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Topological Singularities of Linear Networks (**)

Singolarità topologiche di circuiti lineari

Summary. — The notion of topological singularities of linear networks, i.e. of families of linear components whose members are dependent regardless of the particular values of their parameters, is investigated. Abstract definitions of linear components and dependence are given, which apply to both time-invariant and time-varying networks, and which behave well with unique solvability problems. The notions of socket of a linear component, and of the topological degree of a socket, are introduced, which prove to be the main topological tools for the investigation of topological singularities. The topological characterization and the structure of topological singularities are obtained under purely algebraic and widely general assumptions, which prove to be not only sufficient but also necessary. Examples are given, taken from both time-invariant and time-varying networks.

1. - Introduction

Let N be a linear network, and G the graph associated with N. There are no necessary and sufficient topological conditions for the unique solvability of N. Although these conditions are known for RLC networks without controlled sources [1, 2], no topological conditions can exist for RLC networks with controlled sources [3, 4]. So, whenever N is not uniquely solvable, one has to take the values of parameters of its components into consideration, and may attempt to make slight changes of these values in order to transform N into a uniquely solvable network. This is not always possible; it may be that N has a *topological singularity*, i.e. a subfamily $C_1, \dots, C_r$ of the family of its components, whose members are dependent regardless of the change of the values of their parameters.

Let N be an RLC network with controlled sources; characterizations of topological singularities of N are known in terms of the topology of graphs deduced from G, according to both the behaviour and the connections of its

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components [5]. Let N be an RLC network with memoryless n-ports; characterizations of topological singularities of N are known in terms of the topology of matroids deduced from a system of network equations for N [6, 7].

In this paper we show that characterizations of topological singularities are possible in terms of the topology of G, for an arbitrary linear network N.

We want our results to be valid regardless of whether G is a graph or a matroid [8, 9], regardless of the particular space W of time-functions which describes the state of N, and regardless of the particular space K of endomorphisms of W which describes the analytical behaviour of components. Section 2 gives consequent abstract definitions.

In Subsection 2.1 we introduce the notion of *semigraph*; every result valid for networks associated with a semigraph is a result valid also for networks associated with a graph (see Subsection 2.15 of Reference [10]), or with a matroid (see Subsection 3.2 of Reference [9]).

In Subsection 2.2 we deal with linear components. Our definition is valid regardless of the particular W and K. The topological behaviour of a component C is taken into consideration, and described by the ordered pair (A, M), where A, M are the sets of branches of G whose voltages and whose currents we have respectively related with C; we call (A, M) the *socket* of C. Observe that, in our definition, every component C acts as a single constraint on the voltages and currents of the branches of its socket (A, M).

In Subsection 2.3 we deal with the notion of dependence for linear components. Our definition, although abstract and open to further applications, behaves well with problems of unique solvability. This is shown in Subsection 2.4, where examples are given, taken from both time-invariant and time-varying components.

In Section 3, in order to investigate topological singularities, we deal with components (*parametric components*) which allow arbitrary changes of the values of their parameters. Subsection 3.1 formalizes the problem. Subsection 3.2 introduces the *topological degree* of a socket, which proves to be the main topological tool for the solution. Subsection 3.3 supplies the algebraic background for the proofs.

In Subsection 3.4 we describe purely algebraic and widely general assumptions (*coherence assumptions*) on W, K and on the dependence, and give relating examples taken from both time-invariant and time-varying components. In Subsection 3.5, under coherence assumptions, we give the topological characterization of topological singularities, and describe their structure.

In Theorem 3.5.2 we prove that a family $C_1, \ldots, C_r$ of linear components on the sockets $(A_1, M_1), \ldots, (A_r, M_r)$, respectively, is a topological singularity if and only if it has a subfamily $(C_j | j \in J)$ such that $\deg \bigcup_{j \in J} (A_j, M_j) < |J|$ (the definition of *union* of sockets is straightforward); roughly speaking, if and only if it has a subfamily whose members are all connected to a socket whose topological degree is less than their number.

In Theorem 3.5.4 we prove that a topological singularity $C_1, \ldots, C_r$ has maximal subfamilies without topological singularities; that all these maximal subfamilies have the same number of members; that whenever we add one of the remaining components to one of these maximal subfamilies, then the family so obtained has a unique minimal topological singularity.

Regarding a minimal topological singularity $C_1, \ldots, C_r$, in Proposition 3.5.5 we prove that $\deg \bigcup_{j=1}^{r} (A_j, M_j) = r-1$; roughly speaking, we prove that its members are all connected to a socket whose topological degree is 1 less than their number.

Let n be the number of branches of G; in Corollary 3.5.3 we prove that all families of linear components with more than n members have topological singularities, and that every family without topological singularities may be extended to a family of n components without topological singularities.

In Subsection 3.6 we prove that the coherence assumptions are not only sufficient but also necessary for the above characterization of topological singularities.

When a dependent family $C_1, \ldots, C_r$ of linear components is not a topological singularity, then opportune changes of the values of the parameters transform $C_1, \ldots, C_r$ into an independent family. One may ask whether these changes may be chosen arbitrarily small. A general investigation of the problem involves the topology of K, and is not attempted here. In Subsection 3.7 we give examples of affirmative replies taken from both time-invariant and time-varying components.

2. - Linear components and dependence

In what follows, $\mathbf{R}$ denotes the field of real numbers; $\mathbf{R}^n$ denotes the n-dimensional Euclidean space; $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ denotes the canonical basis of $\mathbf{R}^n$. If H is a subspace of $\mathbf{R}^n$, then $\mathrm{pr}_H \colon \mathbf{R}^n \to \mathbf{R}^n$ will denote the orthogonal projection on H. If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r \in \mathbf{R}^n$, then $[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r]$ will denote the $n \times r$ matrix whose columns are $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$; $\langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_r \rangle$ will denote the subspace of $\mathbf{R}^n$ spanned by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$.

If W is a vector space over $\mathbf{R}$, then W^n will denote the space of the columns with n elements in W.

Let $E = \mathrm{End}\, W$ be the algebra of the $\mathbf{R}$-endomorphisms of W. Let $\boldsymbol{A} = [a_{ij}]$ be an $m \times n$ matrix with elements in E. For every

$$w = [w_1, \ldots, w_n]^T \in W^n$$

(exponent T denotes *transpose*), let

$$\boldsymbol{A}w = \Big[\sum_{j=1}^{n} a_{1j} w_j, \ldots, \sum_{j=1}^{n} a_{mj} w_j\Big]^T.$$

The map $\sigma: W^n \to W^m$ defined by $\sigma(\boldsymbol{w}) = \boldsymbol{A}\boldsymbol{w}$, is obviously R-linear. Let $\sigma: W^n \to W^m$ be any R-linear map; there exists one and only one $m \times n$ matrix $\boldsymbol{A}$, with elements in E, such that $\sigma(\boldsymbol{w}) = \boldsymbol{A}\boldsymbol{w}$ for every $\boldsymbol{w} \in W^n$.

The space of the $m \times n$ matrices with elements in E, will be denoted by $M(m \times n; E)$. Observe that R is, in a canonical way, a subfield of E; this allows us to consider the matrices with elements in R as particular matrices with elements in E. The elements of E will be considered as 1×1 matrices.

If K is a subspace of E, then $M(m \times n; K)$ will denote the space of the $m \times n$ matrices with elements in K; $M(m \times n; K)$ is obviously a subspace of $M(m \times n; E)$. Let $\boldsymbol{A} \in M(m \times n; K)$; for every $\boldsymbol{B} \in M(p \times m; \mathrm{R})$, $\boldsymbol{C} \in M(n \times q; \mathrm{R})$, we have $\boldsymbol{BA} \in M(p \times n; K)$, $\boldsymbol{AC} \in M(m \times q; K)$.

If J is a set, then $|J|$ will denote the cardinality of J. The symbol $(a_j | j \in J)$ will denote the family whose members are the a_j with $j \in J$.

2.1. *Semigraphs*

Let $\{\varrho_1, \dots, \varrho_n\}$ be a set with n elements. An element $\boldsymbol{a} = [a_1, \dots, a_n]^T \in \mathrm{R}^n$ will be considered as a map of $\{\varrho_1, \dots, \varrho_n\}$ on R, namely the map defined by $\boldsymbol{a}(\varrho_\lambda) = a_\lambda$, for $\lambda = 1, \dots, n$.

2.1.1. Definition: An ordered triplet $G = (\{\varrho_1, \dots, \varrho_n\}; V, I)$, where V and I are subspaces of R^n, each being the orthogonal complement of the other, will be called a *semigraph*. The elements $\varrho_1, \dots, \varrho_n$ will be called the *branches* of G; V and I will be called the spaces of the *constant voltages* and of the *constant currents* of G.

2.1.2. Definition: Let $G = (\{\varrho_1, \dots, \varrho_n\}; V, I)$ be a semigraph. For $\lambda = 1, \dots, n$, the orthogonal projections of $\boldsymbol{e}_\lambda$ on V, I will be denoted by $\boldsymbol{\theta}_\lambda$, $\boldsymbol{\gamma}_\lambda$. The matrices of $\mathrm{pr}_V: \mathrm{R}^n \to \mathrm{R}^n$, $\mathrm{pr}_I: \mathrm{R}^n \to \mathrm{R}^n$, with respect to the basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$, will be denoted by $\boldsymbol{\Theta}$, $\boldsymbol{\Gamma}$. Obviously

$$\boldsymbol{\Theta} = [\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_n], \qquad \boldsymbol{\Gamma} = [\boldsymbol{\gamma}, \dots, \boldsymbol{\gamma}_n];$$

moreover $\boldsymbol{\Theta}$ and $\boldsymbol{\Gamma}$ are symmetric idempotent matrices such that $\boldsymbol{\Theta\Gamma} = \boldsymbol{\Gamma\Theta} = \boldsymbol{0}$, $\boldsymbol{\Theta} + \boldsymbol{\Gamma} = \boldsymbol{I}$.

2.2. *Sockets, linear components and equations*

Let $G = (\{\varrho_1, \dots, \varrho_n\}; V, I)$ be a semigraph with n branches. Let W be a vector space over R. Let K be a subspace of End W.

An element $\boldsymbol{w} = [w_1, \dots, w_n]^T \in W^n$ will be considered as a map of $\{\varrho_1, \dots, \varrho_n\}$ on W, namely the map defined by $\boldsymbol{w}(\varrho_\lambda) = w_\lambda$, for $\lambda = 1, \dots, n$.

2.2.1. Definition: The spaces

$$V_w = \{\boldsymbol{w} \in W^n | \boldsymbol{i}^T \boldsymbol{w} = 0, \text{ for every } \boldsymbol{i} \in I\},$$

$$I_w = \{\boldsymbol{w} \in W^n | \boldsymbol{v}^T \boldsymbol{w} = 0, \text{ for every } \boldsymbol{v} \in V\},$$

will be called the spaces of the *W-voltages* and of the *W-currents* of G.

2.2.2. Definition: An ordered pair (Λ, M) of subsets of $\{\varrho_1, \ldots, \varrho_n\}$ will be called a *socket* of G.

2.2.3. Definition: Let (Λ, M) be a socket of G. Let b_λ, c_μ, with $\varrho_\lambda \in \Lambda$, $\varrho_\mu \in M$, be elements of K; the linear map $\sigma \colon V_w \oplus I_w \to \mathbb{W}$ defined by

$$\sigma(\boldsymbol{v}, \boldsymbol{i}) = \sum_{\varrho_\lambda \in \Lambda} b_\lambda \boldsymbol{v}(\varrho_\lambda) + \sum_{\varrho_\mu \in M} c_\mu \boldsymbol{i}(\varrho_\mu),$$

will be called a *linear K-component* on the socket (Λ, M).

2.2.4. Definition: Let (Λ, M) be a socket of G, let σ be a linear K-component on (Λ, M), let $w \in \mathbb{W}$. The ordered pair (σ, w) will be called a *linear K-constraint* on the socket (Λ, M). Let $(\boldsymbol{v}, \boldsymbol{i}) \in V_w \oplus I_w$; if $\sigma(\boldsymbol{v}, \boldsymbol{i}) = w$, then we will say that $(\boldsymbol{v}, \boldsymbol{i})$ *verifies* the constraint (σ, w).

The set of the sockets of G will be denoted by $S(G)$. Let $(\Lambda, M), (\Lambda', M') \in S(G)$; the socket $(\Lambda \cup \Lambda', M \cup M')$ will be called the *union* of $(\Lambda, M), (\Lambda', M')$, and will be denoted by $(\Lambda, M) \cup (\Lambda', M')$; if $\Lambda \subset \Lambda'$ and $M \subset M'$, then we will write $(\Lambda, M) \subset (\Lambda', M')$.

Let σ be a linear K-component on the socket (Λ, M). There are various sockets (Λ', M') such that σ may be considered as a linear K-component on (Λ', M'); for instance, every (Λ', M') such that $(\Lambda, M) \subset (\Lambda', M')$.

Let σ, σ' be linear K-components on the sockets $(\Lambda, M), (\Lambda', M')$, respectively; let $a \in \mathbb{R}$. Obviously the linear maps $\sigma + \sigma'$, $a\sigma \colon \mathbb{W}_w \oplus I_w \to \mathbb{W}$ are linear K-components on the sockets $(\Lambda, M) \cup (\Lambda', M')$, (Λ, M), respectively. Hence the set of all the linear K-components on the sockets of G, is a real vector space; it will be denoted by $C_K(G)$.

2.2.5. Proposition: *Let* $\boldsymbol{w} \in \mathbb{W}^n$. *There exist unique elements* $\boldsymbol{v} \in \mathbb{W}_w$, $\boldsymbol{i} \in I_w$ *such that* $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{i}$. *We have:*

(*a*) $\boldsymbol{v} = \boldsymbol{\Theta w}$, $\boldsymbol{i} = \boldsymbol{\Gamma w}$.

(*b*) $\boldsymbol{v}(\varrho_\lambda) = \boldsymbol{\theta}_\lambda^T \boldsymbol{w}$, $\boldsymbol{i}(\varrho_\lambda) = \boldsymbol{\gamma}_\lambda^T \boldsymbol{w}$, *for* $\lambda = 1, \ldots, n$.

Proof: Write $\boldsymbol{v} = \boldsymbol{\Theta w}$, $\boldsymbol{i} = \boldsymbol{\Gamma w}$. For every $\boldsymbol{a} \in I$, we have $\boldsymbol{a}^T \boldsymbol{v} = \boldsymbol{a}^T \boldsymbol{\Theta w} = \boldsymbol{0 w} = 0$; hence $\boldsymbol{v} \in V_w$. A similar argument proves that $\boldsymbol{i} \in I_w$. Moreover $\boldsymbol{v} + \boldsymbol{i} = (\boldsymbol{\Theta} + \boldsymbol{\Gamma})\boldsymbol{w} = \boldsymbol{I w} = \boldsymbol{w}$.

Let $\boldsymbol{v}' \in V_w$, $\boldsymbol{i}' \in I_w$. If $\boldsymbol{v}' + \boldsymbol{i}' = \boldsymbol{w}$, then $\boldsymbol{z} = \boldsymbol{v}' - \boldsymbol{v} = \boldsymbol{i} - \boldsymbol{i}' \in V_w \cap I_w$. For every $\boldsymbol{a} \in \mathbb{R}^n$, we have $\boldsymbol{a}^T \boldsymbol{z} = (\mathrm{pr}_V\, \boldsymbol{a})^T \boldsymbol{z} + (\mathrm{pr}_I\, \boldsymbol{a})^T \boldsymbol{z} = 0$; hence $\boldsymbol{z} = \boldsymbol{0}$, $\boldsymbol{v}' = \boldsymbol{v}$, $\boldsymbol{i}' = \boldsymbol{i}$.

Since $\boldsymbol{\Theta}, \boldsymbol{\Gamma}$ are symmetric matrices, statement (*b*) follows from statement (*a*), Q.E.D.

2.2.6. Corollary: $V_w + I_w = \mathbb{W}^n$, $V_w \cap I_w = \{\boldsymbol{0}\}$.

Proof: Both statements follow immediately from Proposition 2.2.5, Q.E.D.

2.2.7. Proposition: *Let $\sigma \in C_K(G)$. Then:*

(*a*) *There exists a unique matrix $\tilde{\boldsymbol{\sigma}} \in M(1 \times n; K)$, such that*

$$\sigma(\boldsymbol{v}, \boldsymbol{i}) = \tilde{\boldsymbol{\sigma}}(\boldsymbol{v} + \boldsymbol{i})$$

for every $(\boldsymbol{v}, \boldsymbol{i}) \in V_W \oplus I_W$.

(*b*) *If $b_1, \ldots, b_n, c_1, \ldots, c_n$ are elements of K such that*

$$\sigma(\boldsymbol{v}, \boldsymbol{i}) = \sum_{\lambda=1}^{n} b_\lambda \boldsymbol{v}(\varrho_\lambda) + \sum_{\mu=1}^{n} c_\mu \boldsymbol{i}(\varrho_\mu)$$

for every $(\boldsymbol{v}, \boldsymbol{i}) \in V_W \oplus I_W$, then

$$\tilde{\boldsymbol{\sigma}} = \sum_{\lambda=1}^{n} b_\lambda \boldsymbol{\theta}_\lambda^T + \sum_{\mu=1}^{n} c_\mu \boldsymbol{\Upsilon}_\mu^T.$$

Proof: By (*b*) of Proposition 2.2.5, the matrix $\tilde{\boldsymbol{\sigma}}$ defined in (*b*) verifies the condition described in (*a*). Let $\boldsymbol{\psi} \in M(1 \times n; K)$ verify the same property. Since $V_W + I_W = W^n$ (see Corollary 2.2.6), for every $\boldsymbol{w} \in W^n$ we have $\tilde{\boldsymbol{\sigma}}\boldsymbol{w} = \boldsymbol{\psi}\boldsymbol{w}$; hence $\boldsymbol{\psi} = \tilde{\boldsymbol{\sigma}}$, Q.E.D.

2.2.8. Corollary: *The map $\sigma \to \tilde{\boldsymbol{\sigma}}$ defined in Proposition 2.2.7, is an isomorphism of $C_K(G)$ onto $M(1 \times n; K)$.*

Proof: The map $\sigma \to \tilde{\boldsymbol{\sigma}}$ is obviously linear and injective. Let $\boldsymbol{a} = [a_1, \ldots, a_n] \in M(1 \times n; K)$; let σ be the linear K-component defined by

$$\sigma(\boldsymbol{v}, \boldsymbol{i}) = \sum_{\lambda=1}^{n} a_\lambda \boldsymbol{v}(\varrho_\lambda) + \sum_{\mu=1}^{n} a_\mu \boldsymbol{i}(\varrho_\mu)\,;$$

then $\tilde{\boldsymbol{\sigma}} = \boldsymbol{a}$, Q.E.D.

Let us consider a system $(\sigma_1, w_1), \ldots, (\sigma_r, w_r)$ of linear K-constraints on the sockets $(A_1, M_1), \ldots, (A_r, M_r)$, respectively. If there exists $(\boldsymbol{v}, \boldsymbol{i}) \in V_W \oplus I_W$ which verifies $(\sigma_1, w_1), \ldots, (\sigma_r, w_r)$, then the system $(\sigma_1, w_1), \ldots, (\sigma_r, w_r)$ will be called *consistent*.

2.2.9. Remark: Let $\tilde{\boldsymbol{\sigma}}_1, \ldots, \tilde{\boldsymbol{\sigma}}_r \in M(1 \times n; K)$ be the matrices associated with $\sigma_1, \ldots, \sigma_r$; let N be the matrix defined by

$$N = \begin{bmatrix} \tilde{\boldsymbol{\sigma}}_1 \\ \vdots \\ \tilde{\boldsymbol{\sigma}}_r \end{bmatrix}.$$

If, for every solution $\boldsymbol{u} \in W^n$ of the system

$$N\boldsymbol{u} = [w_1, \ldots, w_r]^T,$$

we construct the ordered pair

$$(v, i) = (\Theta u, \Gamma u),$$

then, by Propositions 2.2.5 and 2.2.7, we obtain the elements of $V_w \oplus I_w$ which verify the constraints $(\sigma_1, w_1), \dots, (\sigma_r, w_r)$.

2.3. *Dependence*

Let $G = (\{\varrho_1, \dots, \varrho_n\}; V, I)$ be a semigraph with n branches. Let W be a vector space over R. Let K, Δ be subspaces of End W.

2.3.1. Definition: A family $\sigma_1, \dots, \sigma_r$ of elements of $C_K(G)$ will be called *Δ-dependent*, if there exist $d_1, \dots, d_r \in \Delta$, not all zero, such that $d_1\sigma_1 + \dots + d_r\sigma_r = 0$.

2.3.2. Definition: A family $\boldsymbol{a}_1, \dots, \boldsymbol{a}_r$ of elements of $M(1 \times n; K)$ will be called *Δ-dependent*, if there exist $d_1, \dots, d_r \in \Delta$, not all zero, such that $d_1\boldsymbol{a}_1 + \dots + d_r\boldsymbol{a}_r = \boldsymbol{0}$.

2.3.3. Proposition: *Let $\sigma_1, \dots, \sigma_r \in C_K(G)$; let $\bar{\sigma}_1, \dots, \bar{\sigma}_r \in M(1 \times n; K)$ be their associated matrices* (see Proposition 2.2.7). *The following conditions are equivalent:*

(*a*) *$\sigma_1, \dots, \sigma_r$ is a Δ-dependent family.*

(*b*) *$\bar{\sigma}_1, \dots, \bar{\sigma}_r$ is a Δ-dependent family.*

Proof: The statement follows from Corollary 2.2.6, Proposition 2.2.7 and Definitions 2.3.1, 2.3.2, Q.E.D.

The information regarding a system of linear constraints which can be deduced by the study of the Δ-dependence, varies according to the choice of Δ. For example, if $\Delta = \{0\}$, the information is void, since every family $\sigma_1, \dots, \sigma_r$ of elements of $C_K(G)$ is Δ-independent; while, if Δ = End W, the Δ-independence of a family $\sigma_1, \dots, \sigma_r$ is equivalent to the consistency of the system $(\sigma_1, w_1), \dots, (\sigma_r, w_r)$ for every $w_1, \dots, w_r \in W$. Further examples are given in the following subsection.

2.4. *Dependence and unique solvability*

Let $G = (\{\varrho_1, \dots, \varrho_n\}; V, I)$ be a semigraph with n branches. The Δ-dependence may give information regarding the unique solvability of a system of linear constraints on G. Here are some examples.

2.4.1. Example: Let W be a vector space over R. The field R is, in a canonical way, a subspace of End W. Let $K = \Delta =$ R.

2.4.2. Example: Let Ω be an open connected subset of $\mathbb{R}$; let $\mathbb{W} = \mathcal{D}'(\Omega)$ be the space of the distributions in Ω. The commutative ring $\mathbb{R}[D]$ of the linear differential operators with constant coefficients, is obviously a subspace of End $\mathbb{W}$. Let $K = \Delta = \mathbb{R}[D]$.

2.4.3. Example: For every $\varepsilon > 0$, let $I_\varepsilon = (-\varepsilon, 0) \cup (0, \varepsilon)$. Let C be the set of the ordered pairs (I_ε, f), where $f \in C^\infty(I_\varepsilon)$. Let $(I_\varepsilon, f), (I_\delta, g) \in C$; if there exists $\nu < \varepsilon, \delta$, such that $f/I_\nu = g/I_\nu$, we will write $(I_\varepsilon, f) \sim (I_\delta, g)$. Let $\mathbb{W} = C/\sim$; obviously $\mathbb{W}$ is a vector space over $\mathbb{R}$.

The space of the germs in 0 of the C^∞-functions defined in some neighborhood of 0, is, in a canonical way, a subspace of $\mathbb{W}$.

Let $\mathcal{M}$ be the field of the germs in 0 of the functions meromorphic in some neighborhood of 0. The non commutative ring $\mathcal{M}[D]$ (for every $f \in \mathcal{M}$ we have $Df = fD + f'$, where f' is the derivative of f) of the linear differential operators with coefficients in $\mathcal{M}$, is obviously a subspace of End $\mathbb{W}$. Let $K = \Delta = \mathcal{M}[D]$.

2.4.4. Lemma: *Let $\mathbb{W}$, K, Δ be as described in Example* 2.4.1, *or in Example* 2.4.2, *or in Example* 2.4.3. *Let $\alpha \colon \mathbb{W} \to \mathbb{W}$ be a non zero element of $\Delta = K$. Then*:

(*a*) $\alpha(\mathbb{W}) = \mathbb{W}$.

(*b*) dim ker $\alpha < \infty$ (*in particular*: dim ker $\alpha = 0$, *in the case of Example* 2.4.1).

Proof: The statements are trivial in the case of Example 2.4.1. In the case of Example 2.4.2, statement (*a*) follows from Theorem 3.6.4 and Corollary 3.6.1 of Reference [14]. Let $u \in \mathcal{D}'(\Omega)$; if $Du = 0$, then u is a constant function; consequently statement (*b*) follows from Subsection 12.59 of Reference [12].

In the case of Example 2.4.3, observe that solving an equation $\alpha(u) = w$, with $w \in \mathbb{W}$, is equivalent to solving separately two opportune linear differential equations with C^∞-coefficients and leading coefficients without zeros, on intervals $(-\varepsilon, 0)$, $(0, \varepsilon)$, respectively. Consequently, the statements follow from Subsection XIII.1 of Reference [15], Q.E.D.

2.4.5. Proposition: *Let $\mathbb{W}$, K, Δ be as described in Example* 2.4.1, *or in Example* 2.4.2, *or in Example* 2.4.3. *Let $\sigma_1, \ldots, \sigma_r \in C_K(G)$.*

(*a*) *If $r > n$, then $\sigma_1, \ldots, \sigma_r$ is a Δ-dependent family.*

(*b*) *If $r < n$, then there exist $d_1, \ldots, d_n \in \Delta$, not all zero, such that, for every $u \in \mathbb{W}$, the ordered pair*

$$(\boldsymbol{v}, \boldsymbol{i}) = (\boldsymbol{\Theta}[d_1 u, \ldots, d_n u]^\tau, \boldsymbol{\Gamma}[d_1 u, \ldots, d_n u]^\tau) \in V_{\mathbb{W}} \oplus I_{\mathbb{W}}$$

verifies the constraints $(\sigma_1, 0), \ldots, (\sigma_r, 0)$.

(c) *If* $r = n$, *and* $\sigma_1, \ldots, \sigma_r$ *is a* Λ*-dependent family, then there exist* $d_1, \ldots, d_n \in \Lambda$, *not all zero, which verify the condition described in* (b).

PROOF: Let $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_r \in M(1 \times n; K)$ be the matrices associated with $\sigma_1, \ldots, \sigma_r$; let

$$N = \begin{bmatrix} \tilde{\sigma}_1 \\ \vdots \\ \tilde{\sigma}_r \end{bmatrix}.$$

Since $\Lambda = K$ is a principal ideal domain (see Subsection 3.1 of Reference [13]), there exist units $U \in M(r \times r; K)$, $V \in M(n \times n; K)$ such that UNV has diagonal form (see Subsection 3.7 of Reference [13]).

If $r > n$, then the last row of UNV is zero; hence also the last row of UN is zero. Let $[d_1, \ldots, d_r]$ be the last row of U; $d_1, \ldots, d_r$ are elements of Λ, not all zero, such that $d_1\tilde{\sigma}_1 + \ldots + d_r\tilde{\sigma}_r = \mathbf{0}$. Then statement (a) follows from Proposition 2.3.3.

If $r < n$, then the last column of UNV is zero; hence also the last column of NV is zero. Let $[d_1, \ldots, d_n]^T$ be the last column of V; $d_1, \ldots, d_n$ are elements of Λ, not all zero, such that $N[d_1, \ldots, d_n]^T = \mathbf{0}$. For every $u \in W$ we have $N[d_1u, \ldots, d_nu]^T = \mathbf{0}$. Then statement (b) follows from Remark 2.2.9.

Let the hypotheses of (c) hold; by Proposition 2.3.3 there exist $\delta_1, \ldots, \delta_n \in \Lambda$, not all zero, such that $[\delta_1, \ldots, \delta_n]N = \mathbf{0}$; obviously $[\delta_1, \ldots, \delta_n]U^{-1}UNV = \mathbf{0}$. Let $[\varepsilon_1, \ldots, \varepsilon_n] = [\delta_1, \ldots, \delta_n]U^{-1}$; there exists λ such that $\varepsilon_\lambda \neq 0$. Since $\Lambda = K$ has no zero divisor, and UNV has diagonal form, then the λ-th column of UNV is zero; hence also the λ-th column of NV is zero. Let $[d_1, \ldots, d_n]^T$ be the λ-th column of V; $d_1, \ldots, d_n$ are elements of Λ, not all zero, such that $N[d_1, \ldots, d_n]^T = \mathbf{0}$. For every $u \in W$ we have $N[d_1u, \ldots, d_nu]^T = \mathbf{0}$. Then statement (c) follows from Remark 2.2.9, Q.E.D.

2.4.6. PROPOSITION: *Let* W, K, Λ *be as described in Example* 2.4.1, *or in Example* 2.4.2, *or in Example* 2.4.3. *Let* $\sigma_1, \ldots, \sigma_n \in C_K(G)$. *Then the following conditions are equivalent*:

(a) $\sigma_1, \ldots, \sigma_n$ *is a* Λ*-independent family.*

(b) *For every* $w_1, \ldots, w_n \in W$, *there exists* $(\mathbf{v}, \mathbf{i}) \in V_w \oplus I_w$ *which verifies the constraints* $(\sigma_1, w_1), \ldots, (\sigma_n, w_n)$.

(c) *The real vector space of the* $(\mathbf{v}, \mathbf{i}) \in V_w \oplus I_w$ *which verify the constraints* $(\sigma_1, 0), \ldots, (\sigma_n, 0)$, *is finite-dimensional* (*in particular: it is* 0*-dimensional in the case of Example* 2.4.1).

Let W, K, Λ *be as described in Example* 2.4.1, *or in Example* 2.4.2; *then* K *is a commutative ring, and the above conditions* (a), (b), (c) *are equivalent to the following condition*:

(*d*) *Let* $\tilde{\boldsymbol{\sigma}}_1, \ldots \tilde{\boldsymbol{\sigma}}_n \in M(1\times n; K)$ *be the matrices associated with* $\sigma_1, \ldots, \sigma_n$*; then*

$$\det \begin{bmatrix} \tilde{\boldsymbol{\sigma}}_1 \\ \vdots \\ \tilde{\boldsymbol{\sigma}}_n \end{bmatrix} \neq 0 .$$

PROOF: Let $\boldsymbol{N}, \boldsymbol{U}, \boldsymbol{V} \in M(n\times n; K)$ be as described in the Proof of Proposition 2.4.5; let $\alpha_1, \ldots, \alpha_n$ be elements of K such that $\boldsymbol{UNV} = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$; let ξ, ψ be the automorphisms of $\mathbb{W}^n$ defined by $\xi(\boldsymbol{w}) = \boldsymbol{U}\boldsymbol{w}$, $\psi(\boldsymbol{w}) = \boldsymbol{V}\boldsymbol{w}$.

For every $\boldsymbol{w} \in \mathbb{W}^n$, we have

$$\begin{aligned} \{\boldsymbol{u} \in \mathbb{W}^n | \boldsymbol{N}\boldsymbol{u} = \boldsymbol{w}\} = \{\boldsymbol{u} \in \mathbb{W}^n | \boldsymbol{UNVV}^{-1}\boldsymbol{u} = \boldsymbol{U}\boldsymbol{w}\} = \\ = \{\boldsymbol{u} \in \mathbb{W}^n | \operatorname{diag}(\alpha_1, \ldots, \alpha_n)\psi^{-1}(\boldsymbol{u}) = \xi(\boldsymbol{w})\} = \\ = \psi\{\boldsymbol{z} \in \mathbb{W}^n | \operatorname{diag}(\alpha_1, \ldots, \alpha_n)\boldsymbol{z} = \xi(\boldsymbol{w})\} . \end{aligned}$$

Since $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n$ are Δ-independent if an only if $\alpha_1 \neq 0, \ldots, \alpha_n \neq 0$, then all the statements follows from Lemma 2.4.4 and from Remark 2.2.9, Q.E.D.

3. - Linear parametric components and topological singularities

3.1. *Linear parametric components*

Let $G = (\{\varrho_1, \ldots, \varrho_n\}; V, I)$ be a semigraph with n branches. Let $\mathbb{W}$ be a vector space over R. Let K, Δ be subspaces of End $\mathbb{W}$.

The following definition formalizes the notion of components which allow arbitrary changes of the values of their parameters.

3.1.1. DEFINITION: Let $(\Lambda, M) \in S(G)$. The space of all linear K-components on the socket (Λ, M) will be called the *linear parametric K-component* on the socket (Λ, M), and will be denoted by $C_K(\Lambda, M)$.

Obviously $C_K(\Lambda, M) \subset C_K(G)$, and $C_K(\{\varrho_1, \ldots, \varrho_n\}, \{\varrho_1, \ldots, \varrho_n\}) = C_K(G)$.

3.1.2. REMARK: Let $(\Lambda, M) \in S(G)$; let $C_K^{\sim}(\Lambda, M)$ be the space of the matrices associated with the elements of $C_K(\Lambda, M)$. By Proposition 2.2.7, we have

$$C_K^{\sim}(\Lambda, M) = \Big\{ \sum_{\varrho_\lambda \in \Lambda} b_\lambda \boldsymbol{\theta}_\lambda^T + \sum_{\varrho_\mu \in M} c_\mu \boldsymbol{\Upsilon}_\mu^T | b_\lambda, c_\mu \in K \Big\} .$$

The following definition formalizes the notion of topological singularities.

3.1.3. DEFINITION: A family $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ of linear parametric K-components will be called *Δ-dependent* if, for every $\sigma_1 \in C_K(\Lambda_1, M_1), \ldots$ $\ldots, \sigma_r \in C_K(\Lambda_r, M_r)$, the family $\sigma_1, \ldots, \sigma_r$ is Δ-dependent.

The topological characterization and the structure of the Λ-dependent families of linear parametric K-components, will be given in Subsection 3.5.

3.2. *The topological degree of a socket*

Let $G = (\{\varrho_1, \dots, \varrho_n\}; V, I)$ be a semigraph with n branches.

3.2.1. Definition: Let X be a subset of $\{\varrho_1, \dots, \varrho_n\}$. If $X \neq \emptyset$, write $X = \{\varrho_{\lambda_1}, \dots, \varrho_{\lambda_r}\}$, with $\varrho_{\lambda_1} < \dots < \varrho_{\lambda_r}$, and let $r_X \colon \mathbb{R}^n \to \mathbb{R}^r$ be the linear map defined by $r_X(\boldsymbol{a}) = [\boldsymbol{a}(\varrho_{\lambda_1}), \dots, \boldsymbol{a}(\varrho_{\lambda_r})]^T$. The spaces $r_X(V)$, $r_X(I)$ will be called the spaces of the *restrictions* of V and I to X, and will be denoted by $V(X)$, $I(X)$. If $X = \emptyset$, we will write $V(X) = I(X) = \{\mathbf{0}\}$.

3.2.2. Definition: Let $(\Lambda, M) \in S(G)$; the non negative integer

$$\deg(\Lambda, M) = \dim V(\Lambda) + \dim I(M)$$

will be called the *topological degree* of the socket (Λ, M).

3.2.3. Remark: Regarding the topological meaning of $\deg(\Lambda, M)$, observe that, if G is the semigraph associated with a graph H, then

$$\deg(\Lambda, M) = \max_{\text{all } T} |\Lambda \cap T| + \max_{\text{all } C} |M \cap C|,$$

where T denotes a tree and C a cotree of H.

3.2.4. Lemma: *Let* $X \subset \{\varrho_1, \dots, \varrho_n\}$. *Then*: $\dim V(X) = \dim \langle \boldsymbol{\theta}_\lambda | \varrho_\lambda \in X \rangle$, $\dim I(X) = \dim \langle \boldsymbol{\gamma}_\lambda | \varrho_\lambda \in X \rangle$.

Proof: If $X = \emptyset$, the statements are trivial. Let $X \neq \emptyset$; write

$$X = \{\varrho_{\lambda_1}, \dots, \varrho_{\lambda_r}\}, \qquad \text{with } \varrho_{\lambda_1} < \dots < \varrho_{\lambda_r}.$$

Obviously $V(X) = \langle r_X \boldsymbol{\theta}_1, \dots, r_X \boldsymbol{\theta}_n \rangle$; hence

$$\dim V(X) = \dim \langle r_X \boldsymbol{\theta}_1, \dots, r_X \boldsymbol{\theta}_n \rangle = \operatorname{rank}[r_X \boldsymbol{\theta}_1, \dots, r_X \boldsymbol{\theta}_n].$$

Since $\boldsymbol{\Theta}$ is symmetric, we have $[r_X \boldsymbol{\theta}_1, \dots, r_X \boldsymbol{\theta}_n] = [\boldsymbol{\theta}_{\lambda_1}, \dots, \boldsymbol{\theta}_{\lambda_r}]^T$; hence

$$\dim V(X) = \operatorname{rank}[\boldsymbol{\theta}_{\lambda_1}, \dots, \boldsymbol{\theta}_{\lambda_r}] = \dim \langle \boldsymbol{\theta}_\lambda | \varrho_\lambda \in X \rangle.$$

A similar argument proves that $\dim I(X) = \dim \langle \boldsymbol{\gamma}_\lambda | \varrho_\lambda \in X \rangle$. Q.E.D.

3.2.5. Proposition: *Let* $(\Lambda, M) \in S(G)$. *Then*: $\deg(\Lambda, M) = \dim \langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in \Lambda, \varrho_\mu \in M \rangle$.

PROOF: Since V and I are each the orthogonal complement of the other, the statement follows immediately from Lemma 3.2.4, Q.E.D.

3.3. *Some algebraic tools*

In this subsection we introduce a notion of dependence for subspaces of $\mathbf{R}^n$, and prove a certain number of relating results, which supply the algebraic background for the proofs in Subsection 3.5.

3.3.1. DEFINITION: A family $E_1, \ldots, E_r$ of subspaces of $\mathbf{R}^n$ will be called *dependent* if every family $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ of elements of $\mathbf{R}^n$, with $\boldsymbol{a}_1 \in E_1, \ldots, \boldsymbol{a}_r \in E_r$, is linearly dependent.

3.3.2. PROPOSITION: *Let $E_1, \ldots, E_r$ be a family of subspaces of $\mathbf{R}^n$. For $j = 1, \ldots, r$, let X_j be a set of generators for E_j. The following conditions are equivalent:*

(*a*) *The family $E_1, \ldots, E_r$ is independent.*

(*b*) *There exists a linearly independent family $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ of elements of $\mathbf{R}^n$, with $\boldsymbol{x}_1 \in X_1, \ldots, \boldsymbol{x}_r \in X_r$.*

PROOF: Let (*a*) hold. Then there exist a linearly independent family $\boldsymbol{a}_1 \in E_1, \ldots, \boldsymbol{a}_r \in E_r$, and an r-multilinear alternating map u: $\mathbf{R}^n \oplus \ldots \oplus \mathbf{R}^n \to \mathbf{R}$, such that $u(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r) \neq 0$ (see Ann. 6.3 of Reference [16]). Suppose that (*b*) doesn't hold; then for every $\boldsymbol{x}_1 \in X_1, \ldots, \boldsymbol{x}_r \in X_r$ we have $u(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r) = 0$; hence $u(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r) = 0$ (see Ann. 6.1.1 of Reference [16]), contradiction. Obviously (*b*) implies (*a*), Q.E.D.

3.3.3. PROPOSITION: *Let $E_1, \ldots, E_r$ be a family of subspaces of $\mathbf{R}^n$. There exist a field F, and a family $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_r$ of elements of F^n, which verify the following condition:*

For every $J \subset \{1, \ldots, r\}$, the family $(E_j | j \in J)$ of subspaces of $\mathbf{R}^n$ is independent if and only if the family $(\boldsymbol{z}_j | j \in J)$ of elements of F^n is linearly independent over F.

PROOF: For $j = 1, \ldots, r$, let $\boldsymbol{A}_j$ be a matrix whose columns are a set of generators for E_j; let α_j be the number of columns of $\boldsymbol{A}_j$. Let

$$(x_{j\lambda_j} | j = 1, \ldots, r;\ 1 \le \lambda_j \le \alpha_j)$$

be a family of independent variables over $\mathbf{R}$ (see Subsection V.3 of Reference [11]); let R be the polynomial ring in these variables, and F the quotient field of R. For $j = 1, \ldots, r$, write $\boldsymbol{x}_j = [x_{j1}, \ldots, x_{j\alpha_j}]^T$, $\boldsymbol{z}_j = \boldsymbol{A}_j \boldsymbol{x}_j \in F^n$.

Let $J \subset \{1, \ldots, r\}$. Without loss of generality, we may clearly assume that $J = \{1, \ldots, s\}$.

Assume that the family $E_1, \ldots, E_s$ is independent. Then there exists a ring homomorphism $f\colon R \to \mathbb{R}$, with $f|\mathbb{R} = id$, such that the family $\boldsymbol{A}_1 f(\boldsymbol{x}_1), \ldots, \boldsymbol{A}_s f(\boldsymbol{x}_s)$ of elements of $\mathbb{R}^n$, is linearly independent over $\mathbb{R}$. Hence there exists an $s \times s$ submatrix $\boldsymbol{M}$ of $[\boldsymbol{A}_1 f(\boldsymbol{x}_1), \ldots, \boldsymbol{A}_s f(\boldsymbol{x}_s)]$, such that $\det \boldsymbol{M} \neq 0$. Let $\boldsymbol{M}'$ be the $s \times s$ submatrix of $[\boldsymbol{A}_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}_s \boldsymbol{x}_s]$ whose image is $\boldsymbol{M}$. Since $f(\det \boldsymbol{M}') = \det \boldsymbol{M} \neq 0$, then $\det \boldsymbol{M}' \neq 0$. This proves that the family $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_s$ of elements of F^n is linearly independent over F.

Conversely, assume that the family $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_s$ of elements of F^n is linearly independent over F. Then there exists an $s \times s$ submatrix $\boldsymbol{M}$ of $[\boldsymbol{A}_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}_s \boldsymbol{x}_s]$ such that $\det \boldsymbol{M} \neq 0$; obviously $\det \boldsymbol{M} \in R$. Since $\mathbb{R}$ is an infinite field, there exists a ring homomorphism $f\colon R \to \mathbb{R}$, with $f|\mathbb{R} = id$, such that $f(\det \boldsymbol{M}) \neq 0$ (see Subsection V.4 of Reference [11]). Hence the family $\boldsymbol{A}_1 f(\boldsymbol{x}_1) \in E_1, \ldots, \boldsymbol{A}_s f(\boldsymbol{x}_s) \in E_s$ is linearly independent over $\mathbb{R}$. This proves that the family $E_1, \ldots, E_s$ is indepenent, Q.E.D.

3.3.4. Proposition: *Let $E_1, \ldots, E_r$ be a minimal dependent family of subspaces of $\mathbb{R}^n$. Then:*

$$\dim \sum_{j=1}^{r} E_j = r - 1 .$$

Proof: Obviously $1 \leq r \leq n+1$. If $r = 1$, or $r = n+1$, then the statement is trivial. Hence we may assume $2 \leq r \leq n$.

Let $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_r$, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$, F be as in the Proof of Proposition 3.3.3. Since $E_1, \ldots, E_r$ is a minimal dependent family, then the family $\boldsymbol{A}_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}_r \boldsymbol{x}_r$ is linearly dependent over F, and minimal with respect to this property.

The rank of $[\boldsymbol{A}_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}_r \boldsymbol{x}_r]$ is $r-1$. Without loss of generality we may assume that the first $r-1$ rows of this matrix are linearly independent over F.

For $j = 1, \ldots, r$, let $\boldsymbol{A}'_j$ be the matrix formed by the first $r-1$ rows of $\boldsymbol{A}_j$. The matrix $[\boldsymbol{A}'_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}'_r \boldsymbol{x}_r]$ has a non singular $(r-1) \times (r-1)$ submatrix. Since there exist $\omega_1, \ldots, \omega_r \in F$, all not zero, such that $\sum_{j=1}^{r} \omega_j \boldsymbol{A}_j \boldsymbol{x}_j = \boldsymbol{0}$, then, for $\xi = 1, \ldots, r$, we have

$$\det [\boldsymbol{A}'_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}'_{\xi-1} \boldsymbol{x}_{\xi-1}, \boldsymbol{A}'_{\xi+1} \boldsymbol{x}_{\xi+1}, \ldots, \boldsymbol{A}'_r \boldsymbol{x}_r] \neq 0 .$$

For $\nu = r, \ldots, n$, let $\boldsymbol{a}_{\nu j}$ be the ν-th row of $\boldsymbol{A}_j$. By our current assumption, there exist $\beta_{\nu 1}, \ldots, \beta_{\nu, r-1} \in F$ such that

$$[\beta_{\nu 1}, \ldots, \beta_{\nu, r-1}][\boldsymbol{A}'_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}'_r \boldsymbol{x}_r] = [\boldsymbol{a}_{\nu 1} \boldsymbol{x}_1, \ldots, \boldsymbol{a}_{\nu r} \boldsymbol{x}_r] .$$

For $\xi = 1, \ldots, r$, since

$$[\beta_{\nu 1}, \ldots, \beta_{\nu, r-1}][\boldsymbol{A}'_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}'_{\xi-1} \boldsymbol{x}_{\xi-1}, \boldsymbol{A}'_{\xi+1} \boldsymbol{x}_{\xi+1}, \ldots, \boldsymbol{A}'_r \boldsymbol{x}_r] =$$
$$= [\boldsymbol{a}_{\nu 1} \boldsymbol{x}_1, \ldots, \boldsymbol{a}_{\nu, \xi-1} \boldsymbol{x}_{\xi-1}, \boldsymbol{a}_{\nu, \xi+1} \boldsymbol{x}_{\xi+1}, \ldots, \boldsymbol{a}_{\nu r} \boldsymbol{x}_r] ,$$

by Cramer's theorem, we have

$$\beta_{\xi 1}, \ldots, \beta_{\xi, r-1} \in \mathrm{R}(\boldsymbol{x}_{j\lambda_j} \mid j = 1, \ldots, \xi - 1, \xi + 1, \ldots, r;\ 1 \leq \lambda_j \leq \alpha_j)\,.$$

Consequently, $\beta_{\xi,1}, \ldots, \beta_{\xi,r-1} \in \mathrm{R}$.

The above argument proves that the rows of $[\boldsymbol{A}_1 | \ldots | \boldsymbol{A}_r]$ are linear combinations, with coefficients in R, of the rows of $[\boldsymbol{A}'_1 | \ldots | \boldsymbol{A}'_r]$. The rows of $[\boldsymbol{A}'_1 \boldsymbol{x}_1, \ldots, \boldsymbol{A}'_r \boldsymbol{x}_r]$ are linearly independent over F; hence the rows of $[\boldsymbol{A}'_1 | \ldots$ $\ldots | \boldsymbol{A}'_r]$ are linearly independent over R. Then the rank of $[\boldsymbol{A}_1 | \ldots | \boldsymbol{A}_r]$ is $r-1$. Since $\sum_{i=1}^{r} E_i$ is the space spanned over R by the columns of $[\boldsymbol{A}_1 | \ldots | \boldsymbol{A}_r]$, our statement is proved, Q.E.D.

3.3.5. Proposition: *Let $E_1, \ldots, E_r$ be a family of subspaces of R^n. The following conditions are equivalent:*

(*a*) *The family $E_1, \ldots, E_r$ is dependent.*

(*b*) *There exists $J \subset \{1, \ldots, r\}$ such that* $\dim \sum_{j \in J} E_j < |J|$.

Proof: Let (*a*) hold. Then there exists $J \subset \{1, \ldots, r\}$ such that $(E_j | j \in J)$ is a minimal dependent family; by Proposition 3.3.4 we obtain $\dim \sum_{j \in J} E_j =$ $= |J| - 1$. Obviously (*b*) implies (*a*), Q.E.D.

3.4. *Coherent spaces of linear operators*

Let W be a vector space over R; let K, Δ be subspaces of End W.

3.4.1. Definition: K will be called a *Δ-coherent* space of linear operators of W if, for every s, $M(1 \times s;\, K)$ verifies the following conditions:

(i) there exist Δ-independent s-tuples $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s$ of elements of $M(1 \times s;\, K)$ (see Definition 2.3.2);

(ii) every r-tuple $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$, with $r > s$, of elements of $M(1 \times s;\, K)$ is Δ-dependent.

Here are some examples of Δ-coherent spaces of linear operators.

3.4.2. Proposition: *Let W, K, Δ be as described in Example 2.4.1, or in Example 2.4.2, or in Example 2.4.3. Then K is a Δ-coherent space of linear operators of W.*

Proof: Obviously $M(1 \times s;\, K)$ verifies condition (i) of Definition 3.4.1. The argument used in the proof of statement (*a*) of Proposition 2.4.5, proves that $M(1 \times s;\, K)$ verifies condition (ii) of Definition 3.4.1, Q.E.D.

3.4.3. Proposition: *Let W, K be as described in Example 2.4.1, or in Example 2.4.2, or in Example 2.4.3; let $\Delta = \mathrm{End}\ W$. Then K is a Δ-coherent space of linear operators of W* (regarding the meaning of End W-coherence, see Subsection 2.3).

PROOF: Obviously $M(1\times s; K)$ verifies condition (i) of Definition 3.4.1. Let $r > s$, and let $a_1, \ldots, a_r \in M(1\times s; K)$; by Proposition 3.4.2, $a_1, \ldots, a_r$ is a K-dependent family, hence $a_1, \ldots, a_r$ is also an End W-dependent family, Q.E.D.

3.4.4. PROPOSITION: *Let W be a finite-dimensional vector space; let K be a subspace of* End W; *let $\Delta =$* End W. *The following conditions are equivalent:*

(*a*) *K is a Δ-coherent space of linear operators of W.*

(*b*) *In K there exists an automorphism of W.*

PROOF: Let (*a*) hold. Since $M(1\times 1; K)$ verifies condition (i) of Definition 3.4.1, there exists $a \in K$ such that, for every non-zero element $d \in$ End W, we have $da \neq 0$; hence $a(W) = W$. Since dim $W < \infty$, then a is an automorphism of W.

Let (*b*) hold, and let $a \in K$ be an automorphism of W. The family $a_1 = [a, 0, \ldots, 0], \ldots, a_s = [0, \ldots, 0, a]$ of elements of $M(1\times s; K)$ is obviously Δ-independent. Let $r > s$, and let $a_1, \ldots, a_r$ be any family of elements of $M(1\times s; K)$; let $\psi\colon W^s \to W^r$ be the linear map defined by

$$\psi(w) = [a_1 w, \ldots, a_r w]^T .$$

Since dim $W^s <$ dim W^r, then ψ is not surjective; hence there exist $d_1, \ldots, d_r \in$ End W, not all zero, such that $d_1 a_1 + \ldots + d_r a_r = 0$, Q.E.D.

3.5. *Dependence theorems under coherence assumptions*

Let $G = (\{\varrho_1, \ldots, \varrho_n\}; V, I)$ be a semigraph with n branches. Let W be a vector space over R; let K, Δ be subspaces of End W.

3.5.1. LEMMA: *Let $(A_1, M_1), \ldots, (A_r, M_r) \in S(G)$. If K is Δ-coherent, then the following conditions are equivalent:*

(*a*) *$C_K(A_1, M_1), \ldots, C_K(A_r, M_r)$ is a Δ-dependent family of linear parametric K-components.*

(*b*) *$\langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in A_1, \varrho_\mu \in M_1\rangle, \ldots, \langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in A_r, \varrho_\mu \in M_r\rangle$ is a dependent family of subspaces of R^n.*

PROOF: Assume (*b*). By Proposition 3.3.5 there exist $j_1, \ldots, j_s$, with $1 \leq j_1 < \ldots < j_s \leq r$, such that

$$\dim\Big(\sum_{\nu=1}^{s} \langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in A_{j_\nu}, \varrho_\mu \in M_{j_\nu}\rangle\Big) < s .$$

Let $\boldsymbol{\theta}_{\lambda_1}, \ldots, \boldsymbol{\theta}_{\lambda_\zeta}, \boldsymbol{\gamma}_{\mu_1}, \ldots, \boldsymbol{\gamma}_{\mu_\eta}$ be a basis of

$$\sum_{\nu=1}^{s} \langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in \Lambda_{i_\nu}, \varrho_\mu \in M_{i_\nu} \rangle ;$$

obviously $\zeta + \eta < s$.

Let $\sigma_1 \in C_K(\Lambda_{i_1}, M_{i_1}), \ldots, \sigma_s \in C_K(\Lambda_{i_s}, M_{i_s})$. By Remark 3.1.2, there exists $\boldsymbol{A} \in M(s \times (\zeta + \eta); K)$ such that

$$\begin{bmatrix} \tilde{\boldsymbol{\sigma}}_1 \\ \vdots \\ \tilde{\boldsymbol{\sigma}}_s \end{bmatrix} = \boldsymbol{A} [\boldsymbol{\theta}_{\lambda_1}, \ldots, \boldsymbol{\theta}_{\lambda_\zeta}, \boldsymbol{\gamma}_{\mu_1}, \ldots, \boldsymbol{\gamma}_{\mu_\eta}]^T .$$

Since K is Δ-coherent, and $\zeta + \eta < s$, there exist $d_1, \ldots, d_s \in \Delta$, not all zero, such that $[d_1, \ldots, d_s]\boldsymbol{A} = \boldsymbol{0}$; then $d_1 \tilde{\boldsymbol{\sigma}}_1 + \ldots + d_s \tilde{\boldsymbol{\sigma}}_s = \boldsymbol{0}$; hence $d_1 \sigma_1 + \ldots + d_s \sigma_s = 0$. We have so proved that (*b*) implies (*a*).

Conversely, assume that the family described in (*b*) is independent. For $j = 1, \ldots, r$, there exists $\boldsymbol{\alpha}_j \in \langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in \Lambda_j, \varrho_\mu \in M_j \rangle$, such that the family $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r$ is linearly independent. Since K is Δ-coherent, then there exists a Δ-independent element $a \in K$. By Remark 3.1.2, we have $a\boldsymbol{\alpha}_1^T \in C_K^\sim(\Lambda_1, M_1), \ldots, a\boldsymbol{\alpha}_r^T \in C_K^\sim(\Lambda_r, M_r)$. Let $d_1, \ldots, d_r$ be elements of Δ such that $d_1 a \boldsymbol{\alpha}_1^T + \ldots + d_r a \boldsymbol{\alpha}_r^T = \boldsymbol{0}$. There exists a non singular $r \times r$ submatrix $\boldsymbol{M}$ of $[\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r]^T$; consequently we have

$$[d_1 a, \ldots, d_r a]\boldsymbol{M} = \boldsymbol{0}\, , \; [d_1 a, \ldots, d_r a] = \boldsymbol{0}\boldsymbol{M}^{-1} = \boldsymbol{0}\, , \; d_1 a = \ldots = d_r a = 0\, .$$

Since a is Δ-independent, we obtain $d_1 = \ldots = d_r = 0$; then $a\boldsymbol{\alpha}_1^T, \ldots, a\boldsymbol{\alpha}_r^T$ is a Δ-independent family. By Proposition 2.3.3, the family $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ is Δ-independent, Q.E.D.

3.5.2. Theorem: *Let* $(\Lambda_1, M_1), \ldots, (\Lambda_r, M_r) \in S(G)$. *If* K *is* Δ-*coherent, then the following conditions are equivalent:*

(*a*) *The family* $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ *is* Δ-*dependent.*

(*b*) *There exists* $J \subset \{1, \ldots, r\}$ *such that* $\deg \bigcup_{j \in J} (\Lambda_j, M_j) < |J|$.

Proof: By Proposition 3.2.5, for every $J \subset \{1, \ldots, r\}$, we have

$$\deg \bigcup_{j \in J} (\Lambda_j, M_j) = \dim \Big\langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in \bigcup_{j \in J} \Lambda_j, \varrho_\mu \in \bigcup_{j \in J} M_j \Big\rangle =$$
$$= \dim \sum_{j \in J} \langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in \Lambda_j, \varrho_\mu \in M_j \rangle\, .$$

Hence, by Proposition 3.3.5, condition (*b*) holds if and only if

$$\langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in \Lambda_1, \varrho_\mu \in M_1 \rangle, \ldots, \langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in \Lambda_r, \varrho_\mu \in M_r \rangle$$

is a dependent family of subspaces of R^n. Then our statement is a consequence of Lemma 3.5.1, Q.E.D.

3.5.3. Corollary: *Let* $(A_1, M_1), \ldots, (A_r, M_r) \in S(G)$. *Let* K *be* Δ*-coherent.*

(a) If $r > n$, *then the family* $C_K(A_1, M_1), \ldots, C_K(A_r, M_r)$ *is* Δ*-dependent.*

(b) If the family $C_K(A_1, M_1), \ldots, C_K(A_r, M_r)$ *is* Δ*-independent, and* $r < n$, *then there exist* $(A_{r+1}, M_{r+1}), \ldots, (A_n, M_n) \in S(G)$ *such that the family* $C_K(A_1, M_1), \ldots$ $\ldots, C_K(A_n, M_n)$ *is* Δ*-independent.*

Proof: The statement is a consequence of Lemma 3.5.1, Q.E.D.

3.5.4. Theorem: *Let* $(A_1, M_1), \ldots, (A_r, M_r) \in S(G)$; *let the family* $C_K(A_1, M_1), \ldots, C_K(A_r, M_r)$ *be* Δ*-dependent. If* K *is* Δ*-coherent, then:*

(a) There exist maximal Δ*-independent subfamilies of* $C_K(A_1, M_1), \ldots, C_K(A_r, M_r)$.

(b) There exists an integer b, *with* $0 \leq b \leq \min(r-1, n)$, *such that, if* $(C_K(A_j, M_j) | j \in J)$ *is a maximal* Δ*-independent subfamily of* $C_K(A_1, M_1), \ldots, C_K(A_r, M_r)$, *then* $|J| = b$.

(c) Let $(C_K(A_j, M_j) | j \in J)$ *be a maximal* Δ*-independent subfamily of* $C_K(A_1, M_1), \ldots, C_K(A_r, M_r)$. *For every* $v \in \{1, \ldots, r\} - J$, *the family* $(C_K(A_j, M_j) | j \in J \cup \{v\})$ *has a unique minimal* Δ*-dependent subfamily.*

Proof: The statement is a consequence of Lemma 3.5.1 and Proposition 3.3.3, Q.E.D.

3.5.5. Proposition: *Let* $(A_1, M_1), \ldots, (A_r, M_r) \in S(G)$; *let* $C_K(A_1, M_1), \ldots$ $\ldots, C_K(A_r, M_r)$ *be a minimal* Δ*-dependent family. If* K *is* Δ*-coherent, then:*

$$\deg \bigcup_{i=1}^{r} (A_i, M_i) = r - 1.$$

Proof: By Lemma 3.5.1,

$$\langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in A_1, \varrho_\mu \in M_1 \rangle, \ldots, \langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in A_r, \varrho_\mu \in M_r \rangle$$

is a minimal dependent family of subspaces of R^n. Since

$$\deg \bigcup_{i=1}^{r} (A_i, M_i) = \dim \sum_{i=1}^{r} \langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in A_i, \varrho_\mu \in M_i \rangle$$

(see the Proof of Theorem 3.5.2), the statement follows from Proposition 3.3.4, Q.E.D.

3.6. *Some remarks regarding coherence assumptions*

In Subsection 3.5 we proved that, if K is a Λ-coherent space of linear operators of W, then the Λ-dependent families of linear parametric K-components are characterized by the topological condition on their sockets described in (*b*) of Theorem 3.5.2. That characterization holds only if K is Λ-coherent, as shown by the following theorem.

3.6.1. Theorem: *Let W be a vector space over* $\mathbf{R}$; *let K, Λ be subspaces of* $\operatorname{End} W$.

Let, for every semigraph G and for every family $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ of linear parametric K-components of G, the following conditions be equivalent:

(*a*) *$C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ is a Λ-dependent family.*

(*b*) *There exists $J \subset \{1, \ldots, r\}$ such that* $\deg \bigcup_{i \in J} (\Lambda_i, M_i) < |J|$.

Then K is Λ-coherent.

Proof: Firstly, assume that there exists s such that every s-tuple of elements of $M(1 \times s; K)$ is Λ-dependent. Then every $a \in K$ is Λ-dependent.

Let $G = (\{\varrho_1, \ldots, \varrho_n\}; V, I)$ be a semigraph; let (Λ, M) be a socket of G, such that $\deg(\Lambda, M) = 1$. By Remark 3.1.2, there exists $\boldsymbol{\alpha} \in \mathbf{R}^n$ such that $C_K^{\sim}(\Lambda, M) = \{a\boldsymbol{\alpha}^T \mid a \in K\}$. Then, the family $C_K(\Lambda, M)$ verifies condition (*a*), but doesn't verify condition (*b*).

Secondly, assume that there exist s and a Λ-independent r-tuple $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ of elements of $M(1 \times s; K)$, with $r > s$.

Let $G = (\{\varrho_1, \ldots, \varrho_n\}; V, I)$ be a semigraph with $n \geq s$; let (Λ, M) be a socket of G such that $\deg(\Lambda, M) = s$. Write $(\Lambda_1, M_1) = \ldots = (\Lambda_r, M_r) = (\Lambda, M)$, and consider the family $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$.

Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s$ be a basis of $\langle \boldsymbol{\theta}_k, \boldsymbol{\gamma}_h \mid \varrho_k \in \Lambda, \varrho_h \in M \rangle$; let

$$\begin{bmatrix} \boldsymbol{\psi}_1 \\ \vdots \\ \boldsymbol{\psi}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_r \end{bmatrix} [\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s]^T.$$

By Remark 3.1.2 we have $\boldsymbol{\psi}_1 \in C_K^{\sim}(\Lambda_1, M_1), \ldots, \boldsymbol{\psi}_r \in C_K^{\sim}(\Lambda_r, M_r)$. Let $d_1, \ldots, d_r$ be elements of Λ such that $d_1\boldsymbol{\psi}_1 + \ldots + d_r\boldsymbol{\psi}_r = \mathbf{0}$. Let

$$[\delta_1, \ldots, \delta_s] = [d_1, \ldots, d_r] \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_r \end{bmatrix};$$

then $[\delta_1, \ldots, \delta_s][\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s]^T = \mathbf{0}$. There exists a non singular $s \times s$ submatrix $\boldsymbol{M}$ of $[\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s]^T$; consequently we have $[\delta_1, \ldots, \delta_s]\boldsymbol{M} = \mathbf{0}$, $[\delta_1, \ldots, \delta_s] = \mathbf{0}\boldsymbol{M}^{-1} = \mathbf{0}$. Since the family $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ is Λ-independent, then $d_1 = \ldots = d_r = 0$. Hence the family $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_r$ is Λ-independent.

By Proposition 2.3.3, the family $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ doesn't verify condition (a), but verifies condition (b), since $\deg \bigcup_{j=1}^{r} (\Lambda_j, M_j) = \deg (\Lambda, M) = s < r$, Q.E.D.

3.7. *Dependence and small changes*

Let $G = (\{\varrho_1, \ldots, \varrho_n\}; V, I)$ be a semigraph with n branches. Let W, K, Λ be as described in Example 2.4.1, or in Example 2.4.2, or in Example 2.4.3.

3.7.1. Definition: *Let* $(\Lambda, M) \in S(G)$. If $|\Lambda| + |M| = 1$, then (Λ, M) will be called an *elementary* socket.

3.7.2. Lemma: *Let*

$$(\Lambda_1, M_1), \ldots, (\Lambda_r, M_r) \in S(G) .$$

If the family $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ *is* Λ*-independent, then there exist elementary sockets* $(\Lambda_1', M_1'), \ldots, (\Lambda_r', M_r')$ *which verify the following conditions:*

(a) $(\Lambda_j', M_j') \subset (\Lambda_j, M_j)$, *for* $j = 1, \ldots, r$;

(b) $\deg \bigcup_{j=1}^{r} (\Lambda_j', M_j') = r$.

Proof: By Proposition 3.4.2, K is Λ-coherent, hence Lemma 3.5.2 applies to $(\Lambda_1, M_1), \ldots, (\Lambda_r, M_r)$; then

$$\langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in \Lambda_1, \varrho_\mu \in M_1 \rangle, \ldots, \langle \theta_\lambda, \gamma_\mu | \varrho_\lambda \in \Lambda_r, \varrho_\mu \in M_r \rangle$$

is an independent family of subspaces of R^n. By Proposition 3.3.2, there exists a linearly independent family $\beta_1, \ldots, \beta_r$ of elements of R^n, such that

$$\beta_j \in \{\theta_\lambda, \gamma_\mu | \varrho_\lambda \in \Lambda_j, \varrho_\mu \in M_j\} \qquad \text{for } j = 1, \ldots, r .$$

If $\beta_j = \theta_{\lambda_j}$, with $\varrho_{\lambda_j} \in \Lambda_j$, write $(\Lambda_j', M_j') = (\{\varrho_{\lambda_j}\}, \emptyset)$; if $\beta_j = \gamma_{\mu_j}$, with $\varrho_{\mu_j} \in M_j$, write $(\Lambda_j', M_j') = (\emptyset, \{\varrho_{\mu_j}\})$. Obviously the family $(\Lambda_1', M_1'), \ldots, (\Lambda_r', M_r')$ verifies conditions (a), (b), Q.E.D.

3.7.3. Theorem: *Let* $(\Lambda_1, M_1), \ldots, (\Lambda_r, M_r) \in S(G)$; *let* $\sigma_1, \ldots, \sigma_r$ *be a family of linear* K*-components on the sockets* $(\Lambda_1, M_1), \ldots, (\Lambda_r, M_r)$, *respectively. If the family* $C_K(\Lambda_1, M_1), \ldots, C_K(\Lambda_r, M_r)$ *is* Λ*-independent, then,*

(i) *for every family* $(\Lambda_1', M_1'), \ldots, (\Lambda_r', M_r')$ *of elementary sockets which verify conditions* (a), (b) *of Lemma* 3.7.2 *with respect to* $(\Lambda_1, M_1), \ldots, (\Lambda_r, M_r)$,

(ii) *for every family* $T_1, \ldots, T_r$ *of subsets of* $C_K(\Lambda_1', M_1'), \ldots, C_K(\Lambda_r', M_r')$, *respectively, such that* $|T_1| > 1, \ldots, |T_r| > 1$, *there exist* $\tau_1 \in T_1, \ldots, \tau_r \in T_r$ *such that the family* $\sigma_1 + \tau_1, \ldots, \sigma_r + \tau_r$ *of linear K-components on the sockets* $(A_1, M_1), \ldots, (A_r, M_r)$, *respectively, is* Δ*-independent.*

PROOF: Let $(A_1', M_1'), \ldots, (A_r', M_r')$ and $T_1, \ldots, T_r$ verify the conditions described in (i), (ii). For $j = 1, \ldots, r$, if $(A_j', M_j') = (\{\varrho_{\lambda_j}\}, \emptyset)$, write $\boldsymbol{a}_j = \boldsymbol{\theta}_{\lambda_j}$, if $(A_j', M_j') = (\emptyset, \{\varrho_{\mu_j}\})$, write $\boldsymbol{a}_j = \boldsymbol{\gamma}_{\mu_j}$. Let $\boldsymbol{a}_{r+1}, \ldots, \boldsymbol{a}_s$ be elements of $\mathbf{R}^n$ such that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r, \boldsymbol{a}_{r+1}, \ldots, \boldsymbol{a}_s$ is a basis of $\sum_{i=1}^{s} \langle \boldsymbol{\theta}_\lambda, \boldsymbol{\gamma}_\mu | \varrho_\lambda \in A_i, \varrho_\mu \in M_i \rangle$.

Let $\tilde{\boldsymbol{\sigma}}_1, \ldots, \tilde{\boldsymbol{\sigma}}_r \in M(1 \times n; K)$ be the matrices associated with $\sigma_1, \ldots, \sigma_r$. By Remark 3.1.2, there are elements $b_{j\nu} \in K$, such that $\tilde{\boldsymbol{\sigma}}_j = \sum_{\nu=1}^{s} b_{j\nu} \boldsymbol{a}_\nu^T$.

Let $T_1^\sim, \ldots, T_r^\sim$ be the sets of matrices associated with the elements of $T_1, \ldots, T_r$, respectively. For every $\tau \in T_j$, by Remark 3.1.2, there exists $b \in K$ such that $\tilde{\boldsymbol{\tau}} = b\boldsymbol{a}_j^T$.

We contend that, for $h = 1, \ldots, r$, there exists a family $b_1, \ldots, b_h$ of elements of K, which verifies the following conditions:

(*a*) $b_1 \boldsymbol{a}_1^T \in T_1^\sim, \ldots, b_h \boldsymbol{a}_h^T \in T_h^\sim$,

(*b*) the family of the rows of

$$\begin{bmatrix} b_{11} & \cdots & b_{1h} \\ \vdots & \ddots & \vdots \\ b_{h1} & \cdots & b_{hh} \end{bmatrix} + \operatorname{diag}(b_1, \ldots, b_h)$$

is Δ-independent.

Let us prove this statement by induction on h. If $h = 1$, the statement is trivial, since $\Delta = K$ is a domain of integrity, and $|T_1| > 1$. Let m be an integer such that $1 \leq m < r$; let us assume the statement proved for $h = m$.

Let the family $b_1, \ldots, b_m$ verify conditions (*a*), (*b*) for $h = m$. For $j = 1, \ldots, m$, let $\boldsymbol{b}_j$ be the j-th row of

$$\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{bmatrix} + \operatorname{diag}(b_1, \ldots, b_m);$$

let $\boldsymbol{b}_{m+1} = [b_{m+1,1}, \ldots, b_{m+1,m}]$. Let

$$H = \left\{ d_{m+1} \in \Delta \mid \text{there exist } d_1, \ldots, d_m \in \Delta \text{ such that } \sum_{j=1}^{m+1} d_j \boldsymbol{b}_j = \mathbf{0} \right\}.$$

Obviously, H is a left ideal of Δ. Since Δ is a principal ideal domain, there exists $d_{m+1}' \in \Delta$ such that $H = \Delta d_{m+1}'$. Since K is Δ-coherent (see Proposition 3.4.2), and $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$ is a Δ-independent family of elements of $M(1 \times m; K)$, then $H \neq \{0\}$; hence $d_{m+1}' \neq 0$. Let $d_1', \ldots, d_m'$ be elements of Δ such that

$\sum_{j=1}^{m+1} d_j' \boldsymbol{b}_j = \mathbf{0}$. Let $d_1, \ldots, d_{m+1}$ be elements of Δ such that $\sum_{j=1}^{m+1} d_j \boldsymbol{b}_j = \mathbf{0}$; there exists $\delta \in \Delta$ such that $d_{m+1} = \delta d_{m+1}'$; hence $\sum_{j=1}^{m} d_j \boldsymbol{b}_j = \sum_{j=1}^{m} \delta d_j' \boldsymbol{b}_j$. Since $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m$ is a Δ-independent family, then, for $j = 1, \ldots, m$, we have $d_j = \delta d_j'$; hence

$$[d_1, \ldots, d_{m+1}] = \delta [d_1', \ldots, d_{m+1}'] .$$

For $j = 1, \ldots, m$, let $\boldsymbol{b}_j'$ be the j-th row of

$$\begin{bmatrix} b_{11} & \cdots & b_{1m} & b_{1,m+1} \\ \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ b_{m1} & \cdots & b_{mm} & b_{m,m+1} \end{bmatrix} + [\operatorname{diag}(b_1, \ldots, b_m) | \mathbf{0}] .$$

If, for every $b \in K$, the family

$$\boldsymbol{b}_1', \ldots, \boldsymbol{b}_m', \boldsymbol{b}_{m+1}' = [b_{m+1,1}, \ldots, b_{m+1,m}, b]$$

is Δ-independent, then, for every $b_{m+1} \in K$ such that $b_{m+1} \boldsymbol{a}_{m+1}^T \in T_{m+1}^{\sim}$, the family $b_1, \ldots, b_m, b_{m+1}$ verifies conditions (a), (b) for $h = m+1$. If there exists $b \in K$ such that the family

$$\boldsymbol{b}_1', \ldots, \boldsymbol{b}_m', \boldsymbol{b}_{m+1}' = [b_{m+1,1}, \ldots, b_{m+1,m}, b]$$

is Δ-dependent, then there exists $\delta \in \Delta$, $\delta \neq 0$, such that $\sum_{j=1}^{m+1} \delta d_j' \boldsymbol{b}_j' = \mathbf{0}$; in particular

$$\delta \Big(\sum_{j=1}^{m} d_j' b_{j,m+1} + d_{m+1}' b \Big) = 0 ;$$

since $\Delta = K$ has no zero-divisor, and $\delta \neq 0$, then $d_{m+1}' b = - \sum_{j=1}^{m} d_j' b_{j,m+1}$. Since $|T_{m+1}| > 1$, $d_{m+1}' \neq 0$, and $\Delta = K$ has no zero-divisor, there exists $b_{m+1} \in K$ such that $b_{m+1} \boldsymbol{a}_{m+1}^T \in T_{m+1}^{\sim}$, and that

$$d_{m+1}' (b_{m+1,m+1} + b_{m+1}) \neq - \sum_{j=1}^{m} d_j' b_{j,m+1} .$$

Then the family $b_1, \ldots, b_m, b_{m+1}$ verifies conditions (a), (b) for $h = m+1$.

The previous argument proves that there exist $b_1, \ldots, b_r \in K$ such that $b_j \boldsymbol{a}_j^T \in T_j^{\sim}$, for $j = 1, \ldots, r$, and that the family of the rows of

$$\begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \vdots\vdots\vdots & \vdots \\ b_{r1} & \cdots & b_{rr} \end{bmatrix} + \operatorname{diag}(b_1, \ldots, b_r)$$

is Δ-indepenent.

For $j = 1, \ldots, r$, let τ_j be the element of T_j such that $\tilde{\boldsymbol{\tau}}_j = b_j \boldsymbol{a}_j^T$. Let $d_1, \ldots, d_r$ be elements of Δ such that $\sum_{j=1}^{r} d_j(\sigma_j + \tau_j) = 0$. Since $\sum_{j=1}^{r} d_j(\tilde{\boldsymbol{\sigma}}_j + \tilde{\boldsymbol{\tau}}_j) = \mathbf{0}$, we have

$$[d_1, \ldots, d_r]\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots\vdots\vdots & \vdots \\ b_{r1} & \cdots & b_{rn} \end{bmatrix} + [\operatorname{diag}(b_1, \ldots, b_r)|\mathbf{0}, \ldots, \mathbf{0}]\right)\begin{bmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_r^T \end{bmatrix} =$$

$$= \mathbf{0} \in M(1 \times n; K).$$

Since the family $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ is linearly independent, there exists $\boldsymbol{N} \in M(n \times s; \mathrm{R})$ such that

$$\begin{bmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_r^T \end{bmatrix} \boldsymbol{N} = \boldsymbol{I} \in M(s \times s; \mathrm{R}).$$

Then

$$[d_1, \ldots, d_r]\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots\vdots\vdots & \vdots \\ b_{r1} & \cdots & b_{rn} \end{bmatrix} + [\operatorname{diag}(b_1, \ldots, b_r)|\mathbf{0}, \ldots, \mathbf{0}]\right)\begin{bmatrix} \boldsymbol{a}_1^T \\ \vdots \\ \boldsymbol{a}_r^T \end{bmatrix} \boldsymbol{N} = \mathbf{0}\boldsymbol{N},$$

$$[d_1, \ldots, d_r]\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \vdots\vdots\vdots & \vdots \\ b_{r1} & \cdots & b_{rn} \end{bmatrix} + [\operatorname{diag}(b_1, \ldots, b_r)|\mathbf{0}, \ldots, \mathbf{0}]\right) = \mathbf{0},$$

$$[d_1, \ldots, d_r]\left(\begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \vdots\vdots\vdots & \vdots \\ b_{r1} & \cdots & b_{rr} \end{bmatrix} + \operatorname{diag}(b_1, \ldots, b_r)\right) = \mathbf{0} \in M(1 \times r; K).$$

Since the family of the rows of

$$\begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \vdots\vdots\vdots & \vdots \\ b_{r1} & \cdots & b_{rr} \end{bmatrix} + \operatorname{diag}(b_1, \ldots, b_r)$$

is Δ-independent, then $d_1 = \ldots = d_r = 0$. Hence the family $\sigma_1 + \tau_1, \ldots, \sigma_r + \tau_r$ is Δ-independent, Q.E.D.

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