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Images of Open Sets Under Certain Multifunctions (**)

Immagini di insiemi aperti mediante certe multifunzioni

SUNTO. — Dati uno spazio topologico X, uno spazio metrico (Y, d) ed una multifunzione F da X in Y a valori non vuoti e siffatta, che per ogni fissato $y \in Y$ la funzione reale $d(y, F(\cdot))$ sia priva di punti di minimo relativo i quali non siano anche di minimo assoluto, in questo lavoro sarà posta in luce una proprietà goduta dalle immagini mediante F dei sottoinsiemi aperti di X.

1. - INTRODUCTION

Here and in the sequel, X is a topological space, (Y, d) is a metric space (if Y is a normed space, d is the metric induced by the norm) and F is a multifunction from X onto Y, with non-empty values.

For every $y \in Y$ and $r > 0$, put $B(y, r) = \{z \in Y: d(y, z) < r\}$ and $S(y, r) = \{z \in Y: d(y, z) = r\}$. Denote by $\mathcal{T}$ the family of all sets $U \subseteq Y$ such that for every $y \in \partial U$ there exists $z \in Y \setminus \{y\}$ such that $d(z, y) = d(z, U)$, where $d(z, U) = \inf\{d(z, u): u \in U\}$. Given a function $f: Y \to]0, +\infty[$, denote by $\mathcal{V}_f$ (resp. $\tilde{\mathcal{V}}_f$) the family of all sets $V \subseteq Y$ such that the set $A_{f,V} = \{z \in Y: \exists y \in V: d(y, z) < f(z)\}$ (resp. $\tilde{A}_{f,V} = \{z \in Y: \exists y \in V: d(y, z) = f(z)\}$) is open.

The present paper can be regarded as a motivation, as well as an invitation, for the study of two problems that, in a generic way, can be stated as follows: If Ω is an open subset of X, when $F(\Omega) \in \mathcal{T}$? If V is a non-empty subset of Y such that $V \in \mathcal{V}_f$ (resp. $V \in \tilde{\mathcal{V}}_f$) for every lower semicontinuous (resp. continuous) function $f: Y \to]0, +\infty[$, when does V possess some other meaningful property?

In fact, we prove that, under suitable hypotheses, for every open set $\Omega \subseteq X$, the set $F(\Omega)$ belongs to the family $\mathcal{V}_f$ (resp. $\tilde{\mathcal{V}}_f$) for every lower semicontinuous (resp. continuous) function $f: Y \to]0, +\infty[$. Moreover, as an example of answer to our second question, we prove that, if Y is a normed space, any $V \subsetneq Y$ such that $V \in \mathcal{T} \cap \mathcal{V}_r$ for every (sufficiently small) constant $r > 0$, is open.

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2. - Results

Our first result characterize belonging to the family $\mathcal{T}$ of certain sets, in the case where Y is a normed space.

Theorem 1: *Assume that Y is a normed space. Then, the following are equivalent:*

(1) *Any continuous linear functional on Y attains its norm on $S(\theta, 1)$, where θ is the null element of Y.*

(2) *Any subset of Y, which is supported at every point of its boundary from closed hyperplanes, belongs to the family $\mathcal{T}$.*

Proof: Let us prove that (1) $\Rightarrow$ (2). Let U be a subset of Y which is supported at every point of its boundary from closed hyperplanes. Let $y^* \in \partial U$ and let f be a continuous linear real functional on Y, with $f \neq 0$, such that the hyperplane $f^{-1}(f(y^*))$ supports U at y^*. Suppose, for instance, that $f(y) < f(y^*)$ for all $y \in U$. By (1), there exists $\bar{y} \in S(\theta, 1)$ such that $f(\bar{y}) = \|f\|$. Now, choose $\lambda > 0$. To finish the proof, it suffices to show that $d(y^* + \lambda\bar{y}, y^*) = d(y^* + \lambda\bar{y}, U)$. Assume the contrary. Therefore, there is $y' \in U$ such that $\|y^* + \lambda\bar{y} - y'\| < \lambda$. Since $f(y') < f(y^*)$, we have

$$|f(y^* + \lambda\bar{y}) - f(y')| = f(y^*) - f(y') + \lambda f(\bar{y}).$$

On the other hand, by the continuity of f, we have $f(y^*) - f(y') + \lambda f(\bar{y}) < \|f\| \|y^* + \lambda\bar{y} - y'\| < \lambda\|f\|$, so that $f(y^*) < f(y')$, a contradiction.

Now let us prove that (2) $\Rightarrow$ (1). Assume that Y is a complex normed space. Let f be any continuous linear functional on Y, with $f \neq 0$. For each $y \in Y$, put $f_1(y) = \mathrm{Re} f(y)$. So that, $f(y) = f_1(y) - i f_1(iy)$. Of course, the set $f_1^{-1}(0)$ is nowhere dense and, by (2), belongs to the family $\mathcal{T}$. Hence, there exists $z \in Y \setminus \{\theta\}$, such that $d(z, f_1^{-1}(0)) = \|z\|$. On the other hand, thanks to a result by Ascoli (see Lemma 1.2 of [1]), we have $d(z, f_1^{-1}(0)) = |f_1(z)|/\|f_1\|$, and so $|f_1(z/\|z\|)| = \|f_1\|$. Then, it is clear that $|f(z/\|z\|)| = \|f\|$. If Y is a real normed space, the proof is analogous.

The next result is about the families $\mathcal{V}_f$ and $\hat{\mathcal{V}}_f$. Of course, we have $\hat{\mathcal{V}}_f \subseteq \mathcal{V}_f$ for every lower semicontinuous function $f \colon Y \to]0, +\infty[$. In certain cases, a partial converse holds. Indeed, we have the following

Theorem 2: *Let V be a connected subset of Y, with more than a point. Then, for every upper semicontinuous function $f \colon Y \to]0, +\infty[$, with $2 \sup_Y f < \mathrm{diam}\,(V)$, the relation $V \in \mathcal{V}_f$ implies $V \in \hat{\mathcal{V}}_f$.*

Proof: Let $V \in \mathcal{V}_f$, with f as in the statement. Let $z^* \in \tilde{A}_{f,V}$. Of course $V \setminus B(z^*, f(z^*)) \neq 0$, since, otherwise, we would have $\mathrm{diam}\,(V) < 2 \sup_Y f < \mathrm{diam}\,(V)$. Therefore, one can choose $\bar{y} \in V$ such that $d(\bar{y}, z^*) > f(z^*)$. Since the function $d(\bar{y},\cdot)-f(\cdot)$ is lower semicontinuous, there is a neighbourhood U of z^* such that $d(\bar{y},z)>f(z)$ for all $z\in U$. Now, observe that, by hypothesis, $A_{f,V}$ is an open neighbourhood of z^*. Hence, if $z\in A_{f,V}\cap U$, we have $d(\bar{y},z)>$ $>f(z)$ and $d(y',z)<f(z)$ for some $y'\in V$. Thus, since V is connected, there is some $y''\in V$ such that $d(y'',z)=f(z)$, and so $z^*\in \operatorname{int}(\bar{A}_{f,V})$.

Now, we prove the following

THEOREM 3: *Let Y be a normed space and $V\subset Y$. If there exists a sequence $\{\alpha_n\}$ in $]0,+\infty[$ such that $\lim_{n\to\infty}\alpha_n=0$ and $V\in\mathcal{T}\cap\mathcal{U}_{\alpha_n}$ for all $n\in\mathrm{N}$, then V is open.*

PROOF: Assume the contrary. Let $y^*\in\partial V\cap V$. Since $V\in\mathcal{T}$, there is $z^*\in Y\setminus\{y^*\}$ such that $d(z^*,y^*)=d(z^*,V)$. Fix $\bar{n}\in\mathrm{N}$ such that $\alpha_{\bar{n}}<\|y^*-z^*\|/2$ and put $\bar{\lambda}=\alpha_{\bar{n}}/\|y^*-z^*\|$. Since $V\in\mathcal{U}_{\alpha_{\bar{n}}}$, there is $\bar{r}\in]0,\|y^*-z^*\|/2[$ such that $B(\bar{\lambda}z^*+(1-\bar{\lambda})y^*,\bar{r})\subseteq A_{\alpha_{\bar{n}},V}$. Now, choose $\tilde{\lambda}\in]\bar{\lambda},\bar{\lambda}+\bar{r}/\|y^*-z^*\|[$. Then, there is $\tilde{y}\in V$ such that $\|\tilde{\lambda}z^*+(1-\tilde{\lambda})y^*-\tilde{y}\|<\alpha_{\bar{n}}$. We have $\|\tilde{y}-z^*\|<\|\tilde{\lambda}z^*+(1-$ $-\tilde{\lambda})y^*-\tilde{y}\|+\|\tilde{\lambda}z^*+(1-\tilde{\lambda})y^*-z^*\|<\alpha_{\bar{n}}+(1-\tilde{\lambda})\|y^*-z^*\|<\alpha_{\bar{n}}+(1-\alpha_{\bar{n}}/\|y^*-$ $-z^*\|)\|y^*-z^*\|=\|y^*-z^*\|$, a contradiction.

Observe that, if Y is not a normed space, Theorem 3 can be false. Indeed, it suffices to take, for instance, $Y=([0,1]\times\{0\})\cup\bigcup_{n\in\mathrm{N}}\{(1,1/n)\}\cup\{(0,2)\}$, with the relative usual R^2-topology, and $V=[0,1]\times\{0\}$.

Our main result on the images of open sets under F is the following

THEOREM 4: *Suppose that, for every $y\in Y$, the real function $d(y,F(\cdot))$ has no local, non-absolute, minimum point. Then, for every lower semicontinuous function $f\colon Y\to]0,+\infty[$ and every open set $\Omega\subseteq X$, the set $F(\Omega)$ belongs to the family $\mathcal{U}_f$. Moreover, if $F(\Omega)\in\mathcal{T}$, then $F(\Omega)$ is open.*

PROOF: Let f and Ω be as in the statement. Let $z_0\in A_{f,F(\Omega)}$. Choose $y_0\in$ $\in F(\Omega)$ such that $d(y_0,z_0)<f(z_0)$. Let $x_0\in\Omega$ be such that $y_0\in F(x_0)$. So that $d(z_0,F(x_0))<f(z_0)$. Since F is onto and $f(z_0)>0$, by hypothesis, it is possible to find $x'\in\Omega$ such that $d(z_0,F(x'))<f(z_0)$. Since the real function $d(\cdot,$ $F(x'))-f(\cdot)$ is upper semicontinuous at z_0, there is a neighbourhood U of z_0 such that $d(y,F(x'))<f(y)$ for all $y\in U$. Then, for each $y\in U$, there is $y'\in F(x')$ such that $d(y,y')<f(y)$. This proves that $U\subseteq A_{f,F(\Omega)}$. Therefore, $A_{f,F(\Omega)}$ is open, and so $F(\Omega)\in\mathcal{U}_f$. Now, let $F(\Omega)\in\mathcal{T}$. Suppose that $F(\Omega)$ is not open. Let $y_1\in\partial F(\Omega)\cap F(\Omega)$. Let $z_1\in Y\setminus\{y_1\}$ be such that $d(z_1,y_1)=d(z_1,F(\Omega))$. Choose $x_1\in\Omega$ such that $y_1\in F(x_1)$. Since $d(z_1,F(x_1))=d(y_1,z_1)>0$, by hypothesis, there is $x_2\in\Omega$ such that $d(z_1,F(x_2))<d(z_1,F(x_1))$. But then there is $y\in F(x_2)$, and so $y\in F(\Omega)$, such that $d(z_1,y)<d(y_1,z_1)$, a contradiction.

As an immediate consequence of Theorem 4, we obtain the following open mapping theorem.

THEOREM 5: *Suppose that, for every $y\in Y$, the real function $d(y,F(\cdot))$ has no local, non-absolute, minimum point and that there exists a base $\mathcal{B}$ of open subsets of X such that $F(\Omega)\in\mathcal{T}$ for all $\Omega\in\mathcal{B}$. Then, the multifunction F is open.*

The next result explains the role of the first assumption of Theorem 5. Let us recall that a set $U \subseteq Y$ is said to be proximinal if for every $y \in Y$ there exists $z \in U$ such that $d(y, z) = d(y, U)$.

THEOREM 6: *Let Y be a normed space. If, for each $x \in X$, the set $F(x)$ is proximinal and the multifunction F is open, then, for every $y \in Y$, the real function $d(y, F(\cdot))$ has no local, non-absolute, minimum point.*

PROOF: Let $x_0 \in X$ and $y_0 \in Y$ be such that $d(y_0, F(x_0)) > 0$. Since $F(x_0)$ is proximinal, there exists $z_0 \in F(x_0)$ such that $\|y_0 - z_0\| = d(y_0, F(x_0))$. Let Ω be any open neighbourhood of x_0. Since $F(\Omega)$ is open, there is $r^* > 0$ such that $B(z_0, r^*) \subset F(\Omega)$. If we choose $\lambda \in]0, r^*/\|y_0 - z_0\|[$, then $\lambda y_0 + (1-\lambda) z_0 \in F(\Omega)$ and $\|\lambda y_0 + (1-\lambda) z_0 - y_0\| < \|y_0 - z_0\|$. Hence $d(y_0, F(\Omega)) < \|y_0 - z_0\|$. But then there is $\bar{x} \in \Omega$ such that $d(y_0, F(\bar{x})) < d(y_0, F(x_0))$. The proof is complete.

With regard to the second assumption of Theorem 5, we present the following

EXAMPLE 1: Let Y be a normed space and X be a dense subset of Y with empty interior. Consider X with the relative topology. Fix $x_0 \in X$ and put

$$F(x) = \begin{cases} \{x\} & \text{if } x \in X \setminus \{x_0\} \\ Y & \text{if } x = x_0 . \end{cases}$$

For every $y \in Y$, $x \in X \setminus \{x_0, y\}$ and every open neighbourhood Ω in Y of x, since X is dense in Y, we have $d(y, \Omega \cap X) = d(y, \Omega)$. On the other hand, since Y is a normed space, we have $d(y, \Omega) < d(x, y)$. Hence, the real function $d(y, F(\cdot))$ has no local, non-absolute, minimum point. However, since the interior of X is empty, the multifunction $F: X \to 2^Y$ is not open.

Now, come back to Theorem 4. If we apply Example 1, by taking $Y = \mathbf{R}$ and $X = \mathbf{Q}$, we realize that, for any open set $\Omega \subseteq \mathbf{R} \setminus \{x_0\}$ and any constant $r > 0$, the set $\bar{A}_{r, F(\Omega \cap \mathbf{Q})}$ is countable. Therefore, $F(\Omega \cap \mathbf{Q}) \notin \tilde{\mathcal{V}}_r$. This shows that in Theorem 4 the conclusion « $F(\Omega) \in \mathcal{V}_r$ » cannot be replaced with the stronger one « $F(\Omega) \in \tilde{\mathcal{V}}_r$ ».

The following final result provides a positive answer in this direction, in the case where F is single-valued.

THEOREM 7. *Let X be locally connected and let g be a continuous function from X onto Y such that, for every $y \in Y$, the real function $d(y, g(\cdot))$ has no local, non-absolute, extremum point. Then, for every open set $\Omega \subseteq X$ and every continuous function $f: Y \to]0, +\infty[$ such that $f(y) < \sup_{x \in X} d(y, g(x))$ for all $y \in Y$, the set $g(\Omega)$ belongs to the family $\tilde{\mathcal{V}}_f$.*

PROOF: Let Ω and f be as in the statement. Let $z_0 \in \bar{A}_{f, g(\Omega)}$. Choose $y_0 \in g(\Omega)$ such that $d(y_0, z_0) = f(z_0)$. Let $x_0 \in \Omega$ be such that $y_0 = g(x_0)$ and let Ω^* be a connected neighbourhood of x_0 contained in Ω. Since g is onto and

$0 < f(z_0) < \sup\limits_{x \in X} d(z_0, g(x))$, by hypothesis, there exist $x', x'' \in \Omega^*$ such that $d(z_0, g(x')) < f(z_0) < d(z_0, g(x''))$. Since f is continuous there is a neighbourhood U of z_0 such that $d(y, g(x')) < f(y) < d(y, g(x''))$ for all $y \in U$. But then, since Ω^* is connected and g is continuous, for each $y \in U$, there is $\bar{x} \in \Omega^*$ such that $d(y, g(\bar{x})) = f(y)$. Therefore, $U \subseteq \bar{A}_{f,g(\Omega)}$ and so $\bar{A}_{f,g(\Omega)}$ is open. Hence, $g(\Omega) \in \widetilde{\mathfrak{V}}_f$.

By means of some examples, the reader can convince himself that none of the hypotheses of Theorem 7 can be dropped out. Here, we limit ourselves to stress that Theorem 7 does not hold, in general, if g is not single-valued. Indeed, we have the following

EXAMPLE 2: Let $X =]0, 2[$, $Y = [0, 2[$ and let $F: X \to 2^Y$ be defined as follows:

$$F(x) = \begin{cases} [0, x] & \text{if } x \in]0, 1] \\ [x-1, x] & \text{if } x \in]1, 2[\,. \end{cases}$$

The multifunction F is continuous and, for every $y \in Y$, the real function $d(y, F(\cdot))$ has no local, non-absolute, extremum point. Moreover, we have $\inf\limits_{y \in Y} \sup\limits_{x \in X} d(y, F(x)) > \frac{1}{3}$. Now, take $\Omega =]0, \frac{1}{2}[$ and $f(y) = \frac{1}{3}$ for all $y \in Y$. Observe that $F(\Omega) = [0, \frac{1}{2}[$, and so $F(\Omega) \notin \widetilde{\mathfrak{V}}_f$. Therefore, F does not satisfy the conclusion of Theorem 7.

REFERENCES

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