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A Nearness Approach to Extension Problems (***) (****)

SOMMARIO. — Usando la nozione di *nearness* si dà una condizione sufficiente perché una funzione continua tra spazi di chiusura di Čech abbia una estensione continua da sottospazi densi senza richiedere la regolarità del codominio. Da questo risultato si deducono i risultati di Rudolf concernenti le estensioni θ-continue di funzioni θ-continue e quelli di Herrmann concernenti gli spazi di convergenza.

Approccio a problemi di estensione mediante la nozione di *Nearness*

ABSTRACT. — By using a nearness approach we give a sufficient condition for a continuous function between Čech closure spaces to have a continuous extension from dense subspaces without requiring the range space to be regular. From this result we deduce the results of Rudolf concerning θ-continuous extensions of θ-continuous functions and of Herrmann concerning convergence spaces.

1. - INTRODUCTION

Suppose X is dense in αX and Y is dense in λY. One of the important problems in Topology/Analysis is to find necessary and/or sufficient conditions for a continuous function $f: X \to Y$ to have a continuous extension $\bar{f}: \alpha X \to \lambda Y$. One of the earliest of such results, which is quite well known, was proved by A. D. Taimanov who considered the special case $\lambda Y = Y$, a compact Hausdorff space. This result of Taimanov can be expressed neatly in the language of proximity spaces. Suppose the compact Hausdorff space Y is assigned the unique compatible EF-proximity δ_0 and X is assigned the

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LO-proximity δ_1 defined by

$$A\delta_1 B \text{ in } X \text{ iff } \mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset \text{ where Cl denotes closure in } \alpha X.$$

(1.1) Taimanov Theorem: A function $f\colon X \to Y$ has a continuous extension $\bar{f}\colon \alpha X \to Y$ if and only if is proximally continuous.

The Taimanov Theorem has been extended by many mathematicians: Engelking [3] to realcompact λY, Herrlich [7] to regular Y, Gagrat and Naimpally [4] to Tychonoff Y with λY, the Smirnov compactification of Y, Naimpally [9] to regular λY using nearness. In other directions Veličko [12], Rudolf [11] extendend the results to θ-continuous functions and Hermann [6] to convergence spaces.

Rudolf [11] has shown how it is necessary, in many cases to replace continuous functions by θ-continuous functions and has obtained results without requiring Y to be regular. However, θ-continuous functions are continuous with respect to the Čech closure spaces associated with the given topological spaces, the association being generated by the θ-closure operator. We propose to unify these two approaches viz. θ-continuous functions and convergence spaces by studying extensions of continuous functions via nearness without requiring Y (or λY) to be regular.

This paper has six sections. In Section 2 we give preliminaries on Čech closure spaces, pretopological convergence spaces and nearness. In Section 3, we exhibit an algorithm which connects the study of θ-continuous functions to one of continuous functions on Čech closure spaces. It also works for pretopological convergence spaces. We consider only the pretological convergence spaces satisfying the following property:

(1.2) $X^- = \alpha X$ and for each nbhd. U of x in αX, there is a nbhd. V of x such that $V^- \subset (U \cap X)^-$.

In Section 4 we prove a very general result on the extensions of continuous functions between Čech closure spaces. Furthermore, we give a sufficient condition for a near map between pretopological T_1 convergence spaces satisfying (1.2) to have a continuous extension from dense subspaces. In Section 5 we show that $f\colon X \to Y$ is a near map if and only if f is weakly admissible in the sense of Hermann [6]. In Section 6 we prove how our result includes Rudolf's results [11] on θ-continuous functions. Finally, when all spaces are topological, αX is regular and $\lambda Y = Y$ is compact Hausdorff, our result is equivalent to the original Taimanov Theorem (1.1).

2. - Preliminaries

(2.1) A *Čech closure operator* $^-$ on the power set of X satisfies

(*a*) $\emptyset^- = \emptyset$,

(*b*) $A \subset A^-$,

(*c*) $(A \cup B)^- = A^- \cup B^-$.

If in addition

(*d*) $(A^-)^- = A^-$,

then $^-$ is the *Kuratowski closure operator*. In this case X is a topological space.

All spaces in this paper are Čech closure spaces unless otherwise stated.

Suppose X is dense in αX. We define a nearness η_0 in αX by

(2.2) $$\eta_0 \mathcal{A} \quad \text{iff} \cap \{A^-: A \in \mathcal{A}\} \neq \emptyset .$$

Clearly η_0 induces a nearness η_1 on X which is a *Čech nearness* associated with $^-$ and satisfies:

(2.3) (*a*) $\cap \mathcal{A} \neq \emptyset$ implies $\eta_1 \mathcal{A}$,

(*b*) $\bar{\eta}_1 \mathcal{A}$ and $\bar{\eta}_1 \mathcal{B}$ implies $\bar{\eta}_1(\mathcal{A} \vee \mathcal{B})$,
where $\mathcal{A} \vee \mathcal{B} = \{A \cup B: A \in \mathcal{A}, B \in \mathcal{B}\}$,

(*c*) $\eta_1 \mathcal{A}$ and for each $B \in \mathcal{B}$, there is an $A \in \mathcal{A}$ such that $A \subset B$, then $\eta_1 \mathcal{B}$,

(*d*) $\emptyset \in \mathcal{A}$ implies $\bar{\eta}_1 \mathcal{A}$.

Note that if η_1 is *associated* with $^-$, then x is in A^- implies $x \eta_1 A$ but the converse need not hold.

If $^-$ is a Kuratowski closure operator, then

(*e*) $\eta_1 \mathcal{A}$ iff $\eta_1 \mathcal{A}^-$ where $\mathcal{A}^- = \{A^-: A \in \mathcal{A}\}$.

This is called *LO-nearness*.

Further if X is an R_0-space

(*f*) $x \in y^-$ iff $y \in x^-$,

then

(*g*) $x \eta_1 A$ iff $x \in A^-$,

and we say that $^-$ is *compatible* with η_1.

Suppose Y is dense in λY and Y is assigned the Čech nearness η_2 induced by η_0 on λY.

(2.4) Definition: A function $f: X \to Y$ is called

(*a*) *continuous* iff $x \in A^-$ implies $f(x) \in f(A)^-$

OR $f(A^-) \subset f(A)^-$,

(*b*) *a near map* iff $\eta_1 \mathcal{A}$ implies $\eta_2 f(\mathcal{A})$.

It is easy to show that if f is a near map and λY is an R_0-topological space, then f is continuous.

Let X be a nonempty set. *A convergence structure* lim on X is a function from the set of all filters on X into $\mathcal{P}(X)$, the power set of X. If $x \in \lim \mathcal{F}$, we say that « $\mathcal{F}$ converges to x » and write $\mathcal{F} \to x$. For each x in X we define the *nbhd. filter* at x by

$$\mathcal{N}(x) = \cap \{\mathcal{G} : \mathcal{G} \to x\} .$$

(2.5) A convergence space $(X, \lim)$ is *pretopological* iff

(*a*) for each x in X, the filter generated by $\{x\}$ converges to x,

(*b*) if $\mathcal{F} \to x$ and $\mathcal{F} \subset \mathcal{F}'$, then $\mathcal{F}' \to x$,

(*c*) if $\mathcal{F} \not\to x$, then there exists $\mathcal{F}' \supset \mathcal{F}$ such that if $\mathcal{F}'' \supset \mathcal{F}'$, then $\mathcal{F}'' \not\to x$,

(*d*) for each x in X, $\mathcal{N}(x) \to x$.

Let X and Y be convergence spaces. If $\mathcal{F}$ is a filter on X, we denote by $f(\mathcal{F}) = \{E \subset Y : f^{-1}(E) \in \mathcal{F}\}$, which is a filter on Y. A function $f : X \to Y$ is continuous if and only if for each x in X and each filter $\mathcal{F}$ on X, $\mathcal{F} \to x$ implies $f(\mathcal{F}) \to f(x)$ in Y.

To each Čech closure operator $^-$ we can associate a pretopological convergence structure as follows: $U \subset X$ is a nbhd. of x in X iff $x \notin (X - U)^-$ and $\mathcal{F} \to x$ iff $\mathcal{F}$ contains all the nbhds. of x. Conversely, to each pretopological space X, we can associate a Čech closure operator $^-$ on X: $x \in A^-$ iff there is a filter $\mathcal{F}$ on X containing A and $\mathcal{F} \to x$.

A filter $\mathcal{F}$ *clusters to* x in X iff $x \in \cap \{F^- : F \in \mathcal{F}\}$. Evidently, if $\mathcal{F} \to x$, then $\mathcal{F}$ clusters to x but the converse does not hold.

3. - θ-CLOSURE

Now we exhibit an algorithm which converts the study of θ-continuous functions into a study of continuous function on Čech closure spaces or equivalently on pretopological convergence spaces.

Let X and Y be topological spaces. A function $f : X \to Y$ is *θ-continuous* iff for each x in X and for each nbhd. V of $f(x)$, there is a nbhd. U of x such that $f(U^-) \subset V^-$. Suppose for each subset A of X we define the *θ-closure* of A as

$$\mathrm{Cl}_\theta A = \{x \in X : U^- \cap A \neq \emptyset \text{ for each nbhd. } U \text{ of } x\} .$$

It is easily seen that Cl_θ is a Čech closure operator on X. The resulting Čech closure space is denoted by θX and is called the θ-closure space associated with the topological space X.

It is easy to see that $f : X \to Y$ is θ-continuous iff $f(\mathrm{Cl}_\theta A) \subset \mathrm{Cl}_\theta f(A)$ for each $A \subset X$. So from (2.4)(*a*) we have:

(3.1) THEOREM: Let X, Y be topological spaces and θX, θY the respective θ-closure spaces. Then a function $f\colon X \to Y$ is θ-continuous if and only if $f\colon \theta X \to \theta Y$ is continuous.

We now recall a few results which either follow easily from the definitions or are implicit in Veličko [13] and Hamlett [5].

(3.2) LEMMA: X is g-H-closed if and only if θX is compact.

(3.3) LEMMA: X is Hausdorff if and only if θX is T_1.

(3.4) LEMMA: X is Urysohn if and only if θX is Hausdorff.

(3.5) COROLLARY: X is H-cosed if and only is θX is compact T_1.

(3.6) COROLLARY: X is θ-compact i.e. H-closed and Urysohn if and only if θX is compact Hausdorff.

Let $\mathcal{F}$ be a filter on X. We say that $\mathcal{F}$ *θ-converges to* x and write $\mathcal{F} \xrightarrow{\theta} x$ iff $\mathcal{F}$ contains the closed nbhds. of x. The following results are immediate:

(3.7) THEOREM: The θ-convergence for filters on X is a pretopological convergence on X.

(3.8) THEOREM: The θ-convergence for filters on X is the natural convergence associated with the Čech closure space θX and vice versa.

(3.9) THEOREM: Closed nbhds. of x generate the θ-convergence nbhd. filter at x.

(3.10) COROLLARY: $f\colon X \to Y$ is θ-continuous iff whenever $\mathcal{F} \xrightarrow{\theta} x$ in X, $f(\mathcal{F}) \xrightarrow{\theta} f(x)$ in Y.

From now on we'll consider a pretopological convergence space satisfying (1.2). In general a pretopological space need not satisfy (1.2), for example: the Ferón Cross pretopology on $\mathbb{R}^2$ and $X = \mathbb{R} \times \mathbb{Q}$ or $\mathbb{Q} \times \mathbb{R}$. Obviously, (1.2) is satisfied by the Kuratowski closure (the topological case) as well as the θ-closure.

Let αX be a topological space and $U(x)$ be the nbhd. filter at x in αX. Consider a closure operator Cl_* on αX such that $\mathrm{Cl}_* A \subset A^-$ for each $A \subset \alpha X$ and the resulting pretopological convergence structure $*$ on αX whose nbhd. filter at x is generated by $\{\mathrm{Cl}_* U\colon U \in U(x)\}$. Then we have the following:

(3.11) THEOREM: $(\alpha X, *)$ satisfies (1.2).

PROOF: Let X be dense in $(\alpha X, *)$ and U be an open nbhd. in $U(x)$. Then $\mathrm{Cl}_* U \subset (\mathrm{Cl}_* U \cap X)^*$. In fact, $y \in \mathrm{Cl}_* U \Rightarrow y \in U^- \Rightarrow U \cap V \neq \emptyset$ for each $V \in U(y) \Rightarrow \mathrm{Cl}_* (U \cap V) \cap X \neq \emptyset$ since X is $*$-dense $\Rightarrow \mathrm{Cl}_* V \cap (\mathrm{Cl}_* U \cap X) \neq \emptyset$ for each $V \in U(y) \Rightarrow y \in (\mathrm{Cl}_* U \cap X)^*$.

We obtain the θ-case when $\mathrm{Cl}_* = {}^-$.

4. - A general result

In this section we prove a rather general result on the extensions of continuous functions. The necessity follows easily whereas the sufficiency requires some additional conditions on f and/or on λY.

(4.1) Theorem (Necessity): If $f: X \to Y$ has a continuous extension $\bar{f}: \alpha X \to \lambda Y$, then f is a near map.

Proof: Suppose $\bar{f}$ is a continuous extension of f and $\eta_1 \mathcal{A}$. Then

$$\cap \mathcal{A}^- \neq 0 \Rightarrow \cap \{\bar{f}(A^-) : A \in \mathcal{A}\} \neq 0$$

in $\lambda Y \Rightarrow \cap \{\bar{f}(A)^- : A \in \mathcal{A}\} \neq 0$ in $\lambda Y \Rightarrow \cap \{f(A)^- : A \in \mathcal{A}\} \neq 0$ in $\lambda Y \Rightarrow \eta_2 f(\mathcal{A})$. Thus f is a near map.

We now pursue *sufficiency* and in view of (4.1), we suppose that f is a near map. If $x \in X$, then $\bar{f}(x) = f(x) \in Y$. On the other hand if $x \in \alpha X - X$, consider

$$\sigma^x = \{E \subset X : x \in E^-\}.$$

(4.2) σ^x is a *clan* i.e.

(*a*) σ^x is a grill i.e. a union of ultrafilters, and

(*b*) $\eta_1 \sigma^x$.

Since f is a near map,

(4.3) $$f^{\#}(\sigma^x) = \{E \subset Y : f^{-1}(E) \in \sigma^x\} \quad \text{is a clan in } (Y, \eta_2).$$

Hence

(4.4) $$\cap f^{\#}(\sigma^x)^- \neq 0 \quad \text{in } \lambda Y.$$

So a candidate for $\bar{f}(x)$ is any point in the set (4.4) i.e.

(4.5) $$\bar{f}(x) \in \cap f^{\#}(\sigma^x)^-.$$

Now we state a simple but essential property of convergence spaces whose proof is straight forward.

(4.6) Lemma: For each x in αX and y in λY,

$$f^{-1}(N(y) \cap Y) \subset \sigma^x \quad \text{iff } f(N(x) \cap X) \subset \sigma^y.$$

Note that in (4.5), $\bar{f}(x)$ need not be unique unless λY is Hausdorff and even then $\bar{f}$ need not be continuous, unless λY is regular. See Herrlich [7]

for an example. However, $\tilde{f}$ in this case (i.e. λY Hausdorff but not regular) is θ-continuous [9].

We now consider several situations in which the function $\tilde{f}$ defined by (4.5) is continuous. Naturally, we are forced to put extra conditions on f since f being a near map is just not enough. Suppose λY is T_2 so that $\tilde{f}(x)$ is uniquely defined for each x in αX when f is a near map. Set

$$\mathcal{N}(\lambda Y) = \{E \subset \lambda Y : E \in \mathcal{N}(y) \text{ for some } y \text{ in } \lambda Y\}.$$

In order that the function $\tilde{f}$, as defined by (4.5), is continuous the following condition must be satisfied:

(4.7) For each x in αX, for each y in λY, for each $A \in \mathcal{N}(\lambda Y)$ and $A \perp \mathcal{N}(y)$, there exists a nbhd. Uy of y in λY such that $f^{-1}(A \cap Y)$ and $f^{-1}(Uy \cap Y)$ do not both belong to σ^*.

(4.8) THEOREM: If f is a near map, then for each x in $\alpha X - X$, $f(\mathcal{N}(x) \cap X)$ clusters in λY. Furthermore, if λY is T_2 and satisfies (4.7), then $f(\mathcal{N}(x) \cap X)$ has a unique cluster point y in λY. Moreover, $f(\mathcal{N}(x) \cap X) \to y$ in λY.

PROOF: Since $x \in (U \cap X)^-$ for each $U \in \mathcal{N}(x)$ and f is a near map,

$$\cap\{f(U \cap X)^- : U \in \mathcal{N}(x)\} \neq \emptyset \qquad \text{for each } x \in \alpha X - X.$$

Since λY is T_2, the above intersection is a singleton. For if y_1, y_2 are in $\cap\{f(U \cap X)^- : U \in \mathcal{N}(x)\}$, then there is a nbhd. U_2 of y_2 such that $U_2 \perp \mathcal{N}(y_1)$. By (4.7), there is a nbhd. U_1 of y_1 such that $x \notin f^{-1}(U_1 \cap Y)^- \cap f^{-1}(U_2 \cap Y)$ which contradicts Lemma (4.6).

Finally, we prove that

$$f(\mathcal{N}(x) \cap X) \to y = \cap\{f(U \cap X)^- : U \in \mathcal{N}(x)\}.$$

If this is not true, then there exists a nbhd. W of y such that

$$f(U \cap X) \cap (Y - W) \neq \emptyset \qquad \text{for each } U \in \mathcal{N}(x).$$

Since $y \notin (Y - W)^-$, we have

$$\cap\{f(U \cap X)^- : U \in \mathcal{N}(x)\} \cap (Y - W)^- = \emptyset.$$

Set

$$U' = f^{-1}\{(Y - W) \cap f(U \cap X)\}.$$

Since f is a near map, $\cap\{U'^- : U \in \mathcal{N}(x)\} = \emptyset$ and so there exist nbhds. U, V of x such that $V \cap U' = \emptyset$. This shows that $f(V \cap U \cap X) \subset W$, a contradiction.

We now prove our main result:

(4.9) Main theorem: Suppose λY is T_2 and $f: X \to Y$ be a near map satisfying (4.7). Then there exists a unique continuous extension $\bar{f}: \alpha X \to \lambda Y$.

Proof: The existence and uniqueness of $\bar{f}$ follows immediately from the preceeding theorem by setting

$$\bar{f}(x) = f(x) \quad \text{for } x \in X,$$
$$= \lim f(\mathcal{N}(x) \cap X) \quad \text{for } x \in \alpha X - X.$$

To prove the continuity of $\bar{f}$ we first note:

$$\text{(4.10)} \qquad \text{for each } U \in \mathcal{N}(\lambda Y), \qquad \bar{f}[f^{-1}(U \cap Y)^-] \subset U.$$

In fact, if $y = \bar{f}(x) \in U$, then by (4.7), $x \notin f^{-1}(U \cap X)^-$.

If $y = f(x)$ for $x \in X$ and $U_y \in \mathcal{N}(y)$, then by the continuity of f and (1.2) there exist U_x, V_x in $\mathcal{N}(x)$ such that

$$V_x \subset (U_x \cap X)^- \subset f^{-1}(U_y \cap Y)^-.$$

So by (4.10), $\bar{f}(V_x^-) \subset U_y$.

Next suppose $x \in \alpha X - X$, $y = \bar{f}(x)$ and $U_y \in \mathcal{N}(y)$. Since λY is Hausdorff, there is a nbhd. V_y of y such that $V_y \subset U_y$ and $\lambda Y - V_y \in \mathcal{N}(\lambda Y)$. By (4.7), $x \notin f^{-1}[(\lambda Y - V_y) \cap Y]^-$. So there is a U_x in $\mathcal{N}(x)$ and by (1.2) and (4.10) a V_x in $\mathcal{N}(x)$ such that

$$U_x \cap X \cap f^{-1}[(\lambda Y - V_y) \cap Y] = \emptyset,$$
$$V_x \subset (U_x \cap X)^- \subset f^{-1}(V_y \cap Y)^- \quad \text{and} \quad \bar{f}(V_x) \subset V_y \subset U_y.$$

5. - Convergence spaces

In this section we show how Hermann's results [6] on extensions of continuous functions on convergence spaces follow from our results. In Hermann's case $\lambda Y = Y$.

(5.1) Definition: $f: X \to Y$ is *weakly admissible* iff for each $\xi \in \alpha X - X$,

$$\cap \{f(U)^-: U \in \mathcal{U}, U \in \mathcal{U}_\alpha(\alpha X, \xi)\} \neq \emptyset$$

where $U = U' \cap X$ with U' an ultrafilter on αX which intersects X and converges to ξ in αX. Evidently, if Y is T_2, then the intersection is a singleton.

Suppose f is weakly admissible. Observe that $f^{\prime}(\sigma^{x})$ is a union of ultrafilters generated by the images of the traces on X of the ultrafilters U in $U_x(\alpha X, \xi)$. It follows that if f is weakly admissible, then $\cap f^{\prime}(\sigma^{x})^{-} \neq 0$.

(5.2) THEOREM: $f\colon X \to Y$ is weakly admissible if and only if is a near map.

PROOF: Suppose f is a near map and consider $U^{\prime} \to \xi$ in αX and $X \in U^{\prime}$. Then $U = U^{\prime} \cap X \subset \sigma^{x}$ and $f(U) \subset f(\sigma^{x}) \subset f^{\prime}(\sigma^{x})$. Since $\eta_2 f^{\prime}(\sigma^{x})$, we have $0 \neq \cap f^{\prime}(\sigma^{x})^{-} \subset \cap \{f(U)^{-}\colon U \in U_x(\alpha X, \xi)\}$.

Conversely, if f is weakly admissible and $\eta_1 \mathcal{A}$, then $\mathcal{A} \in \sigma^{x}$ for some ξ in $\alpha X \Rightarrow f(\mathcal{A}) \in f^{\prime}(\sigma^{x}) \Rightarrow 0 \neq \cap f^{\prime}(\sigma^{x}) \subset \cap \{f(A)^{-}\colon A \in \mathcal{A}\} \Rightarrow \eta_2 f(\mathcal{A})$.

6. - RUDOLF'S RESULTS

We now show how Rudolf's results [11] on extensions of θ-continuous functions follow from ours. In Rudolf's case $\lambda Y = Y$ and he considers θ-continuous functions between Hausdorff spaces. Rudolf's problem is:

(6.1) Given a θ-continuous function $f\colon X \to Y$, where X and Y are Hausdorff, find necessary and/or sufficient conditions that f has a θ-continuous extension $\bar{f}\colon \alpha X \to Y$.

With each topological space X, we associate the Čech closure space θX (which is T_1 if X is T_2 and compact if X is H-closed). Moreover, $f\colon X \to Y$ is θ-continuous iff $f\colon \theta X \to \theta Y$ is continuous. Thus Rudolf's problem (6.1) reduces to:

(6.2) Given T_1-Čech closure spaces X, Y and a continuous function $f\colon X \to Y$ find necessary and/or sufficient conditions for f to have a continuous extension $\bar{f}\colon \alpha X \to Y$.

Rudolf's « α^{*}-proper » condition ([11], p. 177) is a sufficient condition for a θ-continuous function to have a θ-continuous extension. In our terminology, it becomes « αX-proper ».

(6.3) DEFINITION (Rudolf [11]): A θ-continuous function $f\colon X \to Y$ is *αX-proper* iff

(*a*) *proper*: suppose

$$\mathfrak{U}(f, \mathcal{N}(x)) = \\ = \{U\colon U \text{ is open in } Y \text{ and } f(U_x \cap X) \subset U \text{ for some } U_x \text{ in } \mathcal{N}(x)\}.$$

Then for each x in $\alpha X - X$, $\cap\{U^{-}\colon U \in \mathfrak{U}(f, \mathcal{N}(x))\} \neq 0$ i.e. $\mathfrak{U}(f, \mathcal{N}(x))$ clusters in Y, and

(*b*) *αX-free*: for each x in $\alpha X - X$, for each y in Y and each regularly closed set $A \subset Y$ such that $y \notin A$, there is an open nbhd. U_y of y such that $x \notin f^{-1}(U_y^{-}) \cap f^{-1}(A)^{-}$.

Immediately we have:

(6.4) THEOREM: A θ-continuous function $f: X \to Y$ is proper if and only if $f(\mathcal{N}(x) \cap X)$ θ-clusters in Y for each x in $\alpha X - X$.

PROOF: Suppose f is proper, $x \in \alpha X - X$ and $y \in \{U^-: U$ open in Y and $U \supset f(U_\alpha \cap X)$ for some open $U_\alpha \in \mathcal{N}(x)\}$. If $f(\mathcal{N}(x) \cap X)$ does not θ-cluster to y, then there is an open nbhd. U_y of y such that $U_y^- \cap f(U_\alpha \cap X) = \emptyset$ for some U_α in $\mathcal{N}(x)$. This means $y \in (Y - U_y^-)^-$—a contradiction.

In the other direction, suppose x is in $\alpha X - X$ and $\cap \{U^-: U$ open in Y and $U \supset f(U_\alpha \cap X)$ for some U_α in $\mathcal{N}(x)\} = \emptyset$. Then there exists an open set U in Y and an open nbhd. U_y of y such that $U \supset f(U_\alpha \cap X)$ for some U_α in $\mathcal{N}(x)$ and $U^- \cap f(U_\alpha \cap X) = \emptyset$, a contradiction.

Furthermore we also have:

(6.5) THEOREM: Suppose $f: X \to Y$ is θ-continuous (i.e. $f: \theta X \to \theta Y$ is continuous) and $f: \theta X \to \theta Y$ is a near map (equivalently, weakly admissible), then f is proper.

PROOF: Since $x \in \cap \{\mathrm{Cl}_\theta(U_\alpha \cap X): U_\alpha \in \mathcal{N}(x)\}$, $\eta_1(\mathcal{N}(x) \cap X)$. Since f is a near map, $\eta_2 f(\mathcal{N}(x) \cap X)$ and this is equivalent to

$$\cap \{\mathrm{Cl}_\theta f(U_\alpha \cap X): U_\alpha \in \mathcal{N}(x)\} \neq \emptyset$$

which is (6.3)(a).

COMPARISONS

First we observe that each regularly closed set A is a θ-nbhd. of each of its interior points and basic θ-nbhds. of y in Y are closures of open nbhds. of y in Y.

If f has an extension $\bar{f}$ and f is αX-free, then

(6.6) $\qquad \bar{f}[f^{-1}(U^-)^-] \subset U^- \qquad$ for each open set U of Y.

For the θ-continuity of $\bar{f}$, (6.4) plays the same role as (4.10) plays for the continuity if $\bar{f}$ in our Main Theorem (4.9). If f satisfies (4.10) i.e.

$$\bar{f}[\mathrm{Cl}_\theta f^{-1}(U^-)] \subset U^-$$

for each open set U in Y, then it also satisfies (6.6).

Our condition (4.7) is slightly different from αX-freeness—in part stronger and in part weaker. This is obvious because, in the general case, we don't have a subjacent topological space as in the θ-continuous case.

To summarize: if X is dense in αX as in [11], then X is dense in $\theta(\alpha X)$.

If a map is θ-continuous and near, then it is proper in the sense of (6.3)(a). If a θ-continuous map satisfies (4.10), then it also satisfies (6.6).

Rudolf [11] considers T_1 θ-closure spaces whereas we consider T_2 convergence spaces. But our condition (4.7) is weaker than Rudolf's αX-freeness (6.3)(b).

We conclude by considering the case where all spaces are topological.

(6.7) THEOREM: Suppose αX is regular, $\lambda Y = Y$ is compact Hausdorff, $X^{-} = \alpha X$ and $f\colon X \to Y$ is continuous. Then f has a continuous extension $\bar{f}\colon \alpha X \to Y$ if and only if f satisfies (4.7).

PROOF: Since αX is regular, θ-closure in αX coincides with the closure in αX. So αX-freeness is equivalent to (4.7). But when Y is compact Hausdorff, αX-freeness is equivalent to the Taimanov condition (Rudolf [11], p. 174).

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