constants by the same letter C, where it is not confusing.

1. - Existence theorem

Let us consider the following problem in u and p:

$$-\nu\Delta u + \nabla p = f - (u\cdot\nabla)u - \frac{1}{\delta}\beta(u) \qquad \text{in } D\,, \tag{1.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } D\,, \tag{1.2}$$

$$u = 0 \qquad \text{on } S\,, \tag{1.3}$$

$(\delta > 0)$. The operator β above is the penalty operator in $L^2(D)$ related to the constraint $u \in K$, namely

$$\beta(u) = u - P_K(u)$$

where P_K is the projection in $L^2(D)$ on the set K:

$$P_K(u)(x) = \begin{cases} u(x) & \text{if } |u(x)| \le C_1\,, \\ C_1 \dfrac{u(x)}{|u(x)|} & \text{if } |u(x)| > C_1\,. \end{cases}$$

It is easy to see that $|\beta(u)| \le |u|$, $(\beta(u), u) = (|\beta(u)|, |u|)$ and that β is a continuous and monotone operator in $L^2(D)$.

Lemma 1.1: Suppose that $f \in L^2(D)$. Then there exists a solution (u, p) of problem (1.1)-(1.3) such that $u \in V$, $p \in L^q(D)$ for $q < \frac{3}{2}$ and $\int_D p(x)\,dx = 0$. Moreover, the following inequalities hold

$$\|u_x\| \le \frac{C}{\nu}|f|_2\,, \tag{1.4}$$

$$|p|_q \le C\left\{\frac{1}{\nu C_1}|f|_2^2 + |f|_2\left(1 + \frac{1}{\nu}|f|_2\right)\right\} \qquad \text{for } q \in (1, \tfrac{3}{2})\,. \tag{1.5}$$

PROOF: A priori estimates. We multiply both sides of equation (1.1) by u and integrate over D to get

$$\nu\|u\|_1^2+\frac{1}{\delta}(\beta(u),u)=(f,u)\,, \tag{1.6}$$

since $b(u,u,u)=0$. Noticing that $(\beta(u),u)\geq 0$ and that $H_0^1(D)\hookrightarrow L_6(D)$ we get then

$$\nu\|u\|_1^2\leq(f,u)\leq\|f\|_{-1}\cdot\|u\|_1\leq C|f|_{6/5}\|u\|_1\,.$$

Thus, (1.4) is proved.

We fix now u in the right-hand side of (1.1) and consider the linear problem

$$\begin{cases}-\nu\Delta u+\nabla p=\tilde{f} & \text{in } D\,,\\ \operatorname{div} u \qquad\quad =0 & \text{in } D\,,\\ u \qquad\qquad\quad\ =0 & \text{on } S\,,\end{cases}$$

where $\tilde{f}=f-(u\cdot\nabla)u-(1/\delta)\beta(u)$.

By the well known result of Cattabriga [2] we have

$$|p|_q\leq C\|\tilde{f}\|_{W_q^{-1}(D)}\,. \tag{1.7}$$

We shall show that $\|\tilde{f}\|_{W_q^{-1}(D)}$ can be estimated by the right-hand side of (1.5). From (1.4) and (1.6) we conclude that

$$\frac{1}{\delta}(\beta(u),u)\leq\frac{C}{\nu}|f|_{6/5}^2 \tag{1.8}$$

hence

$$\frac{C}{\nu}|f|_{6/5}^2\geq\frac{1}{\delta}(\beta(u),u)=\frac{1}{\delta}(|\beta(u)|,|u|)\geq\frac{C_1}{\delta}|\beta(u)|_1$$

that is

$$\frac{1}{\delta}|\beta(u)|_1\leq\frac{C}{\nu C_1}|f|_{6/5}^2\,. \tag{1.9}$$

Observe that if a distribution v belongs to $L^1(D)$, $D\subset R^3$, $q\in(1,\frac{3}{2})$, $1/q+1/q'=1$ and $\varphi\in\mathring{W}_{q'}^1(D)$, then

$$\|v\|_{W_q^{-1}(D)}=\sup\{|(v,\varphi)|:\ \|\varphi\|_{1,q'}\leq 1\}\leq\sup\{|\varphi|_\infty\cdot|v|_1:\ \|\varphi\|_{1,q'}\leq 1\}\leq$$
$$\leq\sup\{C\|\varphi\|_{1,q'}|v|_1:\ \|\varphi\|_{1,q'}\leq 1\}\leq C|v|_1 \tag{1.10}$$

since $\mathring{W}_{q'}^1(D)\hookrightarrow C^0(D)$ for $q'>3$ [3].

From (1.9) and (1.10) we have

$$(1.11) \qquad \left\| \frac{1}{\delta} \beta(u) \right\|_{W_s^{-1}(D)} \leq \frac{C}{\nu C_1} |f|_2^2 .$$

We shall show now that

$$(1.12) \qquad \|(u \cdot \nabla) u\|_{W_s^{-1}(D)} \leq \frac{C}{\nu} |f|_2^2 .$$

Since $H_0^1(D) \hookrightarrow L^6(D)$, we have

$$\int_D |(u \cdot \nabla) u|^{\frac{3}{2}} \leq C \int_D |u|^{\frac{3}{2}} |\nabla u|^{\frac{3}{2}} \leq C \left(\int_D |u|^6 \right)^{\frac{1}{4}} \left(\int_D |\nabla u|^2 \right)^{\frac{3}{4}} \leq C \left(\int_D |\nabla u|^2 \right)^{\frac{3}{4}} \left(\int_D |\nabla u|^2 \right)^{\frac{3}{4}},$$

hence

$$(1.13) \qquad |(u \cdot \nabla) u|_{\frac{3}{2}} \leq C |\nabla u|_2^2 .$$

On the other hand, by (1.10)

$$(1.14) \qquad \|(u \cdot \nabla) u\|_{W_s^{-1}(D)} \leq C |(u \cdot \nabla) u|_1 \leq C |(u \cdot \nabla) u|_{\frac{3}{2}} .$$

From (1.4), (1.13), (1.14) we get (1.12).

Now, from (1.11), (1.12) and

$$\|f\|_{W_s^{-1}(D)} \leq C |f|_2$$

we conclude (1.5).

To prove the existence of the relevant u and p we proceed as follows. At first we prove the solvability in u of the problem

$$\nu((u, a)) + b(u, u, a) + \frac{1}{\delta} (\beta(u), a) = (f, a), \qquad \text{for all } a \in V .$$

The proof is very similar to that for the stationary Navier-Stokes equations (see [11]) so we omit the details. The existence of a suitable p follows then directly from the main theorem in [2]. Thus, Lemma 1.1 is proved.

Remark: It is easy to show that in fact $p \in W_{\frac{3}{2}}^1(D)$ and $u \in W_{\frac{3}{2}}^2(D)$. It seems difficult, however, to get estimates of u and p—in the norms of these spaces—uniform with respect to δ.

Proof of Theorem 0.1: We shall prove that the solutions $(u, p) = (u^\delta, p^\delta)$ of (1.1)-(1.3) from Lemma 1.1 converge with $\delta \to 0$ to a solution (u, p) of variational inequality (0.6).

We write (1.1)-(1.3) in a weak formulation as follows

$$(1.15)\qquad \nu((u^\delta, a)) + b(u^\delta, u^\delta, a) + \frac{1}{\delta}(\beta(u^\delta), a) - (p^\delta, \operatorname{div} a) = (f, a) \quad \text{for all } a \in H_0^1(D).$$

Let us put $a = u^\delta - \varphi$, $\varphi \in H_0^1(D) \cap K$ in (1.15).

For $\varphi \in K$ we have $\beta(\varphi) = 0$ and by the monotonicity of β

$$\frac{1}{\delta}(\beta(u^\delta), u^\delta - \varphi) = \frac{1}{\delta}(\beta(u^\delta) - \beta(\varphi), u^\delta - \varphi) \geq 0,$$

so we can write

$$(1.16)\qquad \nu((u^\delta, u^\delta - \varphi)) + b(u^\delta, u^\delta, u^\delta - \varphi) - (p^\delta, \operatorname{div}(u^\delta - \varphi)) \leq (f, u^\delta - \varphi).$$

From (1.4), (1.5) and (1.8) we conclude the existence of a subsequence $(u^{\delta'}, p^{\delta'})$, $\delta' \to 0$, such that

$$(1.17)\qquad u^{\delta'} \to u \begin{cases} \text{weakly in } V, \\ \text{strongly in } L^2(D) \text{ and a.e. in } D, \end{cases}$$

$$(1.18)\qquad p^{\delta'} \to p \qquad \text{weakly in } L^q(D),\ q \in (1, \tfrac{3}{2}),$$

where $u \in K$.

We put $(u^\delta, p^\delta) = (u^{\delta'}, p^{\delta'})$ in (1.16) and pass to zero with δ'. Using (1.17), (1.18) and observing that

$$((u, u)) \leq \liminf_{\delta' \to 0} ((u^{\delta'}, u^{\delta'}))$$

we get (0.6). The proof of Theorem 1.1 is complete.

2. - Connections with the Equations of Motion

In this Section we consider the relationship between solutions of variational inequality (0.6) and weak solutions of the boundary-value problem (0.1)-(0.3). We prove Theorems 0.2 and 0.3.

Proof of Theorem 0.2: Suppose that $u \in \operatorname{Int} K$ and that (u, p) satisfies variational inequality (0.6) for all $\varphi \in H_0^1(D) \cap K$. For arbitrary $\xi \in C_0^\infty(D; R^3)$ there exists $\varepsilon_0 > 0$ such that

$$\varphi = u - \varepsilon\xi \in K,$$

provided $|\varepsilon| < \varepsilon_0$. We put φ of the above form into (0.6) to get

$$\varepsilon\{\nu((u,\xi)) + b(u,u,\xi) - (p, \operatorname{div}\xi) - (f,\xi)\} \leq 0$$

independently of the sign of ε. Hence

$$\nu((u,\xi)) + b(u,u,\xi) - (p, \operatorname{div}\xi) = (f,\xi) \tag{2.1}$$

for all $\xi \in C_0^\infty(D; R^3)$ and, in consequence, for all $\xi \in H_0^1(D)$. Thus we have proved that (u, p) is a weak solution of the boundary-value problem (0.1)-(0.3). From the results concerning the regularity of weak solutions of problem (0.1)-(0.3) [2] we conclude that $u \in W_q^2(D)$, $p \in W_q^1(D)$.

Conversely, suppose that (u, p) satisfies (2.1) for all $\xi \in H_0^1(D)$ and that $u \in K$. We can put $\xi = u - \varphi$, $\varphi \in H_0^1(D) \cap K$ in (2.1) to get (0.6). This completes the proof of Theorem 0.2.

Proof of Theorem 0.3: Throughout the proof we adapt much of the argument used lately in [12].

Let $f \in L^q(D)$, $q > 3$ and let

$$R_0 = \left(\frac{1}{\nu}|f|_q + 1\right)|f|_q,$$

$$R_1 = |f|_q + R_0^2,$$

$$A = \left\{v \in V \cap C^0(D) \colon \|v\|_1 \leq \frac{1}{\nu}\|f\|_{V'},\ |v|_q \leq r_0 R_0,\ |v|_\infty \leq r_1 R_1\right\},$$

where r_0, r_1 are some positive constants which will be defined later on.

At first we shall prove two lemmas.

Lemma 2.1: Let $f \in L^q(D)$, $q > 3$ and $v \in A$. Then the problem

$$-\nu\Delta u + \nabla p = f - (v\cdot\nabla)u \qquad \text{in } D, \tag{2.2}$$

$$\operatorname{div} u = 0 \qquad \text{in } D, \tag{2.3}$$

$$u = 0 \qquad \text{on } S, \tag{2.4}$$

has a unique solution (u, p) such that $u \in A \cap W_q^2(D)$, $p \in W_q^1(D)$, $\int_D p(x)\,dx = 0$. Moreover

$$|u|_{2,q} + |p|_{1,q} \leq C(|f|_q + R_1^2). \tag{2.5}$$

Proof: A priori estimates. Multiplying both sides of (2.2) by u and integrating over D we get easily

$$\|u\|_1 \leq \frac{1}{\nu}\|f\|_{V'}. \tag{2.6}$$

From (1.4) and (1.13) we have

$$|(v\cdot\nabla)u|_3 < \frac{C}{\nu}|f|_q^2 .$$

By Cattabriga's estimate for the Stokes problem we get then

$$\|u\|_{2,3} + \|p\|_{1,3} < \left(\frac{C}{\nu}|f|_q + 1\right)|f|_q < CR_0 .$$

Since $W_3^2(D) \to L^q(D)$, there exists a positive constant r_0 such that

(2.7) $$|u|_q < r_0 R_0 .$$

Let

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{3} .$$

We have

$$\int_D |(v\cdot\nabla)u|^r < C\int_D |v|^r |\nabla u|^r < C\left(\int_D |v|^q\right)^{r/q}\left(\int_D |\nabla u|^3\right)^{r/3}$$

so that

$$|(v\cdot\nabla)u|_r < C|v|_q \cdot |\nabla u|_3 < CR_0^2$$

where C does not depend on u, v. Also $|f|_r < C|f|_q$. From Cattabriga's estimate again we have

$$\|u\|_{2,r} + \|p\|_{1,r} < C|f|_q + CR_0^2 < CR_1 .$$

From $W_r^2(D) \to W_q^1(D) \hookrightarrow C^0(D)$ [3] we conclude the existence of a constant r_1 for which

(2.8) $$|u|_\infty < r_1 R_1 .$$

In the end

$$|(v\cdot\nabla)u|_q < C|v|_\infty \cdot |\nabla u|_q < CR_1^2$$

so again by Cattabriga's estimate we get (2.5). Inequalities (2.6), (2.7) and (2.8) give $u \in A$.

To prove the existence of a (unique) solution (u, p), we use the same argument as that described in the proof of Lemma 1.1.

Let us define now the map Φ on A by $\Phi(v) = (u, p)$, where (u, p) is the unique solution of the boundary-value problem (2.2)-(2.4) from Lemma 2.1 and set $\Phi_1(v) = u$.

Lemma 2.2: The map Φ_1: $A \to A$ is continuous with respect to the uniform topology.

Proof: Let $\Phi(v_n)=(u_n, p_n)$, $\Phi(v)=(u, p)$. Because

$$-\nu\Delta u+(v\cdot\nabla)u+\nabla p \quad =f,$$

$$-\nu\Delta u_n+(v_n\cdot\nabla)u_n+\nabla p_n=f,$$

we have

$$(2.9)\qquad -\nu\nabla(u-u_n)+((v-v_n)\cdot\nabla)u_n+(v\cdot\nabla)(u-u_n)+\nabla(p-p_n)=0\,.$$

After multiplying both sides of (2.9) by $u-u_n$ and integrating over D we get

$$\nu|u-u_n|_1^2+b(v-v_n, u_n, u-u_n)=0\,.$$

But

$$|b(v-v_n, u_n, u-u_n)|\le C|v-v_n|_\infty\cdot|u_n|_1\cdot|u-u_n|_1$$

so that, in view of (2.6)

$$\nu|u-u_n|_1\le C|v-v_n|_\infty$$

where C does not depend on u and u_n.

Thus, if $|v_n-v|_\infty\to 0$ as $n\to\infty$ then $u_n\to u$ in V. In consequence

$$((v-v_n)\cdot\nabla)u_n+(v\cdot\nabla)(u-u_n)\to 0 \qquad \text{in } L^2(D)\,,\ \text{as } n\to\infty$$

and from (2.9) and Cattabriga's estimates for the Stokes problem we conclude that $u_n\to u$ in $W_2^2(D)$ and therefore uniformly in $\bar{D}$. Lemma 2.2 is proved.

Now we are in a position to complete the proof of Theorem 0.3 in a few lines. To prove the existence of a solution (u, p) of problem (0.1)-(0.3) as in Theorem 0.3 it suffices to show that the operator Φ_1 has a fixed point. The above considerations imply that $\Phi_1(A)\subset A$, that $\Phi_1(A)$ is a bounded subset of $W_2^2(D)$, hence a relatively compact subset of $C^0(\bar{D})$, and that Φ_1 is continuous in the uniform topology on $\bar{D}$. From Schauder's principle we conclude the existence of $u\in A$ for which $\Phi_1(u)=u$.

From estimate (2.5) and the embeddings $W_q^i(D)\hookrightarrow C^{i-1}(\bar{D})$ $(i=1, 2;$ $q>3)$ [3] it follows that there exists a constant C_2 such that

$$|p(x)|\le C_2\,,\quad |u(x)|\le C_2\,,\quad |\nabla u(x)|\le C_2$$

for almost all $x\in D$. If $|f|_q$ is sufficiently small, we can take $C_2<C_1$. This completes the proof of Theorem 0.3.

3. - Continuous dependence on data and uniqueness

In this Section we prove Theorem 0.4.

Suppose that (u_1, p_1), (u_2, p_2) are two solutions of inequality (0.6) corresponding to data f_1 and f_2 respectively. We write (0.6) for $(u_1, p_1), f_1$ and

$(u_2, p_2), f_2$ respectively, take $\varphi = \frac{1}{2}(u_1 + u_2)$ and add obtained inequalities to get

$$\frac{1}{2}\nu\|u_1-u_2\|_1^2 + \frac{1}{2}b(u_1-u_2, u_2, u_1-u_2) + \frac{1}{2}(f_2-f_1, u_1-u_2) < 0 . \tag{3.1}$$

Since $|u_i| < C$, $i = 1, 2$, then

$$\frac{1}{2}|b(u_1-u_2, u_2, u_1-u_2)| < CC_1\|u_1-u_2\|_1^2 .$$

Also

$$|\frac{1}{2}(f_2-f_1, u_1-u_2)| < C|f_2-f_1|_2\,|u_1-u_2|_2 .$$

From (3.1) and the above inequalities we get

$$(\frac{1}{2}\nu - CC_1)\|u_1-u_2\|_1 < C|f_1-f_2|_2 .$$

If C_1 is such that $\alpha = \frac{1}{2}\nu - CC_1 > 0$ then

$$\|u_1-u_2\|_1 < \frac{C}{\alpha}|f_1-f_2|_2 . \tag{3.2}$$

This proves the first part of Theorem 0.4.

We proceed now to the proof of the second part. If $u_i \in \operatorname{Int} K$, $i = 1, 2$, then by Theorem 0.2

$$-\nu_1 \Delta u_i + \nabla p_i = f_i - (u_i \cdot \nabla) u_i , \qquad i = 1, 2$$

hence

$$\begin{aligned} -\nu\Delta(u_1-u_2) + \nabla(p_1-p_2) &= \\ &= f_1 - f_2 + -(u_1\cdot\nabla)(u_1-u_2) - ((u_1-u_2)\cdot\nabla)u_2 = h . \end{aligned} \tag{3.3}$$

It is easy to see that

$$|h|_2 < |f_1-f_2|_2 + C_1\|u_1-u_2\|_1 + \|u_1-u_2\|_1|f_2|_2 . \tag{3.4}$$

Now, by Cattabriga's estimate for the Stokes problem and inequalities (3.2), (3.4) we get (0.7). The proof of Theorem 0.4 is complete.

REFERENCES

[1] M. Biroli, *Sur l'inéquation d'evolution de Navier-Stokes*, Notes I, II, III, Rend. Acc. Naz. Lincei, VII, vol. LII, 1972.

[2] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Mat. Sem. Univ. Padova, 31 (1961), pp. 308-340.

[3] A. Kufner, O. John - S. Fucik, *Function spaces*, Prague, 1977.

[4] J. L. Lions, *Quelques méthodes de résolution des problèmes avec limites non linéaires*, Dunod, Paris, 1969.

[5] G. Lukaszewicz - B. A. Ton, *On a variational problem associated with non-stationary flows of granulated media* (to appear).

[6] G. Prouse, *Modelli matematici in inquinamento dei fluidi*, Bollettino U.M.I., Analisi Funzionale e Applicazioni, Serie VI, Vol. III-C, No. 1, 1984.

[7] G. Prouse, *Analysis of a three-dimensional mathematical model of water circulation in a basin*, Rend. Acc. Naz. XL, 1981-82, Vol. V, fasc. 2, pp. 13-36.

[8] G. Prouse, *On an inequality related to the motion, in any dimension, of viscous, incompressible fluids*, Notes I, II, III, Rend. Acc. Naz. Lincei, 67, 1979.

[9] G. Prouse - A. Zaretti, *On the inequalities associated to a model of Graffi for the motion of a mixture of two viscous, incompressible fluids*, (to appear in Arch. Rat. Mech. Anal.).

[10] G. Prouse - F. Rolandi - A. Zaretti, *On a mathematical model describing interaction between two gravitational fields*, preprint, Dipartimento di Matematica del Politecnico di Milano, 1985.

[11] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland, Amsterdam, New York and Oxford, 1979.

[12] H. B. da Veiga, *On the stationary motion of granulated media*, preprint, Dipartimento di Matematica, Università degli Studi di Trento, 1985.

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