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On a Variational Problem Associated with non-Stationary Flows of Granulated Media (***)

SUMMARY. — Weak solutions of the equations of motion of granulated media (equations (0.1)-(0.2) below) were considered in [6]. In the present paper we prove the existence of a weak solution (u, p, ω) of a variational problem associated with these equations, when the angular velocity ector of rotation of particles ω is subjected to the constraint $|\omega(x, t)| \leq 1$ in the considered domain D.

Su un problema variazionale associato alle correnti non stazionarie di un mezzo granulare

RIASSUNTO. — In [6] sono considerate soluzioni deboli per le equazioni del moto di un mezzo granulare (cioè per le successive equazioni (0.1)-(0.2)). In questo lavoro noi dimostriamo l'esistenza di una soluzione debole (u, p, ω) per un problema variazionale legato a queste equazioni, e precisamente la dimostriamo nel caso che nel dominio considerato il vettore ω velocità angolare di rotazione delle particelle sia astretto alla limitazione $|\omega(x, t)| \leq 1$.

It is the purpose of this paper to establish the existence of a solution of a variational problem associated with the non-stationary flow of granulated media.

The flow of granulated media of constant density is described by the coupled parabolic-hyperbolic system [1]:

$$
(0.1) \qquad \left| \begin{array}{l} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p - \eta(\omega \times u) = f \\ \nabla \cdot u = 0 \end{array} \right.
$$

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and

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega + F(p)\omega = g. \tag{0.2}$$

The velocity, angular velocity of rotation of particles and the pressure are denoted by u, ω and by p respectively. The positive constants ν, η are the viscosity and the Magnus coefficients. The vector-functions f and g are the exterior mass forces and the density of momentum of the forces, the scalar function $F(p)$ describes the friction between the particles.

We assume that the system (0.1)-(0.2) is satisfied in a domain $Q = G \times (0, T)$, where G is a bounded open connected subset of R^3 with a smooth boundary and $0 < T < \infty$.

The boundary and initial conditions we add to (0.1)-(0.2) are

$$u(x, t) = 0 \quad \text{on } \partial G \times (0, T), \qquad u(x, 0) = u_0(x) \quad \text{in } G \tag{0.3}$$

and

$$\omega(x, 0) = \omega_0(x) \quad \text{in } G. \tag{0.4}$$

The existence of a strong local (in time) solution of (0.1)-(0.4) was established by Antoncev, Kashykov and Monachov [1], Antoncev and Leluch [2]. A weak solution of (0.1)-(0.4) for arbitrary but finite time-interval was shown by Łukaszewicz [6]. Stationary problems were studied by Łukaszewicz [7] and by da Veiga [11].

In this paper we shall consider a variational problem associated with (0.1)-(0.4), namely when the angular velocity ω is subjected to the constraint $|\omega(x, t)| \le 1$ in Q.

The notations, a detailed outline of the paper and the main result are given in Section 1.

1. - Let G be a bounded open subset of R^n with a smooth boundary ∂G an let $W^{k,q}(G)$ be the usual Sobolev space:

$$W^{k,q}(G) = \{v \colon D^\alpha v \text{ in } L^q(G),\ |\alpha| \le k\}$$

with the norm

$$\|v\|_{W^{k,q}(G)} = \Big\{\sum_{|\alpha| \le k} \|D^\alpha v\|^q_{L^q(G)}\Big\}^{1/q}; \quad 1 < q < \infty.$$

The completion of $C_0^\infty(G)$ with respect to the $W^{k,q}(G)$-norm is denoted by $W_0^{k,q}(G)$ and its dual by $W^{-k,s}(G)$ with $1/q + 1/s = 1$.

The dual of $W^{k,q}(G)$ is written as $W^{-k,s}(\bar{G})$.

Let $S = \{u \colon u \in C_0^\infty(G),\ \nabla \cdot u = 0 \text{ in } G\}$. We write V, H as the closure of S in the $W^{1,2}(G)$ and in the $L^2(G)$-norm respectively.

$B^{k,q}(G)$; $0 < k < 1$, $1 < q < \infty$ is the Slobodeckii-space with the norm

$$\|u\|_{B^{k,q}(G)} = \|u\|_{L^q(G)} + \int_G \Big\{\int_G \frac{|u(x) - u(y)|^q}{|x - y|^{3+kq}}\, dy\Big\}^{1/q} dx.$$

$L^s(0, T; W^{r,q}(G))$ is the set of equivalence classes of functions $u(\cdot, t)$ from $(0, T)$ to $W^{r,q}(G)$ which are L^s-integrable over $(0, T)$. It is a reflexive Banach space with the norm

$$\|u\|_{L^s(0,T;W^{r,q}(G))} = \left\{ \int_0^T \|u(\cdot, t)\|^s_{W^{r,q}(G)} dt \right\}^{1/s},$$

$1 < s < \infty$ and $1 < q < \infty$.

We sometimes write u' instead of $\partial u/\partial t$ to denote the time derivative of a function u.

In this paper we shall show the existence of $\{u, p, \omega\}$ such that:

$$(1.1) \quad \left| \begin{array}{l} u' - \nu \Delta u + (u \cdot \nabla) u + \nabla p - \eta(\omega \times u) = f \ \text{ in } Q, \\ \nabla \cdot u = 0 \ \text{ in } Q, \qquad u(x, t) = 0 \ \text{ on } \partial G \times (0, T), \\ u(x, 0) = u_0(x) \ \text{ in } G, \\ \int_G p(x, t)\, dx = 0 \ \text{ for almost all } t \text{ in } (0, T) \end{array} \right.$$

and

$$(1.2) \quad \left| \begin{array}{l} \int_0^T (\varphi', \varphi - \omega)\, dt + \int_0^T ((u \cdot \nabla)\varphi, \varphi - \omega)\, dt + \int_0^T (F(p)\omega, \varphi - \omega)\, dt \geq \\ \qquad \qquad \geq -\frac{1}{2} \|\varphi(0) - \omega_0\|^2_{L^2(G)} + \int_0^T (g, \varphi - \omega)\, dt, \\ |\omega(x, t)| \leq 1 \quad \text{a.e. in } Q \end{array} \right.$$

for all φ in $L^5(0, T; W_0^{1,5}(G))$ with φ' in $L^{5/4}(Q)$ and such that $|\varphi(x, t)| \leq 1$ a.e. in Q.

The main result of the paper is the following theorem.

THEOREM 1.1: *Let $\{f, g, u_0, \omega_0\}$ be in $L^2(Q) \times L^\infty(Q) \times \{H \cap B^{2/5,5/4}(G)\} \times$ $\times L^\infty(G)$ with $|\omega_0(x)| \leq 1$ a.e. in G.*

Let F be a weakly continuous mapping of $L^{5/4}(0, T; W^{1,5/4}(G))$ into $L^{5/4}(0, T;$ $W^{\alpha,5/4}(G))$ for some $0 < \alpha < 1$.

Suppose further that:

i) *$0 \leq F(q)$ a.e. in Q for all q in $L^{5/4}(0, T; W^{1,5/4}(G))$ with $\int_G q(x, t)\, dx = 0$ a.e. in $[0, T]$.*

ii) *there exists a positive, strictly increasing, continuous function ψ with*

$$\|F(p)\|_{L^{5/4}(0,T;W^{\alpha,5/4}(G))} \leq \psi(\|p\|_{L^{5/4}(0,T;W^{1,5/4}(G))}).$$

(iii) *F is continuous from $L^{5/4}(0, T; W^{1,5/4}(G))$ into $L^{5/4}(0, T; L^{5/4}(G))$.*

Then there exists $\{u, p, \omega\}$ in

$$\{L^\infty(0, T; H) \cap L^2(0, T; V)\} \times L^{5/4}(0, T; W^{1,5/4}(G)) \times L^\infty(Q)$$

with $|\omega(x,t)|\leq 1$ *a.e. in* Q, *solution of* (1.1)-(1.2). *Moreover* ω' *is in* $L^{5/4}(0,T;W^{-1,5/4}(G))$ *and* ω *belongs to* $C([0,T];W^{-1,5/4}(G))$ *with* $\omega(x,0)=\omega_0(x)$ *in* G.

In Section 2 we use a discretisation of the time-variable and a singular perturbation method to show the existence of a unique solution of the initial boundary-value problem:

$$
(1.3)\qquad \left|\begin{array}{l}
\omega_\varepsilon'-\varepsilon\Delta\omega_\varepsilon+(v\cdot\nabla)\omega_\varepsilon+F(p)\omega_\varepsilon+\varepsilon^{-1}\beta(\omega_\varepsilon)=g \ \text{ in } Q,\\
\omega_\varepsilon(x,t)=0 \ \text{ on } \partial G\times(0,T), \qquad \omega_\varepsilon(x,0)=\omega_0(x) \ \text{ in } G,\\
\text{with } \beta(\omega_\varepsilon)=(1-|\omega_\varepsilon|^2)^-\omega_\varepsilon .
\end{array}\right.
$$

Appropriate uniform estimates for ω_ε, ω_ε' and for $\varepsilon^{-1}\beta(\omega_\varepsilon)$ are established.
In Section 3 the Schauder fixed point theorem is used to show the existence of a solution of the coupled system:

$$
(1.4)\qquad \left|\begin{array}{l}
u_\varepsilon'-\nu\Delta u_\varepsilon+(u_\varepsilon\cdot\nabla)u_\varepsilon+\nabla p_\varepsilon-\eta(\omega_\varepsilon\times u_\varepsilon)=f, \qquad \nabla\cdot u_\varepsilon=0 \ \text{ in } Q,\\
\omega_\varepsilon'-\varepsilon\Delta\omega_\varepsilon+(u_\varepsilon\cdot\nabla)\omega_\varepsilon+F(p_\varepsilon)\omega_\varepsilon+\varepsilon^{-1}\beta(\omega_\varepsilon)=g \ \text{ in } Q,\\
u_\varepsilon(x,t)=0 \ \text{ on } \partial G\times(0,T), \qquad u_\varepsilon(x,0)=u_0(x), \qquad \omega_\varepsilon(x,0)=\omega_0(x)\\
\text{with } \int_G p_\varepsilon(x,t)\,dx=0 \ \text{ for almost all } t \text{ in } (0,T).
\end{array}\right.
$$

Theorem 1.1 is proved in Section 4 by letting $\varepsilon\to 0$ in (1.4).

2. - In this section we study the initial boundary value problem:

$$
(2.1)\qquad \left|\begin{array}{l}
\omega'-\varepsilon\Delta\omega+(v\cdot\nabla)\omega+F(p)\omega+\varepsilon^{-1}\beta(\omega)=g \ \text{ in } Q,\\
\omega(x,t)=0 \ \text{ on } \partial G\times(0,T), \qquad \omega(x,0)=\omega_0(x) \ \text{ in } G,\\
|\omega_0(x)|\leq 1 \ \text{ a.e. in } G, \qquad \text{with } \beta(\omega)=(1-|\omega|^2)^-\omega .
\end{array}\right.
$$

The function $(1-|\omega|^2)^-$ is equal to 0 if $|\omega|^2\leq 1$ and to $|\omega|^2-1$ if $|\omega|^2>1$. The main result of the section is the following theorem.

THEOREM 2.1: *Let* $\{v,p\}$ *be in* $\{L^2(0,T;V)\cap L^\infty(0,T;H)\}\times L^{5/4}(0,T;W^{1,5/4}(G))$. *Let* $\{g,\omega_0\}$ *and* F *be as in Theorem* 1.1. *Then there exists a unique* $\omega_\varepsilon=\omega$ *in* $L^\infty(Q)\cap L^2(0,T;W_0^{1,2}(G))$, *solution of* (2.1). *Moreover:*

$$
\|\omega\|_{L^\infty(Q)}+\varepsilon^{1/2}\|\omega\|_{L^2(0,T;W_0^{1,2}(G))}+\varepsilon^{-1}\|\beta(\omega)\|_{L^2(Q)}\leq CE(g,\omega_0)
$$

where $E(g,\omega_0)=1+\|\omega_0\|_{L^\infty(G)}+\|g\|_{L^\infty(Q)}$.
C is independent of ε, v, p.

To prove the key estimates of the theorem, namely those of ω in $L^\infty(Q)$ and of $\varepsilon^{-1}\beta(\omega)$ in $L^2(Q)$ respectively we shall use a discretisation of the time-variable and then multiply the approximating equations by nonlinear expressions.

Let N be a large positive integer and let $h = T/N$.

Set:

$$g^k(x) = h^{-1}\int_{kh}^{(k+1)h} g(x, t)\,dt\,; \qquad 0 \leq k \leq N-1\,.$$

In the same way we define v^k and p^k. Consider the nonlinear elliptic boundary-value problems in ω^k:

$$(2.2)\quad \begin{cases} \omega^k - \omega^{k-1} - \mu h \sum_{i=1}^{2} D_i(|D_i\omega^k|^2 D_i\omega^k) - \varepsilon h \Delta\omega^k + h(v^k\cdot\nabla)\omega^k + \\ \qquad\qquad\qquad + hF(p^k)\omega^k + h\varepsilon^{-1}\beta(\omega^k) = hg^k \quad \text{in } G\,, \\ \omega^k = 0 \quad \text{on } \partial G\,, \qquad \omega^0 = \omega_0\,; \qquad 1 \leq k \leq N-1\,. \end{cases}$$

LEMMA 2.1: *Suppose all the hypotheses of Theorem 2.1 are satisfied. Then there exists for each k, a solution $\omega^k_{\varepsilon\mu} = \omega^k$ of (2.2). Moreover:*

$$\|\omega^k\|^2_{L^\infty(G)} + \mu h \sum_{j=1}^{k} \|\nabla\omega^j\|^4_{L^4(G)} + \varepsilon h \sum_{j=1}^{k} \|\nabla\omega^j\|^2_{L^2(G)} \leq CE^2(g, \omega_0)\,.$$

$E(g, \omega_0)$ is as in Theorem 2.1 and C is independent of ε, μ, h, k, p and v.

PROOF: 1) Since G is a bounded open subset of R^2 with a smooth boundary, it follows from the Sobolev imbedding theorem that $W^{1,4}(G) \subset L^\infty(G)$.

$W^{1,4}(G)$ is an algebra with respect to pointwise multiplication and $1-|\omega|^2$ is in $W^{1,4}(G)$. It follows from a result of Stampacchia [10] that $(1-|\omega|^2)^-$ is in $W^{1,4}(G)$ and hence $\beta(\omega) = (1-|\omega|^2)^-\omega$ is in $W^{1,4}_0(G)$.

Let σ be the nonlinear mapping of $W^{1,4}_0(G)$ into its dual $W^{-1,4/3}(G)$ defined by:

$$(\sigma(\omega), \varphi) = \mu \sum_{i=1}^{2} (|D_i\omega|^2 D_i\omega, D_i\varphi) + \varepsilon(\nabla\omega, \nabla\varphi) + (F(p)\omega, \varphi) + \varepsilon^{-1}(\beta(\omega), \varphi)$$

for φ in $W^{1,4}_0(G)$.

Since $F(p) > 0$, β is monotone and $(\beta(\omega), \omega) > 0$. It is not difficult to check that σ is a monotone, hemi-continuous operator taking bounded sets of $W^{1,4}_0(G)$ into bounded sets of $W^{-1,4/3}(G)$. Moreover σ is a coercive operator in $W^{1,4}_0(G)$.

It follows from the standard theory of monotone coercive operators in reflexive Banach spaces that there exists a solution ω^k of (2.2) for each k.

2) We shall now establish the estimates of the lemma. Since $W^{1,4}_0(G)$ is an algebra, $|\omega^k|^{s-2}\omega^k$ is in $W^{1,4}_0(G)$ for each s, $2 \leq s < \infty$. Multiplying (2.2)

by the nonlinear expression $|\omega^k|^{s-2}\omega^k$ and integrating over G we have:

$$(2.3)\quad \|\omega^k\|^s_{L^s(G)} + \mu b \sum_{j=1}^{3} (|D_j\omega^k|^2 D_j\omega^k, D_j\{|\omega^k|^{s-2}\omega^k\}) + \varepsilon b(\nabla\omega^k, \nabla\{|\omega^k|^{s-2}\omega^k\}) +$$
$$+ b((v^k\cdot\nabla)\omega^k, |\omega^k|^{s-2}\omega^k) \leq \|\omega^k\|^{s-1}_{L^s(G)}\|\omega^{k-1}\|_{L^s(G)} + b(g^k, |\omega^k|^{s-2}\omega^k).$$

In the above estimate we have used the facts that:

$$F(p)\geq 0 \quad \text{and} \quad (\beta(\omega^k), |\omega^k|^{s-2}\omega^k)\geq 0.$$

A straightforward calculation yields:

$$((v^k\cdot\nabla)\omega^k, |\omega^k|^{s-2}\omega^k) = 0.$$

We have:

$$\sum_{j=1}^{3} (|D_j\omega^k|^2 D_j\omega^k, D_j\{|\omega^k|^{s-2}\omega^k\}) = \sum_{j=1}^{3} (|D_j\omega_k|^2 D_j\omega_k, |\omega^k|^{s-2} D_j\omega_k) +$$
$$+ \frac{(s-2)}{4}\sum_{j=1}^{3} (|D_j\omega_k|^2 |\omega^k|^{s-4}, (D_j|\omega^k|^2)^2) \geq 0.$$

Similarly:

$$(\nabla\omega^k, \nabla\{|\omega^k|^{s-2}\omega^k\}) \geq 0.$$

Therefore (2.3) becomes:

$$\|\omega^k\|_{L^s(G)} \leq b\|g^k\|_{L^s(G)} + \|\omega^{k-1}\|_{L^s(G)}.$$

Since $\{\omega^k, \omega^{k-1}, g^k\}$ are in $L^\infty(G)$ we may let $s\to+\infty$ and obtain:

$$\|\omega^k\|_{L^\infty(G)} \leq \|\omega^{k-1}\|_{L^\infty(G)} + b\|g^k\|_{L^\infty(G)}.$$

Therefore:

$$(2.4)\qquad \|\omega^k\|_{L^\infty(G)} \leq \|\omega_0\|_{L^\infty(G)} + b\sum_{i=1}^{k}\|g^i\|_{L^\infty(G)} \leq \|\omega_0\|_{L^\infty(G)} + T\|g\|_{L^\infty(Q)}.$$

3) Returning to (2.2), multiplying it by ω^k and integrating over G we get:

$$\|\omega^k\|^2_{L^2(G)} + \mu b\|\nabla\omega^k\|^4_{L^4(G)} + \varepsilon b\|\nabla\omega^k\|^2_{L^2(G)} \leq b\|g^k\|_{L^2(G)}\|\omega^k\|_{L^2(G)} +$$
$$+ \tfrac{1}{2}\|\omega^{k-1}\|^2_{L^2(G)} + \tfrac{1}{2}\|\omega^k\|^2_{L^2(G)}.$$

Taking into account (2.4) we obtain:

$$(2.5)\qquad \mu b\sum_{i=1}^{k}\|\nabla\omega^i\|^4_{L^4(G)} + \varepsilon b\sum_{i=1}^{k}\|\nabla\omega^i\|^2_{L^2(G)} \leq$$
$$\leq \tfrac{1}{2}\|\omega_0\|^2_{L^2(G)} + (\mathrm{mes}\, G)^{1/2}\{\|\omega_0\|_{L^\infty(G)} + T\|g\|_{L^\infty(Q)}\}\, T\|g\|_{L^\infty(Q)}.$$

The lemma is proved.

LEMMA 2.2: *Suppose all the hypotheses of Theorem 2.1 are satisfied. Then there exists for each k, $1<k<N-1$, a solution $\omega_\varepsilon^k = \omega^k$ of the elliptic boundary-value problem:*

$$(2.6)\quad \left|\begin{array}{l} \omega^k-\omega^{k-1}-\varepsilon h\,\Delta\omega^k+h(v^k\cdot\nabla)\omega^k+hF(p^k)\omega^k+h\varepsilon^{-1}\beta(\omega^k)=hg^k \quad \text{in } G, \\ v^k=0 \quad \text{on } \partial G, \qquad (\omega^0=\omega_0(x) \text{ in } G). \end{array}\right.$$

Moreover:

$$\|\omega^k\|^2_{L^\infty(G)}+\varepsilon h\sum_{j=1}^{k}\|\nabla\omega^j\|^2_{L^2(G)}<CE^2(g,\omega_0).$$

C is independent of k, h, ε, p and v. The expression $E(g,\omega_0)$ is as in Theorem 2.1.

PROOF: 1) Let $\omega^k_{\varepsilon\mu}$ which we shall write as ω^k_μ for short, be as in Lemma 2.1. With the estimates of the lemma we get, by taking subsequences if necessary: $\omega^k_\mu\to\omega^k$ weakly in $W_0^{1,2}(G)$ and in the weak*-topology of $L^\infty(G)$, $\mu^{1/4}\omega^k_\mu\to 0$ weakly in $W_0^{1,4}(G)$ as $\mu\to 0$.

The estimates of the Lemma follow from those of Lemma 2.1.

2) We now show that ω^k is a solution of (2.6). Clearly

$$\mu\sum_{i=1}^{2}(|D_i\omega^k_\mu|^2 D_i\omega^k_\mu, D_i\varphi)\to 0$$

as $\mu\to 0$ for all φ in $W_0^{1,4}(G)$.

It remains only to show that there exists a subsequence $\{\mu\}$ such that

$$\beta(\omega^k_\mu)=(1-|\omega^k_\mu|^2)^{-1}\omega^k_\mu\to\beta(\omega^k) \qquad \text{weakly in } L^2(G) \text{ as } \mu\to 0.$$

We know that:

$$(\beta(\omega^k_\mu)-\beta(\varphi),\omega^k_\mu-\varphi)>0 \qquad \text{for all } \varphi \text{ in } W_0^{1,4}(G).$$

On the other hand:

$$\|\beta(\omega^k_\mu)\|_{L^2(G)}<\|\omega^k_\mu\|_{L^\infty(G)}\|1-|\omega^k_\mu|^2\|_{L^2(G)}<C\{1+\|\omega^k_\mu\|^2_{L^\infty(G)}+\|\omega^k_\mu\|_{L^\infty(G)}\}<M.$$

Thus, by taking subsequences we have: $\beta(\omega^k_\mu)\to\psi^k$ weakly in $L^2(G)$. Therefore by applying the Sobolev imbedding theorem, we get:

$$(\psi^k-\beta(\varphi),\omega^k-\varphi)>0 \qquad \text{for all } \varphi \text{ in } L^\infty(G)\cap W_0^{1,2}(G).$$

Take $\varphi=\omega^k+\lambda\chi$ with χ in $L^\infty(G)\cap W_0^{1,2}(G)$. Then:

$$(\psi^k-\beta(\omega^k+\lambda\chi),\chi)<0.$$

Let $\lambda \to 0^+$ and the hemi-continuity of β yields:

$$(\psi^k - \beta(\omega^k), \chi) = 0 .$$

Hence:

$$\beta(\omega^k) = \psi^k .$$

The lemma is proved.

LEMMA 2.3: *Suppose all the hypotheses of Theorem 2.1 are satisfied. Then there exists a unique $\omega_\varepsilon = \omega$ in $L^\infty(Q) \cap L^2(0, T; W_0^{1,2}(G))$ with ω' in $L^2(0, T; W^{-1,2}(G))$ such that:*

$$(2.7)\qquad \begin{cases} \omega' - \varepsilon\Delta\omega + (v\cdot\nabla)\omega + F(p)\omega + \varepsilon^{-1}\beta(\omega) = g & \text{in } Q, \\ \omega(x, t) = 0 \quad \text{on } \partial G \times (0, T), & \omega(x, 0) = \omega_0(x) \quad \text{in } G . \end{cases}$$

Moreover:

$$\|\omega\|_{L^\infty(Q)} + \varepsilon^{1/2}\|\omega\|_{L^2(0,T;W_0^{1,2}(G))} \le CE(g, \omega_0) .$$

C is independent of ε, v, p. The expression $E(g, \omega_0)$ is as in Theorem 2.1.

PROOF: 1) Let $\omega_\varepsilon^k = \omega^k$ be as in Lemma 2.2 and set

$$\omega_N(x, t) = \omega^k(x) \quad \text{for } kh < t < (k+1)h ; \quad k = 0, 1, \dots, N-1 .$$

With the estimates of Lemma 2.2 we obtain:

$$(2.8)\qquad \|\omega_N\|_{L^\infty(Q)} + \varepsilon^{1/2}\|\omega_N\|_{L^2(0,T;W_0^{1,2}(G))} \le CE(g, \omega_0) .$$

We have:

$$h^{-1}\|\omega^k - \omega^{k-1}\|_{W^{-1,2}(G)} \le \|g^k\|_{L^2(G)} + \varepsilon\|\omega^k\|_{W_0^{1,2}(G)} +$$
$$+ C\{\varepsilon^1\|\omega^k\|^2_{L^\infty(G)} + \varepsilon^{-1}\|\omega^k\|_{L^\infty(G)} + \|v^k\|_{L^2(G)}\|\omega^k\|_{L^\infty(G)} + \|\omega^k\|_{L^\infty(G)}\|F(p^k)\|_{L^{6/5}(G)}\} ,$$

since $W^{1,2}(G) \subset L^6(G)$.

Thus,

$$(2.9)\qquad \sum_{k=1}^{N-1} h \left\| \frac{\omega^k - \omega^{k-1}}{h} \right\|^2_{W^{-1,2}(G)} \le M(\varepsilon) .$$

$M(\varepsilon)$ is independent of h.

Let us define functions $\tilde{\omega}_N(t)$ by:

$\tilde{\omega}_N$ = «linear functions on intervals $(kh, (k+1)h)$, $k = 0, \dots, N-1$; continuous on $(0, T)$, such that

$$\tilde{\omega}_N(0) = \omega^0 = \omega_0 , \quad \tilde{\omega}_N(lh) = \omega^{l-1} , \quad l = 1, \dots, N» .$$

We have

$$\int_0^T |\tilde{\omega}_N'(t)|_{W^{-1,2}(G)} = \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} |\tilde{\omega}_N'(t)|_{W^{-1,2}(G)}\, dt = h \cdot \sum_{k=1}^{N-1} \left\| \frac{\omega^k - \omega^{k-1}}{h} \right\|_{W^{-1,2}(G)} < M(\varepsilon)\,.$$

Therefore by taking subsequences we obtain: $\omega_N \to \omega$ weakly in $L^2(0, T; W_0^{1,2}(G))$ and in the weak*-topology of $L^\infty(Q)$, $\tilde{\omega}_N' \to \omega'$ weakly in $L^2(0, T; W^{-1,2}(G))$.

With (2.8)-(2.9) and the discrete analogue of Aubin's theorem we get by taking subsequences: $\omega_N \to \omega$ in $L^2(Q)$ and in $L^r(0, T; W^{-\alpha,5/4}(G))$ for $0 < \alpha < 1$ and $5/4 < r$.

2) We now show that $\beta(\omega_N) = (1 - |\omega_N|^2)^{-} \omega_N \to \beta(\omega)$ weakly in $L^2(Q)$. With the estimate on ω_N, we have by taking subsequences $\beta(\omega_N) \to \chi$ weakly in $L^2(Q)$. But:

$$\int_0^T (\beta(\omega_N) - \beta(\psi), \omega_N - \psi)\, dt \geq 0$$

for all ψ in $L^\infty(Q) \cap L^2(0, T; W_0^{1,2}(G))$.

Passing to the limit in the above inequality we get:

$$\int_0^T (\chi - \beta(\psi), \omega - \psi)\, dt \geq 0\,.$$

Hence, by hemicontinuity of β, $\beta(\omega) = \chi$.

3) Let φ be in $L^6(0, T; W_0^{1,6}(G))$ with φ' in $L^{6/5}(0, T; W^{-1,6/5}(G))$ and let $\varphi^k(x) = \varphi(x, kh)$, $\varphi_N(x, t) = \varphi^k(x)$ for $kh < t < (k+1)h$, $k = 0, 1, \ldots, N-1$.

It is known that $\{g_N, p_N, \varphi_N, v_N\} \to \{g, p, \varphi, v\}$ in

$$L^2(Q) \times L^{5/4}(0, T; W^{1,5/4}(G)) \times L^6(0, T; W^{1,6}(G)) \times L^2(0, T; V)\,.$$

We have:

$$\int_0^T \left(\frac{d\tilde{\omega}_N}{dt}, \varphi_N\right) dt = \sum_{k=1}^{N-1} \int_{kh}^{(k+1)h} \left(\frac{\omega^k - \omega^{k-1}}{h}, \varphi_N\right) dt = \sum_{k=1}^{N-1} (\omega^k - \omega^{k-1}, \varphi^k)\,.$$

Also:

$$\sum_{k=0}^{N-1} h(\nabla\omega^k, \nabla\varphi^k) = \sum_{k=0}^{N-1} \int_{kh}^{(k+1)h} (\nabla\omega^k, \nabla\varphi^k)\, dt = \int_0^T (\nabla\omega_N, \nabla\varphi_N)\, dt\,.$$

Multiplying (2.6) by φ^k, integrating over G and taking the summation from $k=1$ to $N-1$ we obtain:

$$\int_0^T \left(\frac{d\tilde{\omega}_N}{dt}, \varphi_N\right) dt + \varepsilon \int_0^T (\nabla\omega_N, \nabla\varphi_N)\, dt + \int_0^T (F(p_N)\omega_N + \varepsilon^{-1}\beta(\omega_N), \varphi_N)\, dt +$$

$$- \int_0^T ((v_N \cdot \nabla)\varphi_N, \omega_N)\, dt = \int_0^T (g_N, \varphi_N)\, dt + \varepsilon b(\nabla\omega^0, \nabla\varphi^0) + b(F(p^0)\omega^0, \varphi^0) +$$

$$+ \varepsilon^{-1} b(\beta(\omega^0), \varphi^0) - b((v^0 \cdot \nabla)\varphi^0, \omega^0).$$

Let $N \to +\infty$; with our hypothesis on F and the results of the first two parts we have:

$$\int_0^T \langle \omega', \varphi \rangle\, dt + \int_0^T \varepsilon(\nabla\omega, \nabla\varphi)\, dt + \int_0^T (F(p)\omega + \varepsilon^{-1}\beta(\omega), \varphi)\, dt -$$

$$- \int_0^T (v \cdot \nabla\varphi, \omega)\, dt = \int_0^T (g, \varphi)\, dt.$$

Since $\omega \in C([0, T]; L^2(Q))$, the initial condition is satisfied (see the proof of Theorem 2.1 below for more details). The lemma is proved.

Proof of Theorem 2.1: In view of Lemma 2.3 it remains only to show that $\varepsilon^{-1}\|\beta(\omega)\|_{L^2(Q)} \leq C$ where C is a constant independent of ε.
It is the crucial estimate of the paper.

1) We have:

$$(2.10) \qquad \left| \begin{array}{l} \omega' - \varepsilon\Delta\omega = g - (v \cdot \nabla)\omega - F(p)\omega - \varepsilon^{-1}\beta(\omega) \quad \text{in } Q, \\ \omega(x, t) = 0 \quad \text{on } \partial G \times (0, T), \qquad \omega(x, 0) = \omega_0(x) \quad \text{in } G, \\ |\omega_0(x)| \leq 1 \quad \text{a.e. in } G. \end{array} \right.$$

We shall use the Gagliardo-Nirenberg estimate as in [6] to show that the right hand side of (2.10) is in $L^{5/4}(Q)$. The only term which is not obvious is $(v \cdot \nabla)\omega$.

An application of the Hölder inequality gives:

$$\int_G |(v \cdot \nabla)\omega|^{5/4}\, dx \leq C \left(\int_G |v|^{10/3}\, dx\right)^{3/8} \left(\int_G |\nabla\omega|^{5/4 \cdot 8/5}\, dx\right)^{5/8} =$$

$$= C \left(\int_G |v|^{10/3}\, dx\right)^{3/8} \left(\int_G |\nabla\omega|^2\right)^{5/8}.$$

According to the Gagliardo-Nirenberg estimate:

$$\|h\|_{L^{10/3}(G)} \leq C \|h\|^{2/5}_{L^2(G)} \|\nabla h\|^{3/5}_{L^2(G)} \qquad \text{for all } h \text{ in } W^{1,2}_0(G).$$

Thus,

$$\int_G |(v\cdot\nabla)\omega|^{5/4}\,dx \le C\|v(\cdot,t)\|^{1/2}_{L^2(G)}\|\nabla v(\cdot,t)\|^{3/4}_{L^2(G)}\|\nabla\omega(\cdot,t)\|^{3/4}_{L^2(G)}.$$

Hence:

$$\|(v\cdot\nabla)\omega\|^{5/4}_{L^{5/4}(Q)} \le C\|v\|^{1/2}_{L^\infty(0,T;H)}\|\nabla v\|^{3/4}_{L^2(0,T;H)}\|\nabla\omega\|^{3/4}_{L^2(0,T;L^2(G))}.$$

Therefore $\omega'-\varepsilon\Delta\omega$ is in $L^{5/4}(Q)$.

It follows from the theory of linear parabolic equations that ω' is in $L^{5/4}(Q)$

2) With ω in $L^\infty(Q)\cap L^2(0,T;W_0^{1,2}(G))$, $1-|\omega|^2$ is in $L^\infty(Q)\cap L^2(0,T;$ $W_0^{1,2}(G))$ and by a result of Stampacchia, $(1-|\omega|^2)^-$ belongs to $L^\infty(Q)\cap$ $\cap L^2(0,T;W_0^{1,2}(G))$. Therefore: $\beta(\omega)=(1-|\omega|^2)^-\omega$ is also in $L^\infty(Q)\cap$ $\cap L^2(0,T;W_0^{1,2}(G))$. Since $\beta(\omega)$ is in $L^\infty(Q)$, it is also in $L^5(Q)$. We now have:

$$\int_0^T(\omega',\beta(\omega))\,dt+\varepsilon\int_0^T(\nabla\omega,\nabla\beta(\omega))\,dt+\int_0^T(F(p)\omega+(v\cdot\nabla)\omega,\beta(\omega))\,dt+ \tag{2.11}$$
$$+\varepsilon^{-1}\int_0^T\|\beta(\omega)\|^2_{L^2(G)}\,dt=\int_0^T(g,\beta(\omega))\,dt.$$

With ω in $L^\infty(Q)$, ω' in $L^{5/4}(Q)$ it is easy to check that $\omega'\cdot\omega$ is in $L^{5/4}(Q)$ and $\frac{1}{2}(d/dt)|\omega|^2=\omega'\cdot\omega$.

Hence: $1-|\omega|^2$ and $(\partial/\partial t)\{1-|\omega|^2\}$ are in $L^{5/4}(Q)$ and thus, by Stampacchia's result, $(\partial/\partial t)\{1-|\omega|^2)^-\}$ is in $L^{5/4}(Q)$. On the other hand $(1-|\omega|^2)^-$ is in $L^\infty(Q)$, hence in $L^5(Q)$. Therefore $(1-|\omega|^2)^-$ is in $C([0,T];L^2(G))$.

Notice that

$$\int_0^T(\omega',\beta(\omega))\,dt=\sum_{j=1}^3\int_Q\frac{\partial}{\partial t}\omega_j(1-|\omega|^2)^-\omega_j\,dx\,dt=\frac{1}{2}\int_Q\frac{\partial}{\partial t}|\omega|^2(1-|\omega|^2)^-\,dx\,dt=$$
$$=-\frac{1}{2}\int_Q\frac{\partial}{\partial t}(1-|\omega|^2)(1-|\omega|^2)^-\,dx\,dt=\frac{1}{2}\int_Q\frac{\partial}{\partial t}(1-|\omega|^2)^-(1-|\omega|^2)^-\,dx\,dt=$$
$$=\frac{1}{4}\int_G\{((1-|\omega(x,T)|^2)^-\}\,dx\ge 0$$

since $\omega(x,0)=\omega_0$ and $|\omega_0(x)|\le 1$.

Now with v in $L^2(0,T;V)$, $\nabla\cdot v=0$ in G and

$$\int_0^T((v\cdot\nabla)\omega,\beta(\omega))\,dt=\sum_{j,k=1}^3\int_Q v_jD_j\omega_k(1-|\omega|^2)^-\omega_k\,dx\,dt=$$
$$=\frac{1}{2}\sum_{j=1}^3\int_Q v_jD_j|\omega|^2(1-|\omega|^2)^-\,dx\,dt=-\frac{1}{2}\sum_{j=1}^3\int_Q v_jD_j\{(1-|\omega|^2)\}(1-|\omega|^2)^-\,dx\,dt=$$

$$-\frac{1}{2}\sum_{i=1}^{3}\int_Q v_i D_i\{(1-|\omega|^2)^{-1}\}(1-|\omega|^2)^{-1}\,dx\,dt=\frac{1}{4}\sum_{i=1}^{3}\int_Q v_i D_i\{(1-|\omega|^2)^{-1}\}^2\,dx\,dt=0\,.$$

Finally

$$\int_0^T(\nabla\omega,\nabla\beta(\omega))\,dt=\sum_{i,k=1}^{3}\int_Q(D_i\omega_k)^2(1-|\omega|^2)^{-1}\,dx\,dt+$$

$$+\sum_{i,k=1}^{3}\int_Q\omega_k D_i\omega_k D_i\{(1-|\omega|^2)^{-1}\}\,dx\,dt=$$

$$=\sum_{i,k=1}^{3}\int_Q(D_i\omega_k)^2(1-|\omega|^2)^{-1}\,dx\,dt-\frac{1}{2}\sum_{i=1}^{3}\int_Q D_i\{(1-|\omega|^2)^{-1}\}D_i\{(1-|\omega|^2)\}\,dx\,dt=$$

$$=\sum_{i,k=1}^{3}\int_Q(D_i\omega_k)^2(1-|\omega|^2)^{-1}\,dx\,dt+\frac{1}{2}\sum_{i=1}^{3}\int_Q[D_i\{(1-|\omega|^2)^{-1}\}]^2\,dx\,dt\,.$$

So:

$$\int_0^T(\nabla\omega,\nabla\beta(\omega))\,dt\geq 0\,. \tag{2.13}$$

Therefore with (2.11)-(2.13) we obtain:

$$\varepsilon^{-1}\|\beta(\omega)\|^2_{L^2(Q)}\leq\|g\|_{L^2(Q)}\|\beta(\omega)\|_{L^2(Q)}\,.$$

The theorem is proved.

3. – In this section we shall show the existence of a solution of the initial boundary-value problem:

$$\begin{cases} u'-\nu\Delta u+(u\cdot\nabla)u+\nabla p-\eta(\omega\times u)=f \ \text{ in } Q\,, & \nabla\cdot u=0 \ \text{ in } Q\,,\\ u(x,t)=0 \ \text{ on } \partial G\times(0,T)\,, & u(x,0)=u_0(x) \ \text{ in } G\end{cases} \tag{3.1}$$

with the normalizing condition

$$\int_G p(x,t)\,dx=0 \quad \text{for almost all } t \text{ in } (0,T) \tag{3.2}$$

and

$$\left|\begin{array}{l}\omega'-\varepsilon\Delta\omega+(u\cdot\nabla)\omega+F(p)\omega+\varepsilon^{-1}\beta(\omega)=g \ \text{ in } Q\,,\\ \omega(x,t)=0 \ \text{ on } \partial G\times(0,T)\,, \quad \omega(x,0)=\omega_0(x) \ \text{ in } G\,,\\ |\omega_0(x)|<1 \text{ a.e. in } G\,.\end{array}\right. \tag{3.3}$$

$\beta(\omega)$ is the expression $(1-|\omega|^2)^{-1}\omega$.

The main result of the section is the following theorem.

Theorem 3.1: *Suppose all the hypotheses of Theorem* 1.1 *are satisfied. Then there exists* $\{u_\varepsilon, p_\varepsilon, \omega_\varepsilon\}$ *in*

$$\{L^\infty(0,T;H)\cap L^2(0,T;V)\}\times L^{4/3}(0,T;W^{1,4/3}(G))\times$$
$$\times\{L^\infty(Q)\cap L^2(0,T;W_0^{1,2}(G))\}\,,$$

solution of (3.1)-(3.3).

Moreover:

1) $\|u\|_{L^\infty(0,T;H)}+\|u\|_{L^2(0,T;V)}+\|u'\|_{L^{4/3}(0,T;W^{-1,4/3}(G))}+\|p\|_{L^{4/3}(0,T;W^{1,4/3}(G))}\le C_1$.

2) $\|\omega\|_{L^\infty(Q)}+\varepsilon^{1/2}\|\omega\|_{L^2(0,T;W_0^{1,2}(G))}+\varepsilon^{-1}\|\beta(\omega)\|_{L^2(Q)}+\|\omega'\|_{L^{4/3}(0,T;W^{-1,4/3}(G))}\le C_2$.

C_1, C_2 *are independent of* ε.

We shall use the Schauder fixed point theorem to prove the stated result.

Lemma 3.1: *Let* $\{f, u_0\}$ *be as in Theorem* 1.1 *and* $\{v, \bar\omega\}$ *be in* $\{L^\infty(0,T;H)\cap L^2(0,T;V)\}\times L^\infty(Q)$. *Then there exists a unique* $\{u, p\}$ *in* $\{L^\infty(0,T;H)\cap L^2(0,T;V)\}\times L^{4/3}(0,T;W^{1,4/3}(G))$, *solution of the initial boundary-value problem*:

$$(3.4)\qquad \left|\begin{array}{l} u'-\nu\Delta u+(v\cdot\nabla)u+\nabla p-\eta(\bar\omega\times u)=f\,,\quad \nabla\cdot u=0\quad \text{in } Q\,,\\ u(x,t)=0\quad \text{on } \partial G\times(0,T)\,,\qquad u(x,0)=u_0(x)\quad \text{in } G\,,\\ \int_G p(x,t)\,dx=0\qquad \textit{for almost all } t \textit{ in } (0,T)\,.\end{array}\right.$$

Moreover:

1) $\|u\|_{L^\infty(0,T;H)}+\|u\|_{L^2(0,T;V)}\le C_1R(f,u_0)$.

2) $\|p\|_{L^{4/3}(0,T;W^{1,4/3}(G))}\le C_2R(f,u_0)\{1+\|\bar\omega\|_{L^\infty(Q)}+\|v\|^{2/3}_{L^\infty(0,T;H)}\|\nabla v\|^{2/3}_{L^2(0,T;H)}\}$.

3) $\|u'\|_{L^{4/3}(0,T;W^{-1,4/3}(G))}\le C_3R(f,u_0)\{1+\|\bar\omega\|_{L^\infty(Q)}+\|v\|^{2/3}_{L^\infty(0,T;H)}\|\nabla v\|^{2/3}_{L^2(0,T;H)}\}$.

C_1, C_2 *and* C_3 *are independent of* ε, v, $\bar\omega$.

$R(f,u_0)$ *is the expression* $1+\|u_0\|_H+\|u_0\|_{B^{1/2,4/3}(G)}+\|f\|_{L^2(Q)}$.

Proof: Note that $(\bar\omega\times u, u)=0$ for all u in H.

The existence of a unique solution of the linear problem (3.4) is well-known and it is clear that

$$(3.5)\qquad \|u\|_{L^\infty(0,T;H)}+\|u\|_{L^2(0,T;V)}\le C_1R(f,u_0)\,.$$

To estimate p we rewrite (3.4) as:

$$(3.6)\qquad \left|\begin{array}{l} u'-\nu\Delta u+\nabla p=f+\eta(\bar\omega\times u)-(v\cdot\nabla)u\,,\\ \nabla\cdot u=0\quad \text{in } Q\,,\qquad u(x,t)=0\quad \text{on } \partial G\times(0,T)\,,\\ u(x,0)=u_0(x)\quad \text{in } G\,,\qquad \int_G p(x,t)\,dx=0\,.\end{array}\right.$$

From a result of Solonnikov [9] we get:

$$\|p\|_{L^{4/3}(0,T;W^{1,4/3}(G))} \le C\{\|u\|_{B^{1/2,4/3}(G)} + \|f\|_{L^{4/3}(Q)} + \|(v\cdot\nabla)u\|_{L^{4/3}(Q)} + \|\tilde{\omega}\|_{L^\infty(Q)}\|u\|_{L^2(0,T;H)}\}.$$

Taking into account the Gagliardo-Nirenberg estimate as in the proof of Theorem 2.1 we obtain:

$$\|p\|_{L^{4/3}(0,T;W^{1,4/3}(G))} \le C\{\|u_0\|_{B^{1/2,4/3}(G)} + \|f\|_{L^{4/3}(Q)} + \|\tilde{\omega}\|_{L^\infty(Q)}\|u\|_{L^2(0,T;H)} + \\ + \|v\|^{2/3}_{L^\infty(0,T;H)}\|\nabla v\|^{1/3}_{L^2(0,T;H)}\|u\|_{L^2(0,T;V)}\}.$$

With the estimates (3.5) for u we get the desired result. It is now trivial to show that $\|u'\|_{L^{4/3}(0,T;W^{-1,4/3}(G))}$ is as stated in the lemma.

Consider the initial boundary-value problem:

$$(3.7)\qquad \begin{cases} \omega_\varepsilon' - \varepsilon\Delta\omega_\varepsilon + (v\cdot\nabla)\omega_\varepsilon + F(p)\omega_\varepsilon + \varepsilon^{-1}\beta(\omega_\varepsilon) = g \quad \text{in } Q, \\ \omega_\varepsilon(x,t) = 0 \quad \text{on } \partial G\times(0,T), \qquad \omega_\varepsilon(x,0) = \omega_0(x), \\ |\omega_0(x)| \le 1 \text{ a.e. in } G \end{cases}$$

where p is given by Lemma 3.1.

If follows from Theorem 2.1 that there exists a unique ω_ε in $L^\infty(Q)\cap$ $\cap L^2(0,T;W_0^{1,2}(G))$, solution of (3.7). Since

$$((v\cdot\nabla)\omega,\varphi) = -((v\cdot\nabla\varphi,w) \text{ for all } \varphi \text{ in } W_0^{1,2}(G)$$

we get, by taking into account the estimates of Theorem 2.1:

$$(3.8)\quad \|\omega_\varepsilon'\|_{L^{4/3}(0,T;W^{-1,4/3}(G))} \le c_2E(g,\omega_0)\{1 + \|v\|_{L^\infty(0,T;H)} + \gamma(\|p\|_{L^{4/3}(0,T;W^{1,4/3}(G))})\}.$$

c_2 is independent of ε, v, p.

We now proceed to the proof of Theorem 3.1. We know from Theorem 2.1, Lemma 3.1 that for any given $\{v,\tilde{\omega}\}$ in $\{L^\infty(0,T;H)\cap L^2(0,T;V)\}\times L^\infty(Q)$ there exists a unique $\{u,p,\omega\}$, solution of (3.4) and (3.7).

Let:

$$B_1 = \{v : \|v\|_{L^\infty(0,T;H)} + \|v\|_{L^2(0,T;V)} \le C_1R(f,u_0),\ \|v'\|_{L^{4/3}(0,T;W^{-1,4/3}(G))} \le \\ \le C_2R(f,u_0)\{1 + c_1E(g,\omega_0) + C_1R(f,u_0)\}\}$$

where C_1, c_1, C_2, R and E are as in Theorem 2.1 and Lemma 3.1.

Set:

$$B_2 = \{\tilde{\omega} : \|\tilde{\omega}\|_{L^\infty(Q)} \le c_1E(g,\omega_0),\ \|\tilde{\omega}'\|_{L^{4/3}(0,T;W^{-1,4/3}(G))} \le c_2E(g,\omega_0)\cdot \\ \cdot[1 + C_1R(f,u_0) + \gamma(C_2R(f,u_0)\{1 + c_1E(g,\omega_0) + C_1R(f,u_0)\})]\}.$$

Then:

$$B = B_1\times B_2 \subset L^2(0,T;H)\times L^{4/3}(0,T;W^{-\alpha,4/3}(G))$$

where $0 < \alpha < 1$.

Let $\{v, \bar{\omega}\}$ be in B and let A be the mapping of B into $L^2(0, T; H) \times$ $\times L^{3/4}(0, T; W^{-\alpha, 3/4}(G))$, $0 < \alpha < 1$, defined by:

$$A(v, \bar{\omega}) = \{u, \omega\} \tag{3.9}$$

where $\{u, \omega\}$ is the solution of (3.4) and (3.7).

Remark: 1) Strictly speaking $A(v, \bar{\omega}) = \{u, \omega, p\}$. However once v, $\bar{\omega}$ are given the linearized Navier-Stokes equations (3.4) are solved in u and p is determined only after u is known.

2) To define A in terms of $\{u, \omega, p\}$ and in order to apply the Schauder fixed point theorem, we would have to use:

$$A(v, \bar{\omega}, q\} = \{u, \omega, p\}$$

where in (3.7), p is replaced by q. Then A maps $B \times B_2$ into $L^2(0, T; H) \times$ $\times L^{3/4}(0, T; W^{-\alpha, 3/4}(\bar{G})) \times W^{-1, 3/4}(Q)$ with

$$B_2 = \{q : q \in L^{3/4}(0, T; W^{1,3/4}(G)),\ \|q\|_{L^{3/4}(0,T; W^{1,3/4}(G))} <$$
$$< C_2 R(f, u_0)\,[1 + c_1 E(g, \omega_0) + C_1 R(f, u_0)]\} .$$

All the following arguments are valid in that case.

Proof of Theorem 3.1: 1) B is a closed bounded convex subset of $L^2(0, T; H) \times L^{3/4}(0, T; W^{-\alpha, 3/4}(\bar{G}))$ for $0 < \alpha < 1$. We have:

$$W_0^{1,2}(G) \subset W^{\alpha,2}(G) \subset L^2(G) \subset W^{-\alpha, 3/4}(\bar{G}) \subset W^{-1, 3/4}(G) .$$

The injection mapping of $W^{\alpha,2}(G)$ into $L^2(G)$ is compact and hence that of $L^2(G)$ into $(W^{\alpha,2}(G))^* = W^{-\alpha, 3/4}(\bar{G})$ is also compact.

An application of Aubin's theorem shows that B is a compact subset of $L^2(0, T; H) \times L^{3/4}(0, T; W^{-\alpha, 3/4}(G))$.

2) From the estimates of Theorem 2.1 and of Lemma 3.1 we get: $A(B) \subset B$.

We now show that A is continuous. Suppose that $\{v_n, \bar{\omega}_n\}$ is in B and $\{v_n, \bar{\omega}_n\} \to \{v, \bar{\omega}\}$ in $L^2(0, T; H) \times L^{3/4}(0, T; W^{-\alpha, 3/4}(\bar{G}))$. We have to show that:

$$A(v_n, \bar{\omega}_n) = \{u_n, \omega_n\} \to A(v, \bar{\omega}) = \{u, \omega\}$$

in $L^2(0, T; H) \times L^{3/4}(0, T; W^{-\alpha, 3/4}(\bar{G}))$.

From the estimates of Theorem 2.1 and of Lemma 3.1 we get:

$$\|p_n\|_{L^{3/4}(0,T; W^{1,3/4}(G))} + \varepsilon^{1/2} \|\omega_n\|_{L^2(0,T; W_0^{1,2}(G))} < K .$$

K independent of ε, n. Moreover $\{u_n, \omega_n\}$ is in B.

There exists a subsequence such that: $\{u_n, p_n, \omega_n\} \to \{u, p, \omega\}$ weakly in $L^2(0, T; V) \times L^{5/4}(0, T; W^{1,5/4}(G)) \times L^2(0, T; W_0^{2,2}(G))$. Moreover, from the estimates of Lemma 3.1 and from (3.8) we have, by taking subsequences:

$$\{u_n', \omega_n'\} \to \{u', \omega'\} \quad \text{weakly in } L^{5/4}(0, T; W^{-1,5/4}(\bar{G}))$$

and $\omega_n \to \omega$ in the weak*-topology of $L^\infty(Q)$.

Applying Aubin's theorem, we get:

$$\{u_n, \omega_n\} \to \{u, \omega\} \quad \text{in } L^2(0, T; H) \times \{L^r(0, T; W^{-\alpha,5/4}(\bar{G})) \cap L^2(Q)\}$$

for $0 < \alpha < 1$, $5/4 < r$.

We now show that $A(v, \bar{\omega}) = \{u, \omega\}$.

With our hypotheses on F: $F(p_n)\omega_n \to F(p)\omega$ in the distribution sense. On the other hand:

$$\|F(p_n)\omega_n\|_{L^{5/4}(Q)} \leq K .$$

Hence:

$$F(p_n)\omega_n \to F(p)\omega \quad \text{weakly in } L^{5/4}(Q) .$$

We have:

$$\int_0^T (\beta(\omega_n) - \beta(\varphi), \omega_n - \varphi)\, dt \geq 0$$

for all φ in $L^\infty(Q)$. Using the hemi-continuity of β and the fact that $\omega_n \to \omega$ in $L^2(Q)$ with $\beta(\omega_n) \to b$ weakly in $L^2(Q)$, we get: $\beta(\omega_n) \to \beta(\omega)$ weakly in $L^2(Q)$ by taking subsequences. It is now easy to check that indeed there exists a subsequence such that

$$A(v_{n_j}, \bar{\omega}_{n_j}) = \{u_{n_j}, \omega_{n_j}\} \to \{u, \omega\} = A(v, \bar{\omega}) .$$

in $L^2(0, T; H) \times L^{5/4}(0, T; W^{-\alpha,5/4}(\bar{G}))$.

Since the problem (3.4) and (3.6) is uniquely solvable, for each $\{v, \bar{\omega}\}$ in B there is a unique $\{u, \omega\}$ in B with $A(v, \bar{\omega}) = \{u, \omega\}$. Thus the sequence $\{u_n, \omega_n\}$, and not just subsequences converges to $\{u, \omega\}$ in the above norm.

It follows from the Schauder fixed point theorem that there exists $\{u, \omega\}$ in B such that:

$$A(u, \omega) = \{u, \omega\} .$$

The theorem is proved.

4. – We shall now prove the main result of the paper.

Proof of Theorem 1.1: 1) Let $\{u_\varepsilon, p_\varepsilon, \omega_\varepsilon\}$ be the solution of (3.1)-(3.3) given by Theorem 3.1. From the estimates of the theorem we obtain by taking subsequences: $\{u_\varepsilon, p_\varepsilon, u'_\varepsilon\} \to \{u, p, u'\}$ weakly in

$$L^2(0, T; V) \times L^{4/3}(0, T; W^{1,4/3}(G)) \times L^{4/3}(0, T; W^{-1,4/3}(G)).$$

Also: $\{u_\varepsilon, \omega_\varepsilon\} \to \{u, \omega\}$ in the weak*-topology of $L^\infty(0, T; H) \times L^\infty(Q)$ with $\omega'_\varepsilon \to \omega'$ weakly in $L^{4/3}(0, T; W^{-1,4/3}(G))$.

An application of Aubin's theorem gives:

$$\{u_\varepsilon, \omega_\varepsilon\} \to \{u, \omega\} \quad \text{in } L^2(0, T; H) \times L^s(0, T; W^{-\alpha,4/3}(G))$$

for $0 < \alpha < 1$ and $1 < s < \infty$.

Furthermore from Theorem 3.1 we have:

$$\beta(\omega_\varepsilon) = (1 - |\omega_\varepsilon|^2)^- \omega_\varepsilon \to 0 \quad \text{in } L^2(Q).$$

It follows from the monotonicity and the hemi-continuity of β that $\beta(\omega) = 0$ i.e. $|\omega(x, t)| < 1$ a.e. in Q.

2) Let φ be as in the theorem and consider the expression

$$(4.1) \qquad X_\varepsilon = \int_0^T \{(\varphi', \varphi - \omega_\varepsilon) + \varepsilon(\nabla\omega_\varepsilon, \nabla(\varphi - \omega_\varepsilon)) + (u_\varepsilon \cdot \nabla\omega_\varepsilon, \varphi - \omega_\varepsilon) + \\ + (F(p_\varepsilon)\omega_\varepsilon - g, \varphi - \omega_\varepsilon)\}\, dt.$$

We also have:

$$(4.2) \qquad (\omega'_\varepsilon, \varphi - \omega_\varepsilon) + \varepsilon(\nabla\omega_\varepsilon, \nabla(\varphi - \omega_\varepsilon)) + (u_\varepsilon \cdot \nabla\omega_\varepsilon, \varphi - \omega_\varepsilon) + \\ + (\varepsilon^{-1}\beta(\omega_\varepsilon) + F(p_\varepsilon)\omega_\varepsilon, \varphi - \omega_\varepsilon) = (g, \varphi - \omega_\varepsilon).$$

With ω_ε in $L^\infty(Q) \cap L^2(0, T; L^2(G))$ and ω'_ε in $L^{4/3}(0, T; L^{4/3}(G))$, ω_ε is in $C([0, T]; L^2(G))$. We may rewrite (4.1) by taking into account (4.2):

$$X_\varepsilon = \int_0^T \{(\varphi' - \omega'_\varepsilon, \varphi - \omega_\varepsilon) - \varepsilon^{-1}(\beta(\varphi) - \beta(\omega_\varepsilon), \varphi - \omega_\varepsilon)\}\, dt > \\ > \tfrac{1}{2}\|\varphi(T) - \omega_\varepsilon(T)\|^2_{L^2(G)} - \tfrac{1}{2}\|\varphi(0) - \omega_0\|^2_{L^2(G)} > -\tfrac{1}{2}\|\varphi(0) - \omega_0\|^2_{L^2(G)}$$

since $\beta(\varphi) = 0$.

We note that $((u_\varepsilon \cdot \nabla)\omega_\varepsilon, \omega_\varepsilon) = 0$ and

$$((u_\varepsilon \cdot \nabla)\omega_\varepsilon, \varphi) = -((u_\varepsilon \cdot \nabla)\varphi, \omega_\varepsilon).$$

Thus from (4.1)-(4.2) we get:

$$(4.4) \qquad -\tfrac{1}{2}\|\varphi(0) - \omega_0\|^2_{L^2(G)} < X_\varepsilon = \\ = \int_0^T \{(\varphi', \varphi - \omega_\varepsilon) + \varepsilon(\nabla\omega_\varepsilon, \nabla(\varphi - \omega_\varepsilon)) - (u_\varepsilon \cdot \nabla\varphi, \omega_\varepsilon) + (F(p_\varepsilon)\omega_\varepsilon - g, \varphi - \omega_\varepsilon)\}\, dt.$$

Let $\varepsilon \to 0$ and with our hypotheses on F we obtain:

$$(4.5)\qquad -\tfrac{1}{2}\|\varphi(0)-\omega_0\|^2_{L^2(\Omega)} + \liminf_{\varepsilon\to 0}\int_0^T (F(p_\varepsilon)\omega_\varepsilon, \omega_\varepsilon)\,dt \leq$$
$$\leq \int_0^T \{(\varphi', p-\omega) - (u\cdot\nabla\varphi, \omega) + (F(p)\omega, \varphi) - (g, \varphi-\omega)\}\,dt\,.$$

3) It remains to show that

$$\int_0^T (F(p)\omega, \omega)\,dt \leq \liminf_{\varepsilon\to 0}\int_0^T (F(p_\varepsilon)\omega_\varepsilon, \omega_\varepsilon)\,dt\,.$$

From Theorem 2.1 we know that $\{\omega_\varepsilon, \omega'_\varepsilon\}$ is in $L^\infty(Q)\times L^{5/4}(Q)$ and so $(\partial/\partial t)|\omega_\varepsilon|^2$ belongs to $L^{5/4}(Q)$.

Let v be in $W^{1,5}_0(G)$, multiplying (3.3) by $v\omega_\varepsilon$ and integrating over G we get:

$$\frac{1}{2}\left(\frac{\partial}{\partial t}|\omega_\varepsilon|^2, v\right) = (g\omega_\varepsilon, v) - \varepsilon(\nabla\omega_\varepsilon, \omega_\varepsilon\nabla v) - \varepsilon(\nabla\omega_\varepsilon, v\nabla\omega_\varepsilon)\,,$$
$$-(F(p_\varepsilon)|\omega_\varepsilon|^2, v) - \varepsilon^{-1}(\beta(\omega_\varepsilon), \omega_\varepsilon v) + \tfrac{1}{2}\sum_{j=1}^{3}\int_G u_{j\varepsilon}D_j v\cdot|\omega_\varepsilon|^2\,dx\,.$$

Hence:

$$\left|\left(\frac{\partial}{\partial t}|\omega_\varepsilon|^2, v\right)\right| \leq C\|v\|_{W_0^{1,5}(\Omega)}\{\|g\|_{L^2(\Omega)}\|\omega_\varepsilon\|_{L^\infty(\Omega)} + \varepsilon\|\omega_\varepsilon\|_{L^\infty(\Omega)}\|\nabla\omega_\varepsilon\|_{L^2(\Omega)} +$$
$$+ \varepsilon\|\nabla\omega_\varepsilon\|^2_{L^2(\Omega)} + \|F(p_\varepsilon)\|_{L^{5/4}(\Omega)}\|\omega_\varepsilon\|^2_{L^\infty(\Omega)} + \varepsilon^{-1}\|\beta(\omega_\varepsilon)\|_{L^2(\Omega)}\|\omega_\varepsilon\|_{L^\infty(\Omega)} +$$
$$+ \|\omega_\varepsilon\|^2_{L^\infty(\Omega)}\|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}\}\,.$$

With the estimates of Theorem 2.1, we obtain:

$$\left\|\frac{\partial}{\partial t}(|\omega_\varepsilon|^2)\right\|_{L^{5/4}(0,T;\,W^{-1,5/4}(\Omega))} \leq M$$

M independent of ε. Since $\||\omega_\varepsilon|^2\|_{L^\infty(Q)} \leq M_1$, it now follows from Aubin's theorem that by taking subsequences:

$$|\omega_\varepsilon|^2 \to b \quad \text{in the weak*-topology of } L^\infty(Q)$$

and in $L^s(0, T; W^{-\alpha,5/4}(G))$ for any $0<\alpha<1$ and $1<s<\infty$. With our hypotheses on F, we have:

$$(4.6)\qquad \int_Q F(p)\,b\,dx\,dt = \lim_{\varepsilon\to 0}\int_0^T (F(p_\varepsilon), |\omega_\varepsilon|^2)\,dt\,.$$

4) Since $F(p)>0$, it is easy to check that

$$\|\{F(p)\}^{1/2}\omega_\varepsilon\|_{L^2(Q)} < C\|F(p)\|_{L^\infty(Q)}\|\omega_\varepsilon\|_{L^\infty(Q)} < M .$$

M independent of ε. Hence by taking subsequences we get: $\{F(p)\}^{1/2}\omega_\varepsilon \to \{F(p)\}^{1/2}\omega$ weakly in $L^2(Q)$. Moreover:

$$(4.7) \qquad \|\{F(p)\}^{1/2}\omega\|^2_{L^2(Q)} = \int_Q F(p)|\omega|^2\,dx\,dt < \liminf_{\varepsilon\to 0} \|\{F(p)\}^{1/2}\omega_\varepsilon\|^2_{L^2(Q)} < \liminf_{\varepsilon\to 0} \int_Q F(p)|\omega_\varepsilon|^2\,dx\,dt .$$

From (4.6)-(4.7) we have:

$$(4.8) \qquad \int_Q F(p)|\omega|^2\,dx\,dt < \int_Q F(p)b\,dx\,dt = \lim_{\varepsilon\to 0} \int_Q F(p_\varepsilon)|\omega_\varepsilon|^2\,dx\,dt .$$

It follows then from (4.5) and (4.8) that:

$$-\tfrac{1}{2}\|\varphi(0)-\omega_0\| + \int_0^T (F(p)\omega,\omega)\,dt < \int_0^T \{(\varphi',\varphi-\omega) - (u\cdot\nabla\varphi,\omega) + (F(p)\omega,\varphi) - (g,\varphi-\omega)\}\,dt .$$

So:

$$\int_0^T \{(\varphi',\varphi-\omega) + (u\cdot\nabla\varphi,\varphi-\omega) + (F(p)\omega,\varphi-\omega) - (g,\varphi-\omega)\}\,dt > -\tfrac{1}{2}\|\varphi(0)-\omega_0\|^2_{L^2(\Omega)}$$

since $(u\cdot\nabla\varphi,\varphi)=0$ with φ as in the theorem.

The theorem is proved.

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