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Convergence in Homological Algebra (**)

Convergenza in algebra omologica

SOMMARIO. — Si studiano proprietà di convergenza di subquozienti in categorie esatte e la loro compatibilità con i morfismi indotti. Le applicazioni riguardano la convergenza della successione spettrale di un complesso filtrato; saranno estese in un lavoro successivo a teorie omologiche più generali, come il sistema di omologie relative.

0. - INTRODUCTION

0.1. This paper aims to prepare a background for studying convergence properties in homological theories, mainly in the theories which produce spectral sequences (with non-trivial convergence), as the filtered complex, the exact couple [Ms], the system of relative homologies [Ei, De].

Generally (e.g. see [EM]) these properties have been treated in the context of abelian categories with countable products and sums, using the additive structure to transform properties of mappings into properties of objects (the « mapping cylinder » construction) and projective and inductive limits to formulate the hypotheses of convergence.

We study here the « convergence » of subquotients in the more general context of *exact* categories (in the sense of Puppe-Mitchell ([1])): this allows to use the universal model of the above (exact) theories, developped by the

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([1]) There is a zero-object (initial and terminal); each morphism has a kernel and a cokernel, and factors (uniquely) conormal epi-normal mono. The category $\mathfrak{J}$ of sets and partial bijections is an outstanding non-abelian, exact category: it is selfdual and distributive (i.e. its lattices of subobjects are so), it lacks additivity of maps, products, equalizers and kernel-pairs. As proved in [G7], all distributive exact theories (in particular, all the above mentioned theories) have universal model in $\mathfrak{J}$.

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author in [G8, G9] and formalizing the Zeeman diagram [Ze, HW] in the case of the filtered complex. The lack of additivity prevents the use of such constructions as the mapping cylindre; on the other hand, it seems to lead to the essential core of the question.

The appropriate notion of convergence appears here to consist of *unions and intersection of subquotients*, with suitable conditions of regularity: e.g. the convergence theorem for the spectral sequence of a filtered complex A_* (5.4) proves that, under suitable hypotheses on the filtration:

$$\text{(1)} \qquad E^{\infty}_{np} = \bigcap_{r} E^{r}_{np}$$

(« regular », decreasing intersection of subquotients of A_n),

$$\text{(2)} \qquad H_n(A_*) = \bigcup_{p} E^{\infty}_{np}$$

(« regular », telescopic union of subquotients of A_n);

these conditions of regularity allow in particular to obtain « mapping theorems », i.e. to transfer the isomorphism property for the « induced morphisms », from a family of subquotients to its union or intersection.

0.2. Now, let us make precise the notion of subquotient and their order. Let A be an object of the exact category $\boldsymbol{E}$ and $\mathrm{Sub}_{\boldsymbol{E}}(A)$ its modular lattice of subobjects.

A subquotient H/K of A is determined by two subobjects H, K with $H \succ K$, or—diagrammatically—by a bicartesian square:

$$\text{(1)} \qquad \begin{tikzcd} H/K \arrow[d, rightarrowtail, "n"] & H \arrow[l, twoheadrightarrow, "p"'] \arrow[d, rightarrowtail, "m"] \\ A/K & A \arrow[l, twoheadrightarrow, "q"'] \end{tikzcd} .$$

Thus a subquotient of A may be more effectively defined as a subobject $s\colon H/K \to A$ in the involutive category $\boldsymbol{A} = \mathrm{Rel}\,\boldsymbol{E}$ of relations over $\boldsymbol{E}$ ($s = m\tilde{p} = \tilde{q}n$, where $\sim$ denotes the involution of $\boldsymbol{A}$). Accordingly, the subquotients of A are provided with a canonical order $s \prec s'$, characterized in the following terms: $H/K \prec H'/K'$ iff $H \prec H'$ and $K \succ K'$ in the lattice of subobjects of A in $\boldsymbol{E}$. The unions (or intersections) we are speaking of *concern this order*, and are generally calculated by unions of numerators and intersections of denominators (or dually).

Equivalently, but avoiding any problem of choice of representants, a subquotient of A *is a projection* $e\colon A \to A$, i.e. a symmetrical idempotent endorelation of A ($e = \tilde{e} = ee$):

$$\text{(2)} \qquad e = s\tilde{s} = (A \twoheadleftarrow H \twoheadrightarrow H/K \twoheadleftarrow H \rightarrowtail A);$$

the corresponding *canonical order* $e \prec e'$ on the set Prj (A) of projections of A is now characterized as: $e = ee'$ (or equivalently: $e = e'e$).

We shall use both these descriptions of subquotients, the latter (projections in $\boldsymbol{A}$) being more suitable for theorical considerations, the former (subobjects in $\boldsymbol{A}$) being more adapted for applications in homological algebra.

0.3. Thus we introduce in ch. 1 unions and intersections of projections in general involutive categories, together with three conditions of regularity: P-regularity, i.e. consistency with the transfer mappings of projections; D-regularity, i.e. consistency with the relation of domination $e \mathrel{\mathrm{C}} e'$ (1.3); PD-regularity, combining the above and being (probably) the « good » notion.

Chapter 2 recalls the definition of RE-category (slightly extending the categories of relations over exact categories) and characterizes in this case the previous notions, in particular for decreasing filtering intersections (2.7), increasing filtering unions (2.8) and telescopic unions (2.9).

In ch. 3 we consider relations induced on projections and the corresponding mapping theorems for « regular » unions and intersections of projections: in particular for decreasing filtering intersections (3.6), increasing filtering unions (3.5) and telescopic unions or *differences* (3.7). Chapter 4 investigates a condition of regularity for induction, typically occurring in homological theories, which makes it to agree with composition.

Chapter 5 applies these results, yielding the convergence theorem for filtered complexes described above and the corresponding mapping theorem (5.5). Analogous results for the system of relative homologies will be given in [G9].

Finally the appendix (ch. 6) concerns distributive RE-categories (e.g.: the classifying RE-categories of the above mentioned theories), clarifying some aspects of the regularity conditions in this case.

1. - Unions and intersections of projections in RI-categories

In this chapter $\boldsymbol{A}$ will always be a *category with regular involution* (RI-category for short): this means a category provided with an involution $\sim: \boldsymbol{A} \to \boldsymbol{A}$ (a contravariant endofunctor, identical on the objects and involutory) which is *regular*: $a = a\tilde{a}a$ for each morphism a). For example, all categories of relations over exact categories are so [C1, C2; G6].

$\boldsymbol{A}$ is selfdual. The morphism a is monic iff $\tilde{a}a = 1$, iff it has a left inverse; dually for epics; it is an isomorphism iff $\tilde{a}a = 1$ and $a\tilde{a} = 1$.

The considerations of 0.2 should be kept in mind, to substantiate the following arguments.

1.1 *Projections*: For every object A we write $\mathrm{Prj}_{\boldsymbol{A}}(A)$ or Prj (A) the set of *projections* of A, i.e. symmetrical idempotent endomorphisms $e\colon A \to A$

($e = \tilde{e} = ee$), equipped with the canonical (partial) order $\prec$:

(1) $\qquad e \prec f \quad \text{if} \quad e = ef \qquad (\text{iff } e = fe, \text{ iff } e = fef),$

whose greatest element is 1_A.

It is well known that the product of two projections e, f is idempotent, generally not symmetrical. We write « ! » the commutativity relation in $\mathrm{Prj}(\mathcal{A})$:

(2) $\qquad e\,!\,f \quad \text{if} \quad ef = fe \qquad (\text{iff } ef \text{ is a projection}),$

notice that if $e\,!\,f$ then $ef = fe$ is the intersection of e and f in $\mathrm{Prj}(\mathcal{A})$.

Moreover we consider in $\mathrm{Prj}(\mathcal{A})$ the reflexive relation of *domination* $e \sqsubset f$ and the associated symmetrical relation $e \Phi f$:

(3) $\qquad e \sqsubset f \quad \text{if } e = efe, \qquad e \Phi f \quad \text{if } e = efe \text{ and } f = fef$ [8],

which generally are not transitive, together with the idempotent binary operation &:

(4) $\qquad e \,\&\, f = efe$ [8],

which generally is neither associative nor commutative. Clearly:

(5) $\quad e \prec f \text{ iff } e = f \,\&\, e,$

(6) $\quad e \sqsubset f \text{ iff } e = e \,\&\, f,$

(7) $\quad e\,!\,f \text{ iff } e \,\&\, f \prec f, \text{ iff } f \,\&\, e \prec e, \text{ iff } e \,\&\, f \prec f \,\&\, e, \text{ iff } e \,\&\, f = f \,\&\, e,$

(8) $\quad e \prec f \text{ implies } e \sqsubset f,$

(9) $\quad \text{if } e \prec f \sqsubset e \text{ then } e = f,$

(10) $\quad e \,\&\, f \;\Phi\; f \,\&\, e.$

1.2. *Transfer mappings*: For every morphism $a: \mathcal{A} \to \mathcal{A}'$ in $\boldsymbol{A}$ we have the associated *transfer* mappings of projections:

(1) $\qquad a_P: \mathrm{Prj}(\mathcal{A}) \to \mathrm{Prj}(\mathcal{A}'), \qquad a_P(e) = ae\tilde{a},$

(2) $\qquad a^P: \mathrm{Prj}(\mathcal{A}') \to \mathrm{Prj}(\mathcal{A}), \qquad a^P(e') = \tilde{a}e'a,$

[8] If $\boldsymbol{A} = \mathrm{Rel}\,\boldsymbol{E}$, $(H|K) \sqsubset (H'|K')$ means that the canonical correspondence from $H|K$ to $H'|K'$ (induced by 1_A) is monic, while Φ means that the latter is iso (3.2.7-3.2.8). These relations, as well as the operation &, are described by meets and joins in $\mathrm{Sub}(\mathcal{A})$ in 2.3. They are transitive (for every object $\mathcal{A}$) iff the operation & is always associative, iff the category $\boldsymbol{E}$ is distributive (see 6.4).

and two associated projections:

$$c(a) = a^r(1_{A'}) = \bar{a}a \in \operatorname{Prj}(A), \tag{3}$$

$$i(a) = a_r(1_A) = a\bar{a} \in \operatorname{Prj}(A'), \tag{4}$$

respectively simulating the coimage and the image of a. Clearly $a^r = \bar{a}_r$ and $i(a) = c(\bar{a})$.

The procedure $A \mapsto \operatorname{Prj}(A)$, $a \mapsto (a_r, a^r)$ can be formalized as a functor from $\boldsymbol{A}$ into a suitable *RI*-category ([G5], § 7.4).

Remark that, for $e, f \in \operatorname{Prj}(A)$:

$$e \,\&\, f = e_r(f) = e^r(f). \tag{5}$$

1.3. *Null morphisms*: The morphism $a \in \boldsymbol{A}(A, A')$ is said to be *null* if, for every $a' \in \boldsymbol{A}(A', A)$, $aa'a = a$. Null morphisms from an ideal of $\boldsymbol{A}$.

A null projection $e \in \operatorname{Prj}(A)$ is dominated by each parallel projection, while a projection which is dominated by a null one, is null. Transfer mappings preserve null projections.

Two projections $e, f \in \operatorname{Prj}(A)$ are said to be *disjoint* whenever $e \,\&\, f$ (or equivalently $f \,\&\, e$) is null.

1.4. Lemma: Consider, in the *RI*-category $\boldsymbol{A}$, a morphism $a\colon A \to A'$ and projections $e, f, g \in \operatorname{Prj}(A)$, $e' \in \operatorname{Prj}(A')$. Then:

a) if $e \sqsubset f \prec g$ then $e \sqsubset g$,

b) if $e \prec f \sqsubset g$ then $e \sqsubset g$,

c) if $e \prec f$ then: $a_r(e) \prec a_r(f)$, $g \,\&\, e \prec g \,\&\, f$, $e \,\&\, g \sqsubset f \,\&\, g$,

d) if $a_r(e) \prec e'$ and $e \prec \bar{a}a$, then $e \prec a^r(e')$,

e) $e \sqsubset a^r(e')$ iff $(a_r(e) \sqsubset e'$ and $e \sqsubset \bar{a}a)$,

f) if $a_r(e) \sqsubset e'$ then $a^r a_r(e) \sqsubset a^r(e')$.

Proof:

a) $ege = (efe)ge = e(fg)ege = ef(ge)(ge) = efge = efe = e$.

b) $ege = (ef)g(fe) = e(fgf)e = efe = e$.

c) First: $(ae\bar{a})(af\bar{a}) = a(ef)\bar{a}af(\bar{a}a\bar{a}) = ae(f\cdot\bar{a}a)(f\cdot\bar{a}a)\bar{a} = ae(f\cdot\bar{a}a)\bar{a} = af\bar{a}$; the second property follows from the first (by 1.2.5); as to the third:

$$(ege)(fgf)(ege) = egeggege = ege.$$

d) $e = (\bar{a}a)e(\bar{a}a) = a^r a_r(e) \prec a^r(e')$, by *c*).

e) If $e \mathrel{\mathrm{Cl}} a^p(e')$, then $e \mathrel{\mathrm{Cl}} a^p(1) = \tilde{a}a$, by *a*) and *c*); moreover $a_p(e) \mathrel{\mathrm{Cl}} e'$ since:

$$a_p(e)\cdot e'\cdot a_p(e) = (ae\tilde{a})e'(ae\tilde{a}) = a\big(e(\tilde{a}e'a)e\big)\tilde{a} = a\big(e\cdot a^p(e')\cdot e\big)\tilde{a} = ae\tilde{a} = a_p(e)\,.$$

Conversely, when the right-hand property of *e*) holds:

$$e\cdot a^p(e')\cdot e = e(\tilde{a}e'a)e = (e\cdot\tilde{a}a\cdot e)(\tilde{a}e'a)(e\cdot\tilde{a}a\cdot e) = e\tilde{a}\big((ae\tilde{a})e'(ae\tilde{a})\big)ae =$$

$$= e\tilde{a}(ae\tilde{a})ae = (e\cdot\tilde{a}a)(e\cdot\tilde{a}a)e = e\cdot\tilde{a}a\cdot e = e\,.$$

f) $$\big(a^p a_p(e)\big)\big(a^p(e')\big)\big(a^p a_p(e)\big) = (\tilde{a}a\cdot e\cdot\tilde{a}a)(\tilde{a}e'a)(\tilde{a}a\cdot e\cdot\tilde{a}a) =$$

$$= \tilde{a}(ae\tilde{a})\cdot e'\cdot(ae\tilde{a})a = \tilde{a}(ae\tilde{a})a = a^p a_p(e)\,.$$

1.5. Definition: Let $e = \bigcup_i e_i$ $(i \in I)$ in the (canonically) ordered set $\mathrm{Prj}\,(A)$. We say that this union is:

a) *P-regular* if, for each morphism a with domain A, $a_p(e) = \bigcup_i a_p(e_i)$ in $\mathrm{Prj}\,(\mathrm{Cod}\, a)$;

b) *D-regular* if, for each $f \in \mathrm{Prj}\,(A)$, the condition $e_i \mathrel{\mathrm{Cl}} f$ $(\forall i \in I)$ implies $e \mathrel{\mathrm{Cl}} f$;

c) *PD-regular* if, for each morphism a with domain A, $a_p(e) = \bigcup_i a_p(e_i)$ is a D-regular union in $\mathrm{Prj}\,(\mathrm{Cod}\, a)$.

Analogously we define P-, D-, PD-regular intersections $e = \bigcap_i e_i$ $(i \in I)$ in $\mathrm{Prj}\,(A)$; e.g.:

b') the intersection $e = \bigcap_i e_i$ is *D-regular* if, for each $f \in \mathrm{Prj}\,(A)$, the condition $f \mathrel{\mathrm{Cl}} e_i$ $(\forall i \in I)$ implies $f \mathrel{\mathrm{Cl}} e$.

However we shall see in 1.7 that P-regular intersections are always D-regular, and coincide with the PD-regular ones. This does not hold for unions; there are cases which are P- and D-regular, but not PD-regular (2.11). The applications to the convergence of spectral sequences (ch. 5 and [G9]) will show that the good notion is the strongest one: PD-regularity.

1.6. *Elementary properties*:

a) PD-regularity implies D- and P-regularity, for unions as well as for intersections.

b) Associativity and commutativity hold for all these kinds of unions and intersections, as well as the cancellability of repeated elements.

c) If $e = \bigcup_i e_i$ is D- or P- or PD-regular and $e_i \prec f_i \prec e$ for every i, then $e = \cup f_i$ is again so; dually for intersections (for the proof, use 1.4 *a*), *b*)).

d) If $e = \max e_i$ then $e = \bigcup_i e_i$ is PD-regular; dually for intersections.

e) The transfer mappings preserve P- and PD-regular unions or intersections.

f) If $e = \bigcup e_i$ is a D-regular union and e_i is null for each $i \neq j$, then $e = e_j$ (use 1.1.9 and the fact that $e_i \mathrel{\mathrm{Cl}} e_j$ for all i).

1.7. Proposition: If $e, e_i \in \mathrm{Prj}\,(A)$ $(i \in I;\ I \neq \emptyset$ (*)), e is the P-regular intersection of the family (e_i) iff:

$$f \& e = \bigcap_i f \& e_i\,, \qquad \text{for every } f \in \mathrm{Prj}\,(A)\,. \tag{1}$$

In this case, if f commutes with all e_i, it also commutes with e.

P-regular intersections are also D-regular, and coincide with the PD-regular ones.

Proof: The necessity of (1) is obvious, since $f \& e = f_P(e)$. Conversely, assume (1) and take $a \in \boldsymbol{A}(A, A')$. First, $e = \bigcap_i e_i$ (take $f = 1_A$ in (1)). Moreover $e \prec e_i$ implies $a_P(e) \prec a_P(e_i)$; if $e' \prec a_P(e_i)$ for all i:

$$a^P(e') \prec a^P a_P(e_i) = (\tilde{a}a) \& e_i \qquad (i \in I)\,, \tag{2}$$

$$a^P(e') \prec (\tilde{a}a) \& e = a^P a_P(e)\,. \tag{3}$$

Now, as $e' \prec a_P(e_i) \prec a_P(1) = a\tilde{a}$ for *some* $i \in I$ $(I \neq \emptyset)$, by 1.4 *d*) applied to $\tilde{a}$ it follows:

$$e' \prec a_P a^P a_P(e) = a_P(e)\,. \tag{4}$$

Assume now that $e = \bigcap e_i$ is P-regular in $\mathrm{Prj}\,(A)$. If f commutes with all e_i, then (1.1.7) $f \& e_i \prec e_i$ $(i \in I)$ and $f \& e = \bigcap f \& e_i \prec \bigcap e_i = e$, i.e. f commutes with e. In order to prove that $e = \bigcap e_i$ is also D-regular, take $f \mathrel{\mathrm{Cl}} e_i$ $(i \in I)$; then $f = f \& e_i$ and: $f \& e = \bigcap f \& e_i = \bigcap f = f$, i.e. $f \mathrel{\mathrm{Cl}} e$. Finally, since P-regular intersections are preserved by transfer mappings (1.6 *e*)), it follows that P-regular and PD-regular intersections coincide.

1.8. *Commuting projections*: In order to give a similar characterization for P- and PD-regular unions we need some considerations on the commutativity relation « ! » of projections (1.1).

(*) The intersection of the empty family in $\mathrm{Prj}\,(A)$ is 1_A. This intersection is P-regular iff every morphism a of domain A is epic $(a_P(1) = 1)$, which is equivalent to asking that every morphism of domain A is *iso* ($\tilde{a}a : A \to A$ has to be epi). Instead our intersection satisfies (1) iff 1_A is the unique projection of A, iff every morphism of domain A is *monic*, which is a weaker condition.

It may be easily checked that the mapping $- \& f$ does not preserve $\prec$: a fortiori it does not preserve intersections or unions.

Consider on $\boldsymbol{A}$ the following equivalent properties concerning parallel projections (always satisfied for categories of relations over exact categories: 2.3):

(1) if $g \prec e$ and $g \prec f$, then $e \,!\, f$,

(2) $e \,!\, f$ iff there is some projection g such that $g \prec e$ and $g \prec f$,

(3) if $e \,!\, g$ and $g \prec f$, then $e \,!\, f$.

Actually: (1) $\Rightarrow$ (2): if $e \,!\, f$ take $g = e \,\&\, f = ef = fe$; (2) $\Rightarrow$ (3): if $e \,!\, g$ and $g \prec f$, then $ege = geg$ precedes both e and $g \prec f$, hence $e \,!\, f$; (3) $\Rightarrow$ (1): trivial.

Such a category $\boldsymbol{A}$ also verifies (use 1.4 e)):

(4) if $e \,!\, f$ then $a_p(e) \,!\, a_p(f)$, $\qquad$ (for $e, f \in \mathrm{Prj}\,(\mathrm{Dom}\, a)$),

(5) a finite family of parallel projections (e_i) has intersection iff these projections commute; then the intersection is the product and is P-regular.

1.9. Proposition: Assume that the RI-category $\boldsymbol{A}$ satisfies the equivalent conditions 1.8.1-3 and let $e, e_i \in \mathrm{Prj}\,(A)$ $(i \in I;\ I \neq \emptyset)$. Then:

a) e is the P-regular union of (e_i) iff for every $f \in \mathrm{Prj}\,(A)$, $f \,\&\, e = \bigcup f \,\&\, e_i$,

b) e is the PD-regular union of (e_i) iff for every $f \in \mathrm{Prj}\,(A)$, $f \,\&\, e = \bigcup f \,\&\, e_i$ is D-regular.

Moreover, if $e = \bigcup e_i$ (simple union) and there is some $j \in I$ such that $f \,!\, e_j$, then $f \,!\, e$.

Proof:

a) The beginning of the proof is similar to the one of 1.7: we just reverse all $\prec$, $\bigcap$ until we reach 1.7.3, which becomes:

$$a^p(e') \succ a^p a_p(e). \tag{1}$$

Now $e' \succ a_p(e_i)$ and $a_p(e_i) \prec a_p(1) = a\breve{a}$ for some $i \in I$; by 1.8.2, e' commutes with $a\breve{a}$ and:

$$e' \succ e'(a\breve{a})e' = (a\breve{a}) \cdot e' \cdot (a\breve{a}) = a_p a^p(e') \succ a_p a^p a_p(e) = a_p(e). \tag{2}$$

b) Assume that the right-hand condition holds (its necessity is trivial) and take some morphism $a \in \boldsymbol{A}(A, A')$; then $a_p(e) = \bigcup a_p(e_i)$ by *a*), and we have to prove this union is D-regular. Let $a_p(e_i) \mathrel{\mathrm{CI}} e'$ in $\mathrm{Prj}\,(A')$, for each $i \in I$. Then (1.4 f)):

$$(a\breve{a}) \,\&\, e_i = a^p a_p(e_i) \mathrel{\mathrm{CI}} a^p(e') \qquad (i \in I), \tag{3}$$

and, by the hypothesis of D-regularity w.r.t. $\breve{a}a$:

$$(\breve{a}a)\&\, e \sqsubset a^{P}(e') \,. \tag{4}$$

By 1.4 d):

$$a_P(e) = a_P a^P a_P(e) = a_P((\breve{a}a)\&\, e) \sqsubset e' \,. \tag{5}$$

The last remark follows at once from 1.8.3: $f \,!\, e_i$ and $e_i \prec e$.

1.10. *A counterexample*: In order to build a semigroup S, with regular involution, which does not satisfy the conditions 1.8.1-3, just consider the semigroup with regular involution generated by three projections e, f, g under the conditions $g \prec e$, $g \prec f$. The multiplication table of S is the following:

(1)

	e	f	g	ef	fe	efe	fef
e	e	ef	g	ef	efe	efe	ef
f	fe	f	g	fef	fe	fe	fef
g	g	g	g	g	g	g	g
ef	efe	ef	g	ef	efe	efe	ef
fe	fe	fef	g	fef	fe	fe	fef
efe	efe	ef	g	ef	efe	efe	ef
fef	fe	fef	g	fef	fe	fe	fef

while the involution is obvious: $(ef)^{\sim} = fe$, etc.

By adjoining a unit 1 we get a regular involution category $\boldsymbol{A}$ (on one object) which does not satisfy 1.8.1 (since e and f do not commute). Notice that $\boldsymbol{A}$ is idempotent.

2. - Unions and intersections of projections in RE-categories

RE-categories, introduced in [G6], are a slight extension of the categories of relations over exact categories (see also 3.1). We recall here the definition and some properties, and study the unions and intersections of their projections.

2.1. *RO-categories*: An RO-category $\boldsymbol{A} = (A, \sim, <)$ is a category $\boldsymbol{A}$ provided with a regular involution $\sim$ and a with a (partial) order relation $<$ on parallel morphisms, consistent with composition and involution. All sets $\mathrm{Prj}\,(A)$ are thus provided with the canonical order $\prec$ and with the order $<$, which are different (2.3.1-2); unless otherwise stated, all the order notions (e.g. unions and intersections) we consider for these sets *concern the canonical order* $\prec$.

An endomorphism x: $A \to A$ verifying $x \leqslant 1$ is called a *restriction* of A.

Every restriction is a projection; two projections x, x' always commute and:

$$(1) \qquad x \prec x' \quad \text{iff} \quad x < x' .$$

Analogously for *corestrictions*, i.e. endomorphisms $z\colon A \to A$ such that $z > 1$; except that, if z, z' are so:

$$(2) \qquad z \prec z' \quad \text{iff} \quad z > z' .$$

We write $\mathrm{Rst}\,(A)$ and $\mathrm{Crs}\,(A)$ the subsets of $\mathrm{Prj}\,(A)$ of restrictions and corestrictions.

The *proper* morphisms u (verifying $\tilde{u}u > 1$ and $u\tilde{u} < 1$) from a subcategory, $\mathrm{Prp}\,\boldsymbol{A}$; if u and v are so, it is easy to check that $u < v$ implies $u = v$ ([G6], § 1.7).

2.2. *RE-categories*: An *RE*-category $\boldsymbol{A}$ is a *RO*-category satisfying:

(*RE*.1) for every $e \in \mathrm{Prj}\,(A)$ there exists a unique restriction $\boldsymbol{n}(e)$ such that $\boldsymbol{n}(e) \succ e$ and $\boldsymbol{n}(e) < e$ (the *numerator* of e), together with a unique corestriction $\boldsymbol{d}^*(e)$ such that $\boldsymbol{d}^*(e) \succ e$ and $\boldsymbol{d}^*(e) > e$ (the **-denominator* of e) (4);

(*RE*.2) for every object A there is a null restriction ω_A and a null corestriction Ω_A (equivalently: the ordered set $(\boldsymbol{A}(A, A), <)$ has minimum ω_A and maximum Ω_A) (4).

Then ([G6], § 4.7) there is a canonical bijection (**-duality*) between restrictions and corestrictions, which preserves $<$ and reverses $\prec$:

$$(1) \qquad \mathrm{Rst}\,(A) \to \mathrm{Crs}\,(A) , \qquad x \mapsto x^* = \boldsymbol{d}^*(x\,\Omega\,x) ,$$

$$(2) \qquad \mathrm{Crs}\,(A) \to \mathrm{Rst}\,(A) , \qquad z \mapsto z_* = \boldsymbol{n}(z\,\omega\,z) ,$$

and allows to define the *denominator* (4) of a projection e of A:

$$(3) \qquad \boldsymbol{d}(e) = (\boldsymbol{d}^*(e))_* = \boldsymbol{n}(\boldsymbol{d}^*(e)\,\omega\,\boldsymbol{d}^*(e)) = \boldsymbol{n}(e\,\omega\,e) \in \mathrm{Rst}\,(A) .$$

Every morphism $a\colon A \to A'$ determines four restrictions, the *definition*, *annihilator*, *values and indetermination* (corresponding to the usual subobjects

(4) If $\boldsymbol{A} = \mathrm{Rel}\,\boldsymbol{E}$ is a category of relations and e is associated to the subquotient H/K of A:

$$\boldsymbol{n}e = (A \leftarrowtail H \rightarrowtail A) , \qquad \boldsymbol{d}^*(e) = (A \twoheadrightarrow A/K \twoheadleftarrow A) , \qquad \omega_A = (A \leftarrowtail 0 \rightarrowtail A) ,$$

$$\Omega_A = (A \twoheadrightarrow 0 \twoheadleftarrow A) , \qquad \boldsymbol{d}(e) = (A \leftarrowtail K \rightarrowtail A) .$$

The restrictions x of A, characterized by $\boldsymbol{d}x = \omega$, correspond to subjects, while the corestrictions z, characterized by $\boldsymbol{n}z = 1$, correspond to quotients. The *-duality described below corresponds to the kernel-cokernel duality between subobjects and quotients of $\boldsymbol{E}$.

when $\boldsymbol{A}$ is a category of relations):

(4) $\qquad \boldsymbol{def}\, a = \boldsymbol{n}(\tilde{a}a)\,, \qquad \boldsymbol{ann}\, a = \boldsymbol{d}(\tilde{a}a) \qquad \text{in } \mathrm{Rst}\,(\mathcal{A})\,,$

(5) $\qquad \boldsymbol{val}\, a = \boldsymbol{n}(a\tilde{a})\,, \qquad \boldsymbol{ind}\, a = \boldsymbol{d}(a\tilde{a}) \qquad \text{in } \mathrm{Rst}\,(\mathcal{A}')\,,$

which characterize in the usual way monos, epis, isos and proper morphims.

In this chapter, from now on, $\boldsymbol{A}$ *will always be an RE-category.*

2.3. *Calculus of projections*: The following results hold for $e, f \in \mathrm{Prj}\,(\mathcal{A})$ ([G6], §§ 4.4, 6.9):

(1) $\qquad e \prec f$ iff $(\boldsymbol{n}e \prec \boldsymbol{n}f$ and $\mathbf{d}^* e \prec \mathbf{d}^* f)$ iff $(\boldsymbol{n}e < \boldsymbol{n}f$ and $\boldsymbol{d}e > \boldsymbol{d}f)$,

(2) $\qquad e < f$ iff $(\boldsymbol{n}e \prec \boldsymbol{n}f$ and $\mathbf{d}^* e \succ \mathbf{d}^* f)$ iff $(\boldsymbol{n}e < \boldsymbol{n}f$ and $\boldsymbol{d}e < \boldsymbol{d}f)$,

(3) $\qquad \mathrm{Rst}\,(\mathcal{A})$ and $\mathrm{Crs}\,(\mathcal{A})$ are modular lattices with 0 and 1 ,

(4) $\qquad \boldsymbol{n}(e \,\&\, f) = \boldsymbol{n}e \cap (\boldsymbol{n}f \cup \boldsymbol{d}e) = (\boldsymbol{n}e \cap \boldsymbol{n}f) \cup \boldsymbol{d}e\,,$

(5) $\qquad \boldsymbol{d}(e \,\&\, f) = \boldsymbol{n}e \cap (\boldsymbol{d}f \cup \boldsymbol{d}e) = (\boldsymbol{n}e \cap \boldsymbol{d}f) \cup \boldsymbol{d}e\,,$

(6) $\qquad e \,!\, f$ iff $(\boldsymbol{n}e \succ \boldsymbol{d}f$ and $\boldsymbol{n}f \succ \boldsymbol{d}e)$,

(7) $\qquad e \,\square\, f$ iff $(\boldsymbol{n}e \prec \boldsymbol{n}f \cup \boldsymbol{d}e$ and $\boldsymbol{d}e \succ \boldsymbol{d}f \cap \boldsymbol{n}e)$,

(8) $\qquad e$ is null iff $\boldsymbol{n}e = \boldsymbol{d}e$.

In particular, by (6), every *RE*-category satisfies the conditions 1.8.1-3.

There is an isomorphism of ordered sets (actually, modular lattices):

(9) $\quad (\mathrm{Prj}\,(\mathcal{A}), <) \to \mathrm{Rst}_2\,(\mathcal{A}) = \{(x, y) \in \mathrm{Rst}\,(\mathcal{A}) \times \mathrm{Rst}\,(\mathcal{A}) | x > y\}\,, \quad e \mapsto (\boldsymbol{n}e, \boldsymbol{d}e)\,,$

whose reciprocal isomorphism will be written:

(10) $$(x, y) \mapsto x|y = x \cdot y^* = y^* \cdot x\,, \qquad (x > y)\,.$$

Every morphism $a\colon A \to A'$ of $\boldsymbol{A}$ defines increasing *transfer mappings for restrictions*:

(11) $\qquad a_R\colon \mathrm{Rst}\,(\mathcal{A}) \to \mathrm{Rst}\,(\mathcal{A}')\,, \qquad a_R(x) = \boldsymbol{n}(a_P(x)) = \boldsymbol{n}(ax\tilde{a})\,,$

(12) $\qquad a^R\colon \mathrm{Rst}\,(\mathcal{A}') \to \mathrm{Rst}\,(\mathcal{A})\,, \qquad a^R(x') = \boldsymbol{n}(a^P(x')) = \boldsymbol{n}(\tilde{a}x'a)\,,$

satisfying the obvious functoriality conditions and:

(13) $$a_P(x|y) = a_R(x)|a_R(y)\,, \qquad (x > y \text{ in } \mathrm{Rst}\,(\mathcal{A}))\,.$$

2.4. Proposition (*Unions and intersections in RE-categories*): Let e_i be projections of $\mathcal{A}$ in the RE-category $\boldsymbol{A}$ $(i \in I)$.

a) The non-empty family (e_i) has a (simple) union in $\mathrm{Prj}\,(\mathcal{A})$ iff the families $(\boldsymbol{n}e_i)$, $(\boldsymbol{d}e_i)$ have respectively union and intersection in $\mathrm{Rst}\,(\mathcal{A})$; in such a case: $\bigcup e_i = (\bigcup \boldsymbol{n}e_i)/(\bigcap \boldsymbol{d}e_i)$.

b) The family (e_i) has a (simple) intersection in $\mathrm{Prj}\,(\mathcal{A})$ iff the families $(\boldsymbol{n}e_i)$, $(\boldsymbol{d}e_i)$ have respectively intersection and union in $\mathrm{Rst}\,(\mathcal{A})$ and moreover $\bigcap \boldsymbol{n}e_i \succ \bigcup \boldsymbol{d}e_i$; in such a case: $\bigcap e_i = (\bigcap \boldsymbol{n}e_i)/(\bigcup \boldsymbol{d}e_i)$.

Proof:

a) Fix some $j \in I$ $(I \neq \emptyset)$. If the families $(\boldsymbol{n}e_i)$, $(\boldsymbol{d}e_i)$ have respectively union and intersection in $\mathrm{Rst}\,(\mathcal{A})$, then $(\bigcup \boldsymbol{n}e_i) \succ \boldsymbol{n}e_j \succ \boldsymbol{d}e_j \succ (\bigcap \boldsymbol{d}e_i)$, and it is easy to see that the projection $(\bigcup \boldsymbol{n}e_i)/(\bigcap \boldsymbol{d}e_i)$ is the union of (e_i) in $\mathrm{Prj}(\mathcal{A})$ (use 2.3.1). Conversely let $e = x/y = \bigcup e_i$: then $x = \boldsymbol{n}e \succ \boldsymbol{n}e_i$ and $y = \boldsymbol{d}e \prec \boldsymbol{d}e_i$ for each i; moreover if $x' \succ \boldsymbol{n}e_i$ and $y' \prec \boldsymbol{d}e_i$ in $\mathrm{Rst}\,(\mathcal{A})$, for every i, then $x' \succ \boldsymbol{n}e_j \succ \boldsymbol{d}e_j \succ y'$ and we can consider $e' = x'/y' \in \mathrm{Prj}\,(\mathcal{A})$; trivially, $e' \succ e_i$ for all i, hence $e' \succ e$, i.e. $x' \succ x$ and $y' \prec y$.

b) The right-hand condition is clearly sufficient. Conversely, let $e = x/y = \bigcap e_i$; then $x \prec \boldsymbol{n}e_i$ and $y \succ \boldsymbol{d}e_i$ for each i. Assume that $x' \prec \boldsymbol{n}e_i$ and $y' \succ \boldsymbol{d}e_i$ in $\mathrm{Rst}\,(\mathcal{A})$, for all i; as $x \cup x' \succ x = \boldsymbol{n}e \succ \boldsymbol{d}e = y \succ y \cap y'$, we may consider the projection $f = (x \cup x')/(y \cap y') \prec e_i$ $(i \in I)$: it follows that $f \prec e$, i.e. $x \cup x' \prec x$ and $y \cap y' \succ y$, i.e. $x' \prec x$ and $y' \succ y$.

2.5. Proposition (*Unions and intersections of restrictions*): Let $e \in \mathrm{Prj}\,(\mathcal{A})$ and $x, x_i \in \mathrm{Rst}\,(\mathcal{A})$, for $i \in I$. Then.

(1) $e = \bigcup x_i$ in $\mathrm{Prj}\,(\mathcal{A})$ iff $e \in \mathrm{Rst}\,(\mathcal{A})$ and $e = \bigcup x_i$ in $\mathrm{Rst}\,(\mathcal{A})$,

(2) if $x = \bigcup x_i$ in $\mathrm{Rst}\,(\mathcal{A})$, the union is D-regular in $\mathrm{Prj}\,(\mathcal{A})$ iff for every $y \in \mathrm{Rst}\,(\mathcal{A})$, $y \cap x_i = \omega$ for all i implies $y \cap x = \omega$.

Further, the following conditions are equivalent:

a) $x = \bigcup x_i$ is P-regular in $\mathrm{Prj}\,(\mathcal{A})$,

b) $x = \bigcup x_i$ is PD-regular in $\mathrm{Prj}\,(\mathcal{A})$,

c) $x = \bigcup x_i$ is a *distributive union* in the lattice $\mathrm{Rst}\,(\mathcal{A})$ (i.e. $x \cap y = \bigcup (x_i \cap y)$, for all $y \in \mathrm{Rst}\,(\mathcal{A})$),

d) $a_R(x) = \bigcup a_R(x_i)$ in $\mathrm{Rst}\,(\mathcal{A}')$, for every $a \in \boldsymbol{A}(\mathcal{A}, \mathcal{A}')$.

A dual result holds for intersection of restrictions. Distributive unions and intersections of restrictions are preserved by the transfer mappings a_R, as well as by $- \cap y$ and $- \cup y$ $(y \in \mathrm{Rst}\,(\mathcal{A}))$.

PROOF: Property (1) follows from 2.4 a), with $e_i = x_i = x_i/\omega$; (2) is also trivial, since $x \sqsubset x'/y$ iff $x \prec x'$ and $x \cap y = \omega$ (2.3.7).

$a) \Rightarrow d)$ If $x = \bigcup x_i$ is P-regular, $a_P(x) = \bigcup a_P(x_i)$ in $\mathrm{Prj}\,(\mathcal{A}')$. By 2.3.12 and 2.4: $a_R(x) = \boldsymbol{n}(a_P(x)) = \bigcup \boldsymbol{n}(a_P(x_i)) = \bigcup a_R(x_i)$.

$d) \Rightarrow b)$ Take a morphism $a\colon \mathcal{A} \to \mathcal{A}'$; then, by 2.4, $a_P(x) = \bigcup a_P(x_i)$ in $\mathrm{Prj}\,(\mathcal{A}')$:

$$\text{(3)} \qquad \boldsymbol{n}a_P(x) = a_R(x) = \bigcup a_R(x_i) = \bigcup \boldsymbol{n}a_P(x_i)\,,$$

$$\text{(4)} \qquad \boldsymbol{d}a_P(x) = a_R(\omega) = \bigcap a_R(\omega) = \bigcap \boldsymbol{d}a_P(x_i)\,.$$

In order to prove that the previous union is D-regular in $\mathrm{Prj}\,(\mathcal{A}')$, assume that $a_P(x_i) \sqsubset f = x'/y'$ in $\mathrm{Prj}\,(\mathcal{A}')$ $(i \in I)$; by the characterization of domination in 2.3.7 this means that:

$$\text{(5)} \qquad a_R(x_i) \prec x' \cup a_R(\omega)\,, \qquad a_R(\omega) \succ y' \cap a_R(x_i) \qquad (i \in I)\,,$$

$$\text{(6)} \qquad a_R(x) = \bigcup a_R(x_i) \prec x' \cup a_R(\omega)\,,$$

$$\text{(7)} \qquad y' \cap a_R(x) = y'_R a_R(x) = \bigcup y'_R a_R(x_i) = \bigcup (y' \cap a_R(x_i)) \prec a_R(\omega)\,,$$

so that, again by 2.3.7, $a_P(x) \sqsubset f$.

$b) \Rightarrow a)$ Obvious.

$d) \Rightarrow c)$ For each $y \in \mathrm{Rst}\,(\mathcal{A})$, $y_R(x) = \boldsymbol{n}(yxy) = xy = x \cap y$; analogously for x_i.

$c) \Rightarrow a)$ We use the characterization 1.9 of P-regular unions (the condition 1.8.1 holds, as remarked in 2.3); if $f \in \mathrm{Prj}\,(\mathcal{A})$, by 2.3.4-5:

$$\text{(8)} \qquad \boldsymbol{n}(fxf) = \boldsymbol{n}f \cap (x \cup \boldsymbol{d}f) = \boldsymbol{n}f \cap (\bigcup x_i \cup \boldsymbol{d}f) =$$
$$= \bigcup (\boldsymbol{n}f \cap (x_i \cup \boldsymbol{d}f)) = \bigcup \boldsymbol{n}(fx_if)\,,$$

$$\text{(9)} \quad \boldsymbol{d}(fxf) = \boldsymbol{d}f \cup (\omega \cap \boldsymbol{n}f) = \boldsymbol{d}f\,, \qquad \boldsymbol{d}(fx_if) = \boldsymbol{d}f \cup (\omega \cap \boldsymbol{n}f) = \boldsymbol{d}f\,,$$

so that, by 2.4: $f \,\&\, x = \bigcup f \,\&\, x_i$.

Now a)-d) are equivalent. For the intersection case, use the $({}^{*}, \prec)$-duality, which preserves projections and restrictions but reverses $\prec$.

The last remark follows from the functoriality of $a \mapsto a_P$, together with the following facts: $- \cap y = y_R$, $- \cup y = (1/y)_R$ (from 2.3).

2.6. THEOREM (*P-regular unions and intersections*): Let $e_i \in \mathrm{Prj}\,(\mathcal{A})$, $i \in I \neq \emptyset$.

a) The family (e_i) has a P-regular union iff $(\boldsymbol{n}e_i)$ and $(\boldsymbol{d}e_i)$ have respectively distributive union and distributive intersection in $\mathrm{Rst}\,(\mathcal{A})$; in such a case: $\bigcup e_i = (\bigcup \boldsymbol{n}e_i)/(\bigcap \boldsymbol{d}e_i)$.

b) The family (e_i) has a *P*-regular intersection iff $(\boldsymbol{ne}_i)$ and $(\boldsymbol{de}_i)$ have respectively distributive intersection and distributive union in $\mathrm{Rst}(\mathcal{A})$ and moreover the former is greater than the latter; in such a case: $\bigcap e_i = (\bigcap \boldsymbol{ne}_i)/(\bigcup \boldsymbol{de}_i)$.

c) *P*-, *D*- and *PD*-regular intersections of projections coincide.

d) If there exist $x = \max \boldsymbol{ne}_i$ and $y = \min \boldsymbol{de}_i$, then x/y is the *P*-regular union of (e_i) (this union need not be *PD*-regular: see 2.11).

e) If there exist $x = \min \boldsymbol{ne}_i$ and $y = \max \boldsymbol{de}_i$, with $x \succ y$, then x/y is the *PD*-regular intersection of (e_i).

Proof:

a) If (e_i) has a *P*-regular union e, for each $a\colon \mathcal{A} \to \mathcal{A}'$, $a_P(e) = \bigcup a_P(e_i)$; by 2.4, this means that:

$$(1) \qquad a_R(\boldsymbol{ne}) = \bigcup a_R(\boldsymbol{ne}_i), \qquad a_R(\boldsymbol{de}) = \bigcap a_R(\boldsymbol{de}_i) \qquad \text{in } \mathrm{Rst}(\mathcal{A}'),$$

hence, by 2.5, $\boldsymbol{ne} = \bigcup \boldsymbol{ne}_i$, $\boldsymbol{de} = \bigcap \boldsymbol{de}_i$ distributively in $\mathrm{Rst}(\mathcal{A})$. Conversely, when the last property holds (e_i) has a union $e = (\bigcup \boldsymbol{ne}_i)/(\bigcap \boldsymbol{de}_i)$ by 2.4, which is easily seen to the *P*-regular by reversing the above argument.

b) Reverse all unions and intersections in the proof of *a*).

c) We already know that, as to intersections of projections, *P*-regular implies *D*-regular and coincides with *PD*-regular (1.7). Therefore, let $e = \bigcap e_i$ be *D*-regular in $\mathrm{Prj}(\mathcal{A})$: by 2.4, $x = \bigcap x_i$ and $y = \bigcup y_i$ in $\mathrm{Rst}(\mathcal{A})$, where $x = \boldsymbol{ne}$, $y = \boldsymbol{de}$ and so on; by *b*), we just need to prove this intersection and union to be distributive.

Let $\mathfrak{z} \in \mathrm{Rst}(\mathcal{A})$: first we want to show that $x \cup \mathfrak{z} = \bigcap (x_i \cup \mathfrak{z})$; we can assume that $\mathfrak{z} \succ x$: otherwise, by setting $\bar{\mathfrak{z}} = x \cup \mathfrak{z} \succ x$, it follows that

$$x \cup \mathfrak{z} = \bar{\mathfrak{z}} = \bigcap (x_i \cup \bar{\mathfrak{z}}) = \bigcap (x_i \cup x \cup \mathfrak{z}) = \bigcap (x_i \cup \mathfrak{z}).$$

Take also some restriction $\mathfrak{z}' \prec x_i \cup \mathfrak{z}$ $(i \in I)$. Then $(\mathfrak{z} \cup \mathfrak{z}')/\mathfrak{z} \mathrel{\mathrm{Cl}} e_i$ $(i \in I)$ by 2.3.7, since:

$$(2) \qquad \mathfrak{z} \cup \mathfrak{z}' \prec x_i \cup \mathfrak{z}, \qquad y_i \cap (\mathfrak{z} \cup \mathfrak{z}') \prec y \prec x \prec \mathfrak{z}.$$

By the hypothesis of *D*-regularity of $e = \bigcap e_i$, $(\mathfrak{z} \cup \mathfrak{z}')/\mathfrak{z} \mathrel{\mathrm{Cl}} e$, hence $\mathfrak{z} \cup \mathfrak{z}' \prec x \cup \mathfrak{z}$ and $\mathfrak{z}' \prec x \cup \mathfrak{z}$.

Second, we have to prove that $y \cap \mathfrak{z} = \bigcup (y_i \cap \mathfrak{z})$; hence we assume $\mathfrak{z} \prec y$ (otherwise, consider $\bar{\mathfrak{z}} = y \cap \mathfrak{z}$) and choose some $\mathfrak{z}' \succ y_i \cap \mathfrak{z}$, for all *i*. Now we have $\mathfrak{z}/(\mathfrak{z} \cap \mathfrak{z}') \mathrel{\mathrm{Cl}} e_i$ $(i \in I)$, since:

$$(3) \qquad \mathfrak{z} \prec y \prec x \prec x_i \cup (\mathfrak{z} \cap \mathfrak{z}'), \qquad y_i \cap \mathfrak{z} \prec \mathfrak{z} \cap \mathfrak{z}';$$

again by the D-regularity of $e=\bigcap e_i$, $z/(z\cap z')\sqsubset e$. Finally $y\cap z\prec z\cap z'$ and $z'\succ y\cap z$.

d) Clearly $x/y=\bigcup e_i$ (2.4); since transfer mappings preserve $\prec$, hence maxima and minima of indexed families, the union is P-regular.

e) Analogous to the proof of *d*), combined with *c*).

2.7. Corollary (*Intersection of decreasing filtering families*): Let (e_i) be a decreasing family of projections of A, indexed on a non-empty, filtering set I (i.e. an ordered set where every finite subset has some upper bound). If $x=\bigcap \boldsymbol{ne}_i$ and $y=\bigcup \boldsymbol{de}_i$ exist in Rst (A), then $x\succ y$ and $x/y=\bigcap e_i$; the last intersection is P-regular iff it is D-regular, iff it is PD-regular, iff $x=\bigcap \boldsymbol{ne}_i$ and $y=\bigcup \boldsymbol{de}_i$ are distributive in Rst (A).

Proof: We only need to prove the inequality $x\succ y$, as the rest follows from 2.4 *b*) and 2.6 *b*), *c*). Actually, if $i<j$ in I: $\boldsymbol{ne}_i\succ \boldsymbol{ne}_j\succ \boldsymbol{de}_j$; thus, for all i, $\boldsymbol{ne}_i\succ \bigcup_{j>i} \boldsymbol{de}_j=\bigcup_{j\in I} \boldsymbol{de}_j$ (the set $\{j: j>i\}$ is cofinal in I), and $x=\bigcap \boldsymbol{ne}_i\succ y=$ $=\bigcup \boldsymbol{de}_i$.

2.8. Theorem (*Union of increasing filtering families*): Let I be a non-empty, filtering set I. For increasing I-families of projections, P-regular unions are D-regular and coincide with the PD-regular ones.

Proof: Let (e_i) be an increasing I-family of projections of A and assume that $e=\bigcup e_i$ is P-regular: by 2.6, $\boldsymbol{ne}=\bigcup \boldsymbol{ne}_i$ and $\boldsymbol{de}=\bigcap \boldsymbol{de}_i$ are distributive in Rst (A); consider now a projection $f=x'/y'$ such that $e_i\sqsubset f$ $(i\in I)$, i.e.:

$$\boldsymbol{ne}_i\prec x'\cup \boldsymbol{de}_i\,,\qquad y'\cap \boldsymbol{ne}_i\prec \boldsymbol{de}_i\,,\qquad (i\in I)\,. \tag{1}$$

Now, for $i<j$ in I:

$$\boldsymbol{ne}_i\prec \boldsymbol{ne}_j\prec x'\cup \boldsymbol{de}_j\,,\qquad \boldsymbol{de}_i\succ \boldsymbol{de}_j\succ y'\cap \boldsymbol{ne}_j\,; \tag{2}$$

from the cofinality of $\{j: j>n\}$ in I, it follows that:

$$\boldsymbol{ne}_i\prec \bigcap_{j>i} x'\cup \boldsymbol{de}_j=\bigcap_j x'\cup \boldsymbol{de}_j=x'\cup\left(\bigcap \boldsymbol{de}_j\right)=x'\cup \boldsymbol{de}\,, \tag{3}$$

$$\boldsymbol{de}_i\succ \bigcup_{j>i} y'\cap \boldsymbol{ne}_j=\bigcup_j y'\cap \boldsymbol{ne}_j=y'\cap\left(\bigcup \boldsymbol{ne}_j\right)=y'\cap \boldsymbol{ne}\,, \tag{4}$$

$$\boldsymbol{ne}=\bigcup \boldsymbol{ne}_i\prec x'\cup \boldsymbol{de}\,,\qquad \boldsymbol{de}=\bigcap \boldsymbol{de}_i\succ y'\cap \boldsymbol{ne}\,, \tag{5}$$

that is, $e\sqsubset f$.

Finally, since transfer mappings preserve P-regular unions and increasing families (1.4 *c*)), the above argument also proves that every P-regular union is also PD-regular.

2.9. Theorem (*Union of telescopic families*):

a) Let:

(1) $$x_0 \prec x_1 \prec \ldots \prec x_n, \qquad (n \geq 1),$$

be a finite increasing family in $\mathrm{Rst}(\mathcal{A})$ and consider the associated *telescopic* family of projections $e_i = x_i/x_{i-1}$ $(i = 1, 2, \ldots, n)$. Then:

(2) $$x_n/x_0 = \bigcup e_i \qquad (PD\text{-regular in } \mathrm{Prj}(\mathcal{A})).$$

b) Let $(x_i)_{i \in I}$ be an increasing family in $\mathrm{Rst}(\mathcal{A})$, indexed on an interval I of $\mathbf{Z}$ (with at least two elements); consider the associated telescopic family of projections $e_i = x_i/x_{i-1}$ $(i \in I'$, where $I' = \{i \in I : i - 1 \in I\})$.

The following conditions are equivalent:

(3) (e_i) has a P-regular union e,

(4) (e_i) has a PD-regular union e,

(5) (x_i) has union x^+ and intersection x^-, distributively in $\mathrm{Rst}(\mathcal{A})$,

and in such a case: $e = x^+/x^-$.

Telescopic unions are preserved by transfer mappings α_p and in particular by mappings $f\&-$.

Proof:

a) By 2.6*d*), $x_n/x_0 = \bigcup e_i$ is P-regular; since transfer mappings of projections trivially preserve telescopic families, we only need to prove that this union is D-regular. Assume that $e_i \,\mathbf{\triangleleft}\, f = x/y$ $(i = 1, \ldots, n)$; then:

(6) $$x_i \prec x \cup x_{i-1}, \qquad y \cap x_i \prec x_{i-1},$$

(7) $$x_n \prec x \cup x_{n-1} \prec x \cup (x \cup x_{n-2}) \prec \ldots \prec x \cup x_0,$$

(8) $$x_0 \succ y \cap x_1 \succ y \cap (y \cap x_1) \succ \ldots \succ y \cap x_n,$$

which precisely means that $x_n/x_0 \,\mathbf{\triangleleft}\, f$.

b) We may assume that $I = \mathbf{Z}$ $(= I')$, possibly by repeating elements in (x_i). By 2.6, the family (e_i) has P-regular union $e = x^+/x^-$ iff:

(9) $$x^+ = \bigcup \boldsymbol{n} e_i = \bigcup_{i \in \mathbf{Z}} x_i, \qquad x^- = \bigcap \boldsymbol{d} e_i = \bigcap_{i \in \mathbf{Z}} x_{i-1} \qquad \text{(distributive)}.$$

If this is the case, the union is also PD-regular, by the following argument: for every $i \in \mathbf{N}$, we have a projection:

(10) $$f_i = x_i/x_{-i} = \bigcup_{-i+1}^{i} (x_j/x_{j-1}) \qquad (PD\text{-regular union by } a)),$$

and we get an increasing family (f_i) on $\mathbf{N}$, which has (2.6, 2.8) PD-regular union $\left(\bigcup_{i\in\mathbf{N}} x_i\right)/\left(\bigcap_{i\in\mathbf{N}} x_{-i}\right) = x^+/x^- = e$; by associativity and cancellability of repeated elements (1.6), the union of (e_i) is PD-regular.

2.10. *Some examples in categories of modules:*

a) If $\boldsymbol{E}$ is the abelian category of modules on some unitary ring R, every increasing union $H = \bigcup H_i$ of submodules of A is clearly distributive:

$$H \cap K = \bigcup (H_i \cap K), \qquad \text{for all submodules } K; \tag{1}$$

indeed, if $x \in H \cap K$ then $x \in H_i$ for a suitable index i and $x \in \bigcup (H_i \cap K)$.

b) In the category of abelian groups there exist decreasing intersections of subgroups which are not distributive.

E.g. in $\mathbf{Z}$, the family $H_n = 2^n \cdot \mathbf{Z}$ $(n > 0)$ has null intersection, while for each n, $2^n \cdot \mathbf{Z} \vee 3\mathbf{Z} = \mathbf{Z}$.

Notice also that, for decreasing families (H_n) of subgroups of the abelian group A, the conditions:

i) $\bigcap H_n = 0$, distributively in $\mathrm{Sub}(A)$,

ii) $\lim_{\leftarrow} (A/H_n) = A$,

are independent. First, take:

$$A = \mathbf{Z}^{\mathbf{N}}, \qquad H_n = \{(k_i) \in \mathbf{Z}^{\mathbf{N}} : k_0 = k_1 = \ldots = k_n = 0\}, \tag{2}$$

so that ii) holds and i) does not: for the subgroup $K = \mathbf{Z}^{(\mathbf{N})}$ of quasi-null sequences of integers, $K \vee H_n = A$ for all n.

On the other hand, let p_i denote the sequence of all prime integers and set:

$$A = \mathbf{Z}, \qquad H_n = p_1^n \cdot p_2^n \cdot \ldots \cdot p_n^n \cdot \mathbf{Z}; \tag{3}$$

since H_n is eventually contained in each subgroup of $\mathbf{Z}$, the condition i) is satisfied, while it is not difficult to see that ii) is not.

2.11. *A counterexample:* Last we build an example, in the category $\boldsymbol{A}$ of relations over abelian groups, of a P-regular, D-regular union $1 = e_1 \cup e_2$ which is not PD-regular.

Thus, for unions of projections in RE-categories: P- and *D-regular does not imply PD-regular*; moreover *P-regular does not imply D-regular* (otherwise P-regularity should coincide with PD-regularity, as it happens for intersections: 1.7).

Let A be an abelian group whose lattice of subgroups is a four-element chain: $0 \prec Y \prec X \prec A$; e.g. $A = \mathbf{Z}/8\mathbf{Z}$. Thus the lattice $\mathrm{Rst}(A)$ of restrictions of A in $\boldsymbol{A}$ is the chain: $\omega \prec y \prec x \prec 1$, where $y = (A \twoheadleftarrow Y \rightarrowtail A)$ and analogously for x. Consider the projections of A:

(1) $$y = y/\omega\,, \qquad x' = 1/x\,,$$

corresponding to the subquotients Y and A/X.

By 2.6 *d*), $y \cup x' = 1/\omega = 1$ is *P*-regular. It is easy to see that the union is also *D*-regular: the unique projection f of A such that $y, x' \,\mathrm{C}\, f$ is 1.

Finally we show that $y \cup x' = 1$ is not *PD*-regular. Consider the projection $f = x/y$ and the transfer mapping $f_p = f\& -$:

(2) $$f\& y = (x \cap (y \cup y))/(x \cap (\omega \cup y)) = y/y\,,$$

(3) $$f\& (1/x) = (x \cap (1 \cup y))/(x \cap (x \cup y)) = x/x\,,$$

now the union $(y/y) \cup (x/x) = f\& 1 = f$ is not *D*-regular, since the projections (2) and (3) are null (i.e. dominated by ω), while f is not so.

3. - Induction in *RI*-categories

We treat here the morphisms induced on projections by the morphisms of $\boldsymbol{A}$ and their connections with unions and intersections of projections; in order to avoid the condition that $\boldsymbol{A}$ be factorizing, the induced morphisms will live in the associated factorizing category $\boldsymbol{B} = \mathrm{Fct}\,\boldsymbol{A}$ recalled in 3.1. $\boldsymbol{A}$ is always an *RI*-category, with $a \in \boldsymbol{A}(A, A')$, $e \in \mathrm{Prj}\,(A)$ and $e' \in \mathrm{Prj}\,(A')$.

3.1. *Factorizing RI-categories.* The *RI*-category $\boldsymbol{A}$ is said to be *factorizing* when each morphism has an essentially unique epi-mono factorization (actually the uniqueness follows easily from the existence of a regular involution). Equivalent condition: for each projection e there exists a mono s such that $e = s\tilde{s}$.

In particular, an *RE*-category $\boldsymbol{A}$ is factorizing iff its category of proper morphisms $\boldsymbol{E}$ is componentwise exact; in such case $\boldsymbol{A} \simeq \mathrm{Rel}\,\boldsymbol{E}$ ([G6], §§ 6.1, 6.4).

Every *RI*-category $\boldsymbol{A}$ has an associated factorizing *RI*-category $\boldsymbol{B} = \mathrm{Fct}\,\boldsymbol{A}$ ([G6], § 3.5): its objects are all the projections of $\boldsymbol{A}$, while a morphism $a\colon e \to f$ is given by any $a \in \boldsymbol{A}(\mathrm{Dom}\,e, \mathrm{Dom}\,f)$ such that $a = fae$ (or equivalently: $a = ae = fa$); the composition and involution are those of $\boldsymbol{A}$ and $1_e = (e\colon e \to e)$. Thus:

(1) $a\colon e \to f$ is monic in $\boldsymbol{B}$ iff $\tilde{a}a = e$ (in $\boldsymbol{A}$); it is epic iff $a\tilde{a} = f$.

There is an obvious fully faithful embedding:

(2) $$\boldsymbol{A} \to \mathrm{Fct}\,\boldsymbol{A}\,, \qquad A \mapsto 1_A\,.$$

If $s\colon S \to A$ is monic in $\boldsymbol{A}$, 1_s is isomorphic in $\boldsymbol{B}$ to the associated projection $e = s\tilde{s}\colon A \to A$, through:

$$s\colon 1_s \to e\,. \tag{3}$$

Thus (1) is an equivalence of categories iff $\boldsymbol{A}$ itself is factorizing; in this case the reciprocal equivalence is: $e \mapsto \operatorname{Im} e$ (for some choice of images in $\boldsymbol{A}$).

If $\boldsymbol{A}$ is an *RO*-category, $\boldsymbol{B}$ is also so (with the « same » order $<$); $\boldsymbol{A}$ is an *RE*-category iff $\boldsymbol{B}$ is so ([G6], § 6.5).

3.2. Definition: We say that the morphism $a\colon A \to A'$ *induces* from $e \in \operatorname{Prj}(A)$ to $f \in \operatorname{Prj}(A')$ the following morphism of $\boldsymbol{B}(e, f)$:

$$fae\colon e \to f\,. \tag{1}$$

Analogously, if $s\colon S \to A$ and $t\colon T \to A'$ are monic in $\boldsymbol{A}$, we say that a induces from s to t the morphism $a' = \tilde{t}as\colon S \to T$ of $\boldsymbol{A}$. The second notion is a particular case of the first one (3.1.3) and the two notions are equivalent when $\boldsymbol{A}$ is factorizing. Therefore we develop mainly the former, but we shift to the latter in the applications of ch. 5.

Beware, this induction, generally, does not agree with composition ([Ma], p. 53; see also 6.3).

We say that *a induces a mono* (resp. *an epi*) from e to f whenever the (trivially) equivalent properties (2)-(5) (resp. (2')-(5')) are satisfied:

(2) $fae\colon e \to f$ is mono in $\boldsymbol{B}$, (2') $fae\colon e \to f$ is epi in $\boldsymbol{B}$,

(3) $e\tilde{a}fae = e$ in $\boldsymbol{A}$, (3') $fae\tilde{a}f = f$ in $\boldsymbol{A}$,

(4) $e = e \,\&\, a^P(f)$ in $\boldsymbol{A}$, (4') $f = f \,\&\, a_P(e)$ in $\boldsymbol{A}$,

(5) $e \sqsubset a^P(f)$ in $\boldsymbol{A}$, (5') $f \sqsubset a_P(e)$ in $\boldsymbol{A}$;

accordingly, we say that *a induces an iso* from e to f when (2) and (2') both hold; by 1.4 *e*) this is equivalent to:

$$e \,\Phi\, a^P(f) \qquad \text{and} \qquad f \,\Phi\, a_P(e) \qquad \text{in } \boldsymbol{A}\,. \tag{6}$$

Notice that the properties (3)-(5), (3')-(5') and (6) do not require the construction of $\operatorname{Fct} \boldsymbol{A}$. Thus, from now on, *we work directly in* $\boldsymbol{A}$.

Trivially, a induces a mono from e to f iff $\tilde{a}$ induces an epi from f to e. If e and f are null projections, a induces always an iso from e to f.

For two parallel projections $e_1, e_2 \in \operatorname{Prj}(A)$:

$$e_1 \sqsubset e_2 \text{ iff } 1_A \text{ induces a mono from } e_1 \text{ to } e_2, \tag{7}$$

$$e_1 \,\Phi\, e_2 \text{ iff } 1_A \text{ induces an iso from } e_1 \text{ to } e_2, \tag{8}$$

and in the last case e_1 and e_2 may be said to be *canonically isomorphic*. Recall that this is not an equivalence relation, generally (1.2; 6.3-6.4).

3.3. *Mapping theorem for D-regular unions*: Let $e_i \prec e$ in Prj (A) and $f_i \prec f$ in Prj (A'), for $i \in I$.

a) If a induces a mono from e_i to f_i (for all i) and $e = \bigcup e_i$ is D-regular in Prj (A), then a induces a mono from e to f.

b) If a induces an epi from e_i to f_i (for all i) and $f = \bigcup f_i$ is D-regular in Prj (A'), then a induces an epi from e to f.

Proof: *a*) By hypothesis, for each i: $e_i \Subset a^*(f_i) \prec a^*(f)$; by 1.4 *a*), $e_i \Subset a^*(f)$ and, by D-regularity, $e \Subset a^*(f)$. The property *b*) follows from *a*) and $\sim$-duality.

3.4. *Mapping theorem for D-regular intersections*: Let $e \prec e_i$ in Prj (A) and $f \prec f_i$ in Prj (A'), for $i \in I \neq \emptyset$.

a) If a induces a mono from e_i to f_i (for all i) and $f = \bigcap f_i$ is D-regular in Prj (A'), then a induces a mono from e to f.

b) If a induces an epi from e_i to f_i (for all i) and $e = \bigcup e_i$ is D-regular in Prj (A), then a induces an epi from e to f.

Proof: Also here it suffices to check *a*). For each i: $e \prec e_i \Subset a^*(f_i)$, hence (1.4 *b*)): $e \Subset a^*(f_i)$; by 1.4 *e*) it follows that $a_*(e) \Subset f_i$, for all i. By D-regularity, $a_*(e) \Subset f$. Now, for some $i \in I \neq \emptyset$: $e \Subset a^*(f_i) \prec a^*(1) = \partial a$, hence $e \Subset \partial a$ and, again by 1.4 *d*), $a_*(e) \Subset f$ implies $e \Subset a^*(f)$.

3.5. *Mapping theorem for filtered unions in RE-categories*: Let $\boldsymbol{A}$ be an *RE*-category and $a\colon A \to A'$. Let be given a filtered, non-empty set I together with projections:

$$(1) \qquad e = x|y, \qquad e_i = x_i|y_i \text{ in Prj}(A), \qquad (i \in I),$$

$$(2) \qquad f = x'|y', \qquad f_i = x'_i|y'_i \text{ in Prj}(A'), \qquad (i \in I),$$

with (e_i) and (f_i) increasing and $e_i \prec e$, $f_i \prec f$ $(i \in I)$.

a) If a induces a mono from e_i to f_i (for all i) and $x = \bigcup x_i$, $y = \bigcap y_i$ distributively in Rst (A), then a induces a mono from e to f.

b) If a induces an epi from e_i to f_i (for all i) and $x' = \bigcup x'_i$, $y' = \bigcap y$ distributively in Rst (A'), then a induces an epi from e to f.

Proof: By 2.6 *a*), 2.8 and 3.3.

3.6. *Mapping theorem for filtered intersections in RE-categories*: In the same general hypotheses of 3.5, assume now that (e_i) and (f_i) are *decreasing* and $e \prec e_i$, $f \prec f_i$ $(i \in I)$.

a) If a induces a mono from e_i to f_i (for all i) and $x' = \bigcap x'_i$, $y' = \bigcup y'_i$ distributively in Rst $(\mathcal{A}')$, then a induces a mono from e to f.

b) If a induces an epi from e_i to f_i (for all i) and $x = \bigcap x_i$, $y = \bigcup y_i$ distributively in Rst $(\mathcal{A})$, then a induces an epi from e to f.

Proof: By 2.7 and 3.4.

3.7. *Mapping theorem for telescopic unions and differences in RE-categories*: Let $\boldsymbol{A}$ be an *RE*-category, $a\colon \mathcal{A} \to \mathcal{A}'$ and I an interval of $\mathbf{Z}$ (with at least two elements); let (x_i) and (y_i) be *increasing* families of Rst $(\mathcal{A})$ and Rst $(\mathcal{A}')$ and assume that:

$$x^+ = \bigcup x_i\,, \qquad x^- = \bigcap x_i \qquad \text{are distributive in Rst}\,(\mathcal{A})\,, \tag{1}$$

$$y^+ = \bigcup y_i\,, \qquad y^- = \bigcap y_i \qquad \text{are distributive in Rst}\,(\mathcal{A}')\,. \tag{2}$$

Write $I' = \{i \in I : i - 1 \in I\}$, choose a fixed $j \in I'$ and set:

$$e_i = x_i | x_{i-1} \in \mathrm{Prj}\,(\mathcal{A})\,, \qquad f_i = y_i | y_{i-1} \in \mathrm{Prj}\,(\mathcal{A}) \qquad (i \in I')\,. \tag{3}$$

a) If a induces a mono (resp. epi, iso) from e_i to f_i, for all $i \in I'$, then the same holds from $e = x^+ | x^-$ to $f = y^+ | y^-$.

b) If a induces a *mono* from e to f and an *epi* from e_i to f_i for all $i \in I'$, $i \neq j$, then it induces a *mono* from e_j to f_j.

b') If a induces an *epi* from e to f and a *mono* from e_i to f_i for all $i \in I'$, $i \neq j$, then it induces an *epi* from e_j to f_j.

b'') If a induces an *iso* from e to f as well as from e_i to f_i for all $i \in I'$, $i \neq j$, then the same holds from e_j to f_j.

Proof: First notice that $e = \bigcup e_i$ and $f = \bigcup f_i$ are *PD*-regular unions, by 2.9. Thus *a*) is a straightforward consequence of 3.3 and we only need to prove *b*): the rest follows by $\sim$-duality.

By hypothesis a induces a mono from e to f, that is:

$$e_j < e \triangleleft a^r(f) = \bigcup_{i \in I'} a^r(f_i)\,;$$

thus $e_j \triangleleft \bigcup_{i \in I'} a^r(f_i)$, i.e.:

$$e_j = e_j \,\&\, \Big(\bigcup_{i \in I'} a^r(f_i)\Big) = \bigcup_{i \in I'} (e_j \,\&\, a^r(f_i))\,. \tag{3}$$

We prove now that e_j is *disjoint* from $a^r(f_i)$ (i.e. $e_j \,\&\, a^r(f_i)$ is null) for all $i \in I'$, $i \neq j$. This will complete the proof, since by (3) and 1.6 *f*) it implies that $e_j = e_j \,\&\, a^r(f_j)$, i.e. our thesis: $e_j \triangleleft a^r(f_j)$.

Actually, take $i \in I'$, $i \neq j$: the morphism a induces an epi from e_i to f_i, i.e. $f_i \sqsubset a_r(e_i)$, hence (1.4 e)) $a^r(f_i) \sqsubset e_i$ and (2.3.7):

$$a^n(y_i) \prec x_i \cup a^n(y_{i-1}), \tag{4}$$

$$a^n(y_{i-1}) \succ x_{i-1} \cap a^n(y_i), \tag{5}$$

so that, if $i < j$, by (4) and 2.3.4-2.3.5:

$$\begin{aligned} \boldsymbol{n}(e_j \& a^r(f_i)) = x_j \cap (a^n(y_i) \cup x_{j-1}) \prec x_j \cap (x_i \cup a^n(y_{i-1}) \cup x_{j-1}) = \\ = x_j \cap (a^n(y_{i-1}) \cup x_{j-1}) = \boldsymbol{d}(e_j \& a^r(f_i)), \end{aligned} \tag{6}$$

and the projection $e_j \& a^r(f_i)$ is null (2.3.8).

Analogously, if $i > j$, by (5) and 2.3.4-2.3.5:

$$\begin{aligned} \boldsymbol{d}(e_j \& a^r(f_i)) = (x_j \cap a^n(y_{i-1})) \cup x_{j-1} \succ (x_j \cap x_{i-1} \cap a^n(y_i)) \cup x_{j-1} = \\ = (x_j \cap a^n(y_i)) \cup x_{j-1} = \boldsymbol{n}(e_j \& a^r(f_i)). \end{aligned} \tag{7}$$

4. - Regular induction

We consider here a notion of « regular induction » which agrees with composition and naturally appears in transformations of models in homological algebra. It should be noticed, however, that non-regular induction may compose with the regular one in regular and useful ways (e.g. in RO-inductive squares: 4.8, 5.2), so that we may not restrict our attention to the regular situation.

$\boldsymbol{A}$ is always an RO-category (2.1) and $\boldsymbol{B} = \mathrm{Fct}\, \boldsymbol{A}$ the associated factorizing RO-category (3.1).

4.1. *RO-squares*: In the theory of RO-categories the following square diagrams, called RO-squares in [G6], § 2.2, are of more interest than the commutative ones:

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow u & < & \downarrow v \\ A' & \xrightarrow{b} & B' \end{array} \tag{1}$$

(2) u and v are proper and $va < bu$ (or equivalently: $\hat{v}\tilde{a} < \bar{b}v$),

e.g. they appear in the definition of RO-transformations [G6] or, here, of regular induction. The RO-squares of $\boldsymbol{A}$ form a RO-category, w.r.t. the obvious vertical composition and involution.

4.2. Definition: We say that $a\colon A \to B$ *induces regularly* (R-induces for short) from $e \in \mathrm{Prj}\,(A)$ to $f \in \mathrm{Prj}\,(B)$ (the morphism $a' = fae\colon e \to f$ of $\boldsymbol{B}$ (see 3.2)) if:

$$ae < fa\,. \tag{1}$$

R-induction agrees with composition: if also $b\colon B \to C$ R-induces from f to $g \in \mathrm{Prj}\,(C)$ (the morphism $b' = gbf\colon f \to g$ of $\boldsymbol{B}$), then $ba\colon A \to C$ R-induces from e to g the composed $b'a'$ of the induced morphisms, since:

$$(ba)e < bfa < g(ba)\,, \tag{2}$$

$$g(ba)e = gb(ae)e < gb(fa)e = (gbf)(fae) = b'a'\,, \tag{3}$$

$$g(ba)e = g(gb)ae > g(bf)ae = (gbf)(fae) = b'a'\,. \tag{4}$$

Moreover, if a R-induces *also* from e' to f', it is easy to see that the same happens from $e \& e'$ to $f \& f'$.

4.3. *RO-induction*: Since we are mostly interested in the *proper* case, we say that $u\colon A \to B$ *RO-induces* from e to f (the morphism $fue\colon e \to f$ of $\boldsymbol{B}$) whenever u is proper and R-induces from e to f ($ue < fu$). This condition, which amounts to saying that a suitable square diagram is RO, will be characterized in 4.5 for categories of relations.

a) If $u\colon A \to B$ is proper, it RO-induces from e to f iff the following equivalent conditions hold:

$$u_P(e) < f\,, \tag{1}$$

$$e < u^P(f)\,; \tag{2}$$

indeed, if $ue < fu$ we have: $u_P(e) = ue\tilde{u} < fu\tilde{u} < f$; if (1) holds: $e < \tilde{u}u \cdot e \cdot \tilde{u}u = u^P u_P(e) < u^P(f)$; last, if (2) is satisfied: $ue < u\tilde{u}fu < fu$.

b) The morphism $fue\colon e \to f$ RO-induced by u is itself *proper* (in $\boldsymbol{B}$), as:

$$(fue)^{\sim} \cdot (fue) = e\tilde{u}fue = e\tilde{u}(fu)e > e\tilde{u}ue > e\,, \tag{3}$$

and analogously $(fue) \cdot (fue)^{\sim} < f$.

4.4. *RO-induction in RE-categories*: If $\boldsymbol{A}$ is an RE-category and $u\colon A \to B$ is proper, the following conditions are equivalent (by 2.5-2.6):

a) $u\colon A \to B$ RO-induces from e to f,

b) $u_R(\mathbf{n}e) \prec \mathbf{n}f$, $u_R(\mathbf{d}e) \prec \mathbf{d}f$,

c) $\mathbf{n}e \prec u^R(\mathbf{n}f)$, $\mathbf{d}e \prec u^R(\mathbf{d}f)$;

moreover the induced (proper) morphism is mono (resp. epi) iff *d*) (resp. *e*)) holds:

d) $\mathbf{d}e = \mathbf{n}e \cap u^*(\mathbf{d}f)$,

e) $\mathbf{n}f = \mathbf{d}f \cup u_*(\mathbf{n}e)$.

4.5. *RO-induction in categories of relations*: If $\boldsymbol{A} = \mathrm{Rel}\,\boldsymbol{E}$ is the category of relations over $\boldsymbol{E}$ (exact), $u\colon A \to B$ is proper and $s\colon H|K \to A$, $t\colon H'|K' \to B$ are subquotients of A and B (i.e., monorelations (0.2)), the following conditions are equivalent:

a) u *RO*-induces a (proper) morphism w from s to t,

b) $u(H) \prec H'$ and $u(K) \prec K'$ in $\mathrm{Sub}_{\boldsymbol{E}}(B)$,

c) there is a *RO*-square in $\boldsymbol{A}$:

(1)
$$\begin{tikzcd} A \arrow[r, "u"] & B \\ H|K \arrow[u, "s"] \arrow[r, "w"] \arrow[ur, phantom, "<"] & H'|K' \arrow[u, "t"'] \end{tikzcd} \qquad us < tw \;\; (w\tilde{s} < \tilde{t}u),$$

d) there is a commutative diagram of $\boldsymbol{E}$:

(2)
$$\begin{tikzcd} A \arrow[r, "u"] & B \\ H \arrow[u, tail, "m"] \arrow[r, "v"] \arrow[d, two heads, "p"] & K \arrow[u, tail, "n"'] \arrow[d, two heads, "q"'] \\ H|K \arrow[r, "w"] & H'|K' \end{tikzcd} \qquad s = m\tilde{p}, \; t = n\tilde{q}.$$

W.r.t. projections, the induced morphism from $e = s\tilde{s}$ to $f = t\tilde{t}$ is $tw\tilde{s}\colon e \to f$.

PROOF: *a*) $\Rightarrow$ *b*): by 4.4. *b*) $\Rightarrow$ *d*): by well-known properties of exact categories. *d*) $\Rightarrow$ *c*): the commutative squares of (2) are trivially *RO*-squares; by vertical involution and composition one gets (1). *c*) $\Rightarrow$ *a*): $ue = us\tilde{s} < tw\tilde{s} < t\tilde{t}u = ue$.

Finally, the proper morphism w in (1) and (2) is the induced one:

$$w = ws\tilde{s} < (\tilde{t}us) < \tilde{t}tw = w, \tag{3}$$

$$w = wp\tilde{p} = qv\tilde{p} = q\tilde{n}nv\tilde{p} = q\tilde{n}\cdot u\cdot m\tilde{p} = \tilde{t}us. \tag{4}$$

4.6. *RO-induction and P-regular unions of intersections*: Let $\boldsymbol{A}$ be an *RE*-category; if $e = \bigcup e_i$ and $f = \bigcup f_i$ are *P*-regular, respectively in $\mathrm{Prj}\,(A)$ and $\mathrm{Prj}\,(B)$ and $u\colon A \to B$ *RO*-induces from e_i to f_i, for all i, the same holds from e to f. Analogously for *P*-regular intersections of projections.

The proof is a straigthforward application of 4.4 and 2.5-2.6. E.g., in the case of unions:

(1) $$u_R(x) = u_R(\bigcup x_i) = \bigcup u_E(x_i) < \bigcup x'_i = x',$$

(2) $$u_R(y) = u_R(\bigcap y_i) = \bigcap u_R(y_i) < \bigcap y'_i = y'.$$

4.7. *RO-inductive squares*: Let $\boldsymbol{A}$ be an *RO*-category: the following is again a situation typically produced by transformations of models of homological theories.

The diagram of $\boldsymbol{A}$, equipped with the projections e, f, e', f':

(1) $$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow{\scriptstyle a} & < & \downarrow{\scriptstyle b} \\ A' & \xrightarrow{u'} & B' \end{array} \qquad \begin{array}{l} e \in \mathrm{Prj}(A),\ f \in \mathrm{Prj}(B), \\ e' \in \mathrm{Prj}(A'),\ f' \in \mathrm{Prj}(B'), \end{array}$$

will be said to be a *RO-inductive* square if:

(2) $u'a < bu$,

(3) u *RO*-induces from e to f,

(4) u' *RO*-induces from e' to f',

(5) $e'ae\colon e \to e$ and $f'bf\colon f \to f$ are proper in $\boldsymbol{B}$.

Then the morphisms induced by $bu\colon A \to B'$ and by $u'a\colon A \to B'$ (from e to f'), are proper and equal and coincide with the composition of the morphisms induced by u and b, as well as with the composition of the morphisms induced by a and u':

(6) $$f'(bu)e = f'(u'ae)e = (f'bf)(fue) = (f'u'e')(e'ae).$$

Indeed, by (2)-(4):

(7) $$(f'u'e')(e'ae) < f'u'ae < f'bue < (f'bf)(fue);$$

now, by 4.3 *b*) and (3)-(5), the first and the last term of (7) are proper morphisms (in $\boldsymbol{B}$), hence coincide (2.1).

5. - Convergence for spectral sequences

We give here an application, concerning the spectral sequence of the Z-filtered complex.

$\boldsymbol{E}$ is an exact category and $\boldsymbol{A} = \mathrm{Rel}\,\boldsymbol{E}$ its *RE*-category of relations. By the usual abuse of notation, if $H \succ K$ in the lattice Sub (A) of subobjects of A in $\mathbf{E}$, the subquotient H/K will stand also for the $\mathbf{A}$-subobject $s: H/K \to A$ described in 0.2, or equivalently for the projection $e = s\bar{s}: A \to A$.

5.1. *The spectral sequence of a filtered complex*: Let be given a $\mathbf{Z}$-filtered complex in the exact category $\mathbf{E}$:

$$A_* = ((A_n), (\partial_n), (F_p A_n)), \qquad n, p \in \mathbf{Z}, \tag{1}$$

where, for each n, $\partial_n: A_n \to A_{n-1}$ is a morphism of $\mathbf{E}$, $\partial_{n-1} \cdot \partial_n = 0$ and $(F_p A_n)$ is an increasing filtration of A_n, with the coherence condition: $\partial_*(F_p A_n) \prec F_p A_{n-1}$ (*).

Consider the following subquotients of A_n corresponding to the homology, the graduations associated to the filtrations and the terms of the spectral sequence (use 2.3.1, 2.3.4, 2.3.5):

$$H_n = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}, \tag{2}$$

$$G_{np} = F_p A_n / F_{p-1} A_n, \tag{3}$$

$$\begin{aligned} E^r_{np} &= \\ &= (F_p A_n \cap (\partial^* F_{p-r} A_{n-1} \cup F_{p-1} A_n)) / (F_p A_n \cap (\partial_* F_{p+r} A_{n+1} \cup F_{p-1} A_n)) = \\ &= G_{np} \,\&\, (\partial^* F_{p-r} A_{n-1}) / (\partial_* F_{p+r} A_{n+1}) \prec G_{np} \qquad (r > 0), \end{aligned} \tag{4}$$

$$\begin{aligned} E^\infty_{np} &= (F_p A_n \cap (\partial^* 0 \cup F_{p-1} A_n)) / (F_p A_n \cap (\partial_* A_{n+1} \cup F_{p-1} A_n)) = \\ &= G_{np} \,\&\, H_n \prec G_{np}, \end{aligned} \tag{5}$$

$${}'E^\infty_{np} = H_n \,\&\, G_{np} \prec H_n, \tag{6}$$

$${}'E^\infty_{np} \,\Phi\, E^\infty_{np}. \tag{7}$$

5.2. *Morphisms*: A morphism $u: A_* \to B_*$ of filtered complexes is a family of morphisms $u_n: A_n \to B_n$ in $\mathbf{E}$, commuting with the differentials and *RO*-inducing from $F_p A_n$ to $F_p B_n$:

$$u_*(F_p A_n) \prec F_p B_n. \tag{1}$$

It is easy to see that the morphisms u_n *RO*-induce w.r.t. the subquotients considered above in 5.1.2-5.1.6 (in particular, use the preservation of *RO*-induction by the &-product: 4.2). Moreover, since ${}'E^\infty_{np} \,\Phi\, E^\infty_{np}$ and because of the *RO*-inductive square lemma (4.7), the morphism $u: A_* \to B_*$ induces a mono, or an epi, or an iso from $E^\infty_{np}(A_*)$ to $E^\infty_{np}(B_*)$ iff it does from ${}'E^\infty_{np}(A_*)$ to ${}'E^\infty_{np}(B_*)$.

(*) ∂_* denotes the direct images of subobjects by ∂; ∂^* the counterimages of subobjects. The degree n is dropped when no confusion may arise.

5.3. Definition: Say that the filtered complex A_* *converges* in degree n if:

$$(1) \qquad \bigcap_{p\in\mathbf{Z}} F_pA_n = 0\,, \qquad \bigcup_{p\in\mathbf{Z}} F_pA_n = A_n\,, \qquad \text{distributively in } \mathrm{Sub}\,(A_n)\,.$$

5.4. Convergence theorem for filtered complexes:

a) If the filtered complex A_* converges in degree n, then:

$$(1) \qquad H_n = \bigcup_{p\in\mathbf{Z}}{}' E^\infty_{np} \qquad (PD\text{-regular, telescopic union in } \mathrm{Sub}_A\,(A_n))\,.$$

b) If A_* converges in degrees $n-1$ and $n+1$, then:

$$(2) \qquad E^\infty_{np} = \bigcap E^r_{np} \qquad (PD\text{-regular, decreasing intersection in } \mathrm{Sub}_A\,(A_n))\,.$$

Proof:

a) By 2.9 and the hypothesis, there is a PD-regular, telescopic union of subquotients of A_n:

$$(3) \qquad A_n = \bigcup_p (F_pA_n/F_{p-1}A_n) = \bigcup_p G_{np}\,,$$

which is preserved by applying $H_n\&-$.

b) By 2.5 and the hypothesis, there is a PD-regular, decreasing intersection of subquotients of A_n:

$$(4) \qquad H_n = \bigcap_r (\partial^* F_{p-r}A_{n-1})/(\partial_* F_{p+r}A_{n+1}))\,,$$

which gives the thesis, by applying $G_{np}\&-$.

5.5. Mapping lemma for filtered complexes: Let $u\colon A_* \to B_*$ be a morphism of filtered complexes. Let $r, n \in \mathbf{Z}$.

a) If, for all $p \in \mathbf{Z}$:

$$(1) \qquad E^r_{ip}(u)\colon E^r_{ip}(A_*) \to E^r_{ip}(B_*) \quad \text{is mono for } i = n-1 \text{ and epi for } i = n\,,$$

then the «same» holds for all $s > r$ and $p \in \mathbf{Z}$.

b) If, for all $p \in \mathbf{Z}$:

$$(2) \qquad E^r_{ip}(u)\colon E^r_{ip}(A_*) \to E^r_{ip}(B_*)$$

is mono for $i = n-1$, iso for $i = n$ and epi for $i = n+1$,

then the «same» holds for all $s > r$ and $p \in \mathbf{Z}$: in particular $E^\infty_{np}(u)$ is iso ($p \in \mathbf{Z}$).

Proof: We just have to prove a). Consider the differential of the spectral sequence of A_*:

$$\partial = \partial^s_{np}(A_*)\colon E^s_{np}(A_*) \to E^s_{n-1,p-s-1}(A_*) \qquad (s \geq r), \tag{3}$$

together with its « cycles » and « bords »:

$$Z^s_{np}(A_*) = \operatorname{Ker} \partial\,, \qquad B^s_{np}(A_*) = \operatorname{Im} \partial\,. \tag{4}$$

We prove now that, for $s \geq r$:

(5) $E^s_{np}(f)$ is epi and $E^s_{n-1,p}(f)$ is mono for all $p \in \mathbf{Z}$,

by induction on s. Since for $s = r$ this is the hypothesis (1), we assume it holds for some $s \geq r$ and check it for $s+1$.

The commutative diagram:

$$\tag{6} \begin{tikzcd} E^s_{np}(A_*) \arrow[r, two heads] \arrow[d, "E^s_{np}(f)"] & B^s_{np}(A_*) \arrow[r, tail] \arrow[d, "B^s_{np}(f)"] & E^s_{n-1,p-s-1}(A_*) \arrow[d, tail, "E^s_{n-1,p-s-1}(f)"] \\ E^s_{np}(B_*) \arrow[r, two heads] & B^s_{np}(B_*) \arrow[r, tail] & E^s_{n-1,p-s-1}(B_*)\,. \end{tikzcd}$$

shows that $B^s_{np}(f)$ is iso (for all $p \in \mathbf{Z}$). Thus, the commutative diagrams (with exact rows):

$$\tag{7} \begin{tikzcd} Z^s_{np}(A_*) \arrow[r, tail] \arrow[d] & E^s_{np}(A_*) \arrow[r, two heads] \arrow[d] & B^s_{np}(A_*) \arrow[d, tail] \\ Z^s_{np}(B_*) \arrow[r, tail] & E^s_{np}(B_*) \arrow[r, two heads] & B^s_{np}(B_*)\,, \end{tikzcd} \qquad \begin{tikzcd} Z^s_{np}(A_*) \arrow[r, two heads] \arrow[d] & E^{s+1}_{np}(A_*) \arrow[d] \\ Z^s_{np}(B_*) \arrow[r, two heads] & E^{s+1}_{np}(B_*) \end{tikzcd}$$

show, respectively, that $Z^s_{np}(f)$ is epi and $E^{s+1}_{np}(f)$ too. Dually, working on the right-hand side of (6), one proves that $E^{s+1}_{n-1,p-s-1}(f)$ is mono; by the arbitrariness of $p \in \mathbf{Z}$, the proof is complete.

5.6. Mapping theorem for filtered complexes: Let $u\colon A_* \to B_*$ be a morphism of filtered complexes and $r, n \in \mathbf{Z}$.

a) If A_* converges in degrees $n-1$ and $n+1$, B_* converges in degree n (5.3) and:

(1) $E^r_{n-1,p}(u)$ is mono and $E^r_{np}(u)$ is epi, for all $p \in \mathbf{Z}$,

then $H_n(u)\colon H_n(A_*) \to H_n(B_*)$ is epi, as well as $E^\infty_{np}(u)$, for all $p \in \mathbf{Z}$.

b) If A_* converges in degree n, B_* converges in degrees $n-1$ and $n+1$ and:

(2) $E^r_{np}(u)$ is mono and $E^r_{n+1,p}(u)$ is epi, for all $p \in \mathbf{Z}$,

then $H_n(u)\colon H_n(A_*) \to H_n(B_*)$ is mono, as well as $E^\infty_{np}(u)$, for all $p \in \mathbf{Z}$, *c*) If the filtered complexes A_* and B_* both converge in degrees $n-1$, n, $n+1$ and, for all $p \in \mathbf{Z}$:

$$(3) \qquad E^r_{ip}(u)\colon E^r_{ip}(A_*) \to E(B_*)$$

is mono for $i=n-1$, iso for $i=n$ and epi for $i=n+1$,

then $H_n(u)\colon H_n(A_*) \to H_n(B_*)$ is iso (as well as $E^\infty_{np}(u)$, for all $p \in \mathbf{Z}$).

Proof: We prove *a*), since *b*) is its dual and *c*) follows from both. By 5.5 *a*), $E^s_{np}(u)$ is epi for all $s>r$ and all $p \in \mathbf{Z}$. By 5.4 *b*), the subquotient $E^\infty_{np}(A_*)$ is the *PD*-regular intersection of $E^s_{np}(A_*)$, for $s>r$; by the mapping theorem for *D*-regular intersections (3.4 *b*)), $E^\infty_{np}(u)$ is epi for all $p \in \mathbf{Z}$. Finally, by 5.4 *a*), $H_n(B_*)$ is the telescopic, *PD*-regular union of the subquotients ${}'E^\infty_{np}(B_*)$, for $p \in \mathbf{Z}$; by the mapping theorem for *D*-regular unions (3.3 *b*)), $H_n(u)$ is epi.

6. - Appendix: the orthodox and distributive case

In order to clarify the behaviour of the relations $\sqsubset$ and Φ, we recall here briefly some results from papers on orthodox involutive categories and distributive exact categories ([G1, G2, G3, G4]).

6.1. *Orthodox and inverse RI-categories*: An *RI*-category $\boldsymbol{A}$ is said to be *orthodox* [G2] when its idempotent endomorphisms are stable for composition (*).

A category $\boldsymbol{K}$ is *inverse* ([G1]; [Sc]) whenever each morphism $a\colon A \to A'$ has a unique generalized inverse $\tilde{a}$ (*); then the mapping $a \mapsto \tilde{a}$ defines a regular involution in $\boldsymbol{K}$ (clearly the only one). It is not difficult to prove (extending a well known result for semigroups) that a category is inverse iff it is ϱ-regular (*) and its idempotents commute. iff it has a regular involution and its projections commute.

In an inverse category $\boldsymbol{K}$ all sets $\mathrm{Prj}\,(A)$ are semilattices, the meet being the product. The operation & and the relation $\sqsubset$ between projections coincide respectively with the product and the canonical order $\prec$.

6.2. *The canonical preorder*: An orthodox *RI*-category is provided with a *canonical preorder* (or *domination*) $a \sqsubset b$ on parallel morphisms, consistent with

(*) More generally, extending orthodox semigroups, a category is said to be *orthodox* [G1] if it is *ϱ-regular* (each morphism $a\colon A \to A'$ has some *generalized inverse* $a'\colon A' \to A$, with: $aa'a = a$ and $a'aa' = a'$) and its idempotent endomorphisms are stable for composition. Obviously, an *RI*-category is always ϱ-regular.

composition and involution, defined by the following equivalent properties [G2]:

(1) $\quad a = a\tilde{b}a\,,$

(2) $\quad a = a\tilde{a}\cdot b\cdot\tilde{a}a\,,$

(3) $\quad$ there exist idempotents e, f such that: $a = fbe\,,$

(4) $\quad$ there exist projections e, f such that: $a = fbe\,.$

This preorder extends the domination of projections, which is thus, in the orthodox case, a preorder.

The associated congruence Φ yields a quotient $\boldsymbol{A}/\Phi$ which is an inverse category, provided with its *canonical order* $\sqsubset$ (extending the well-known canonical order of inverse semigroups).

6.3. Theorem [G3]: The *RI*-category $\boldsymbol{A}$ is orthodox iff in $\boldsymbol{A}$ *induced isomorphisms are preserved by composition.* The last properties means that, if $a \in \boldsymbol{A}(A, B)$ induces an iso from $e \in \mathrm{Prj}\,(\boldsymbol{A})$ to $f \in \mathrm{Prj}\,(B)$ and $b \in \boldsymbol{A}(B, C)$ induces an iso from f to $g \in \mathrm{Prj}\,(C)$ then ba induces the composed iso from e to g: $(gbf)(fae) = g\cdot ba\cdot e$.

6.4. Theorem ([G4], § 3.17): Let $\boldsymbol{A} = \mathrm{Rel}\,\boldsymbol{E}$ be the category of relations over an exact category $\boldsymbol{E}$; the following conditions are equivalent:

a) $\boldsymbol{A}$ is orthodox;

b) for every object A, the modular lattice $\mathrm{Rst}\,(A)$ is distributive;

c) $\boldsymbol{E}$ is distributive (i.e. all its lattices of subobjects are so);

d) the relation of domination $\sqsubset$ is transitive (a preorder) on each set $\mathrm{Prj}\,(A)$;

e) the relation Φ is transitive (hence an equivalence) on each set $\mathrm{Prj}\,(A)$;

f) the operation & is associative on each set $\mathrm{Prj}\,(A)$;

g) for every $a\colon A \to B$ in $\boldsymbol{A}$, the mapping a_r preserves the operation &;

h) for every $a\colon A \to B$ in $\boldsymbol{A}$, the mapping a_g preserves binary meets and joins.

Analogous results ([G6]; § 7.4) hold more generally for any *RE*-category $\boldsymbol{A}$ and its associated componentwise exact category $\boldsymbol{E} = \mathrm{Prp\,Fct}\,\boldsymbol{A}$.

6.5. *Distributive RE-categories*: Thus an orthodox *RE*-category is also called a *distributive* one, when we want to stress the properties *b*) and *c*) of the previous characterization.

Examples of distributive exact categories: cyclic groups; sets and partial bijections; the distributive expansion Dst $\boldsymbol{E}$ of any exact category ([G6], § 7.10). Their categories of relations are distributive.

6.6. *Unions and intersections of projections in inverse categories*: Let $\boldsymbol{K}$ be an inverse category (with its unique *RI*-structure). Then:

a) every intersection of projections is *PD*-regular,

b) every union of projections is *D*-regular,

c) *P*-regular unions of projections coincide with *PD*-regular unions, as well as with *distributive* unions (w.r.t. the intersection, or product, in the semilattice $\mathrm{Prj}\,(A)$).

Indeed *a*) and *b*) follow from 1.7 and from the last remark in 6.1. As to *c*), *P*-regular unions are also *PD*-regular by *a*), and coincide with distributive ones by 1.9 (the condition 1.8.1 being trivially satisfied when all projections commute).

6.7. *Unions and intersections of projections in orthodox RI-categories*: Let $\boldsymbol{A}$ be an orthodox *RI*-category; write $\boldsymbol{K} = \boldsymbol{A}/\Phi$ the associated inverse category and $a \mapsto \bar{a}$ the quotient functor. Then:

a) if $e = \bigcap e_i$ in $\mathrm{Prj}_A(A)$, the intesection is *D*-regular iff $\bar{e} = \bigcap \bar{e}_i$ in $\mathrm{Prj}_K(A)$,

b) if $e = \bigcup e_i$ in $\mathrm{Prj}_A(A)$, the union is *D*-regular iff $\bar{e} = \bigcup \bar{e}_i$ in $\mathrm{Prj}_K(A)$,

c) if $e = \bigcup e_i$ is *PD*-regular in $\mathrm{Prj}_A(A)$, then $\bar{e} = \bigcup \bar{e}_i$ is distributive in $\mathrm{Prj}_B(A)$.

Indeed, *a*) and *b*) follow trivially from the fact that $e \,\mathrm{Cl}\, f$ in $\boldsymbol{A}$ iff $\bar{e} \prec \bar{f}$ in $\boldsymbol{B}$. As to *c*), it is easy to see that, if $e = \bigcup e_i$ is *PD*-regular in $\mathrm{Prj}_A(A)$, then $\bar{e} = \bigcup \bar{e}_i$ is *P*-regular in $\mathrm{Prj}_K(A)$, hence (6.6 *c*)) distributive.

6.8. *Remark*: Last we notice that the example considered in 2.11 (a *P*-regular, *D*-regular union $1 = e_1 \cup e_2$ which is not *PD*-regular) actually lives in the (orthodox) category of relations over the *distributive* exact category $\boldsymbol{E}$ of *cyclic* groups. Its image $1 = \bar{e}_1 \cup \bar{e}_2$ in $\mathrm{Rel}\,(\boldsymbol{E})/\Phi$ shows a (*D*-regular) union in an inverse category which is not *P*-regular.

REFERENCES

[C1] M. S. Calenko, *Correspondences over a quasi-exact category*, Dokl. Akad. Nauk SSSR, 155 (1964), pp. 292-294; Soviet Math. Dokl., 5 (1964), pp. 216-218.

[C2] M. S. Calenko, *Correspondences over a quasi-exact category*, Mat. Sbornik, 73 (1967), pp. 564-584; Math. USSR Sbornik, 2 (1967), pp. 501-519.

[De] R. Deheuvels, *Topologie d'une fonctionnelle*, Ann. Math., 61 (1955), pp. 13-72.
[Ei] S. Eilenberg, *La suite spectrale. - I: Construction générale*, Sém. Cartan, 1950-51, exp. 8.
[EM] S. Eilenberg - J.C. Moore, *Limits and spectral sequences*, Topology, 1 (1961), pp. 1-23.
[G1] M. Grandis, *Canonical preorder in orthodox semigroups and categories (Orthodox Categories 1)*, Boll. Un. Mat. Ital., 13-B (1976), pp. 634-650.
[G2] M. Grandis, *Regular and orthodox involution categories (Orthodox Categories 2)*, Boll. Un. Mat. Ital., 14-A (1977), pp. 38-48.
[G3] M. Grandis, *Induction in orthodox involution categories (Orthodox Categories 3)*, Ann. Mat. Pura Appl., 116 (1978), pp. 87-99.
[G4] M. Grandis, *Exact categories and orthodox symmetrizations (Orthodox Symmetrizations 3)*, Boll. Un. Mat. Ital., 16-B (1979), 178-195.
[G5] M. Grandis, *Transfer functors and projective spaces*, Math. Nachr., 118 (1984), pp. 147-165.
[G6] M. Grandis, *On distributive homological algebra. - I: RE-categories*, Cahiers Top. Géom. Diff. Catég., 25 (1984), pp. 259-301.
[G7] M. Grandis, *On distributive homological algebra. - II: Theories and models*, Cahiers Top. Géom. Diff. Catég., 25 (1984), pp. 353-379.
[G8] M. Grandis, *On distributive homological algebra. - III: Homological theories*, Cahiers Top. Géom. Diff. Catég., 26 (1985), pp. 169-213.
[G9] M. Grandis, *On distributive homological algebra. - IV: The exact system and the exact couple*, to appear.
[HW] P. J. Hilton - S. Wylie, *Homology Theory*, Cambridge University Press, 1962.
[Ma] S. Mac Lane, *Homology*, Springer, 1963.
[Mi] B. Mitchell, *Theory of Categories*, Academic Press, 1965.
[Ms] W. S. Massey, *Exact couples in algebraic topology*, Ann. Math., 56 (1952), pp. 363-396.
[Pu] D. Puppe, *Korrespondenzen in abelschen Kategorien*, Math. Annalen, 148 (1962), pp. 1-30.
[Sc] E. Schwab, *Über das verallgemeinerte Inverse eines Morphismus*, Math. Nachr., 81 (1978), pp. 243-257.
[Ze] E. C. Zeeman, *On the filtered differential group*, Ann. Math., 66 (1957), pp. 557-585.

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