Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
104° (1986), Vol. X, fasc. 20, pagg. 223-239

CLELIA MARCHIONNA (*)

Motion of a Vibrating String with an End Point Fixed on a Continuous Obstacle (**)

Moto di una corda vibrante vincolata ad un estremo contro un ostacolo continuo

SUNTO. — Si consideri il problema dell'urto di una corda contro una parete. È noto che tale problema presenta difficoltà diverse a seconda che la parete sia concava o convessa, o se è presente una forza esterna che tende ad avvicinare la corda all'ostacolo. Nei molti lavori sull'argomento è stata posta inoltre l'ipotesi che gli estremi della corda siano *staccati* dall'ostacolo.

In questa nota si mostra in quali ipotesi (cfr. anche [6]) si possa studiare il problema della corda vincolata ad un estremo su un ostacolo continuo in modo sostanzialmente analogo a quello dei lavori precedenti.

Si illustra inoltre con un controesempio (in presenza di ostacolo *piano* e di forza esterna *nulla*) come si possa presentare il fenomeno di infiniti archi d'urto in tempo finito, se non si richiede la monotonia a tratti della traccia della soluzione sulle caratteristiche.

1. - INTRODUCTION

The motion of a finite string, vibrating against a rigid wall, is generally studied with the hypothesis that the end points never touch the obstacle. An exception is given by paper [5], assuming however very particular initial conditions (convex parabolic obstacle, string plucked at its midpoint and with vanishing initial velocity).

In the present note we show that, under suitable hypotheses, the theory in [1], [2], [6] can be adapted in order to obtain the solution also in this more general case. Furthermore we present a critical example of a infinite number of impact arcs in a finite time, that gives evidence to the cases that produce failure in the usual extension techniques.

(*) Dipartimento di Matematica del Politecnico di Milano. L'autrice aderisce al G.N.A.F.A. del C.N.R.

(**) Memoria presentata il 21 febbraio 1986 da Luigi Amerio, uno dei XL. Lavoro parzialmente finanziato con fondi M.P.I.

ISSN-0392-4106

We study the following problem in the semistrip

$$Z = \{(x, t) | 0 < x < l, 0 < t < +\infty\}. \tag{1.1}$$

We look for $y(x, t)$ satisfying the conditions

$$\square y = 2F(x, t) + 2J \quad \text{in } \mathfrak{D}'(\mathring{Z}), \tag{1.2}$$

$$y(x, 0) = A_0(x) > 0, \quad 0 < x < l, \; A_0 \in W^{1,1}(0, l), \tag{1.3}$$

$$y_t(x, 0) = A_1(x), \quad A_1 \in L^1(0, l), \tag{1.4}$$

$$y(0, t) = 0, \quad y(l, t) = k > 0, \tag{1.5}$$

$$y > 0 \quad \text{in } Z, \quad y \in C^0(Z), \tag{1.6}$$

$$J > 0, \quad \operatorname{Supp} J \subset \{(x, t) | y(x, t) = 0\}; \tag{1.7}$$

Moreover, let us set the compatibility conditions

$$A_0(0) = 0, \quad A_0(l) = k, \tag{1.8}$$

and require that

(1.9) y satisfies the extension laws with respect to the elementary problems of Cauchy, Darboux and Goursat (see for ex. [2], § 2).

To this problem we can reduce also the problem with a more general obstacle where the unilateral condition is $y > \psi(x)$, transferring by the change $Y = y - \psi(x)$, the effect of the obstacle on the force F.

We can study various cases, corresponding to different assumptions on F; for ex. $F = 0$ (see [1], [2]), $F \in C^0(Z)$, $F < 0$ or that it changes sign (see [6]), $F = f + \sum_1^n \alpha_k \delta_{x_k}$ with $f \in C^0(Z)$ or $f(x) \in L^1(0, l)$ (see [7]), posing the relevant hypotheses on data and on the admissible functions set where we look for the solution.

The construction of the first line of influence and its properties, is as in [1], [2], with the only difference that the origin $(0, 0)$ can belong to it.

Let $P_0 = (0, t_0)$ the intersection of the first line of influence with the t axis.

The above mentioned works allow to extend the solution in the curved triangle T_0 in fig. 1.1 (or part of it, if the force is directed towards the wall). It is sufficient to guarantee the extension in a characteristic triangle T_1 with the lower vertex in P_0, in order to extend the solution beyond the first line of influence.

We shall consider first the case $F = 0$ (§ 2) and then the case of F changing sign. The example shown in § 3 justifies the hypothesis (1.12) on the set of admissible functions to which the solution belongs.

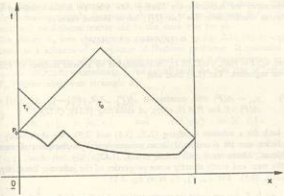

Fig. 1.1

Let $\xi = (x + t)/\sqrt{2}$, $\eta = (-x + t)/\sqrt{2}$ be the characteristic coordinates and $f(\xi, \eta) = F(x, t)$. In characteristic form (1.2) becomes

$$y_{\xi\eta} = f(\xi, \eta) + J \,. \tag{1.10}$$

y belongs to the following set of admissible functions

(1.11) $y \in C^0(Z)$, $y(\xi, \eta)$ is ξ-absolutely continuous $\forall \eta$ and η-absolutely continuous $\forall \xi$;

(1.12) let r_ξ be an arbitrary characteristic segment $\eta = \eta_1$, $\xi_0 < \xi < \xi_1$ in Z. It is then possible to divide r_ξ in a finite number of segments where $y(\xi, \eta_1)$ is strictly increasing (↑), or strictly decreasing (↓) or constant; the same property is true also on the η-characteristic segments.

If $f = 0$, these hypotheses are always satisfied by the solution, under the single condition that the data belong to a suitable class. If f changes sign, y is required to belong to the smaller admissible class described in [6], § 2.

2. - String not subject to external forces

We study first the case

$$f(\xi, \eta) = 0 \,. \tag{2.1}$$

Furthermore we substitute the Cauchy data with an initial condition on a piecewise characteristic line (see [2]); let us assume exactly

$$Z = \{(x, t) \mid 0 < x < l,\ t > \tau(x)\} \tag{2.2}$$

where $\sigma_0 : t = \tau(x)$, $\tau \in C_0[0, l]$, is constituted by a finite number of characteristic segments. Let (1.5) hold and

(2.3) $\quad y|_{\sigma_0} = A(P)$ with continuous $A(P)$, $A(0, \tau(0)) = 0$, $A(l, \tau(l)) = k$, $A(P) > 0$ for $P \neq (0, \tau(0))$, A satisfying (1.11), (1.12).

We look for a solution satisfying (2.3), (1.6) and (1.9). We shall verify that a solution can be constructed in an unique way by subsequent elementary problems. Moreover it will satisfy (1.11), (1.12).

We state and verify directly some properties of the influence lines, implied by (2.3), (1.5). Let T_0, T_1, T_2 as in fig. 2.1:

a) If T_0 has non empty intersection with the first line of influence, then the latter is constituted by a finite number of space-like arcs and characteristic segments. Furthermore the solution in T_0 satisfies (1.11), (1.12).

Let $\zeta(\xi, \eta)$ be the solution of the free problem ($\zeta_{\xi\eta} = 0$) in T_0, and γ_0 the first line of influence.

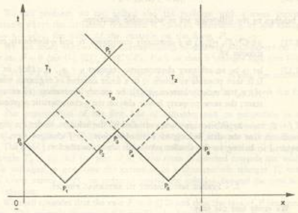

Fig. 2.1

We divide σ_0 by a finite number of points P_k, so that the datum $A(P)$ is stricly increasing (strictly decreasing, constant) in any segment P_kP_{k+1} (with respect to ξ on ξ-characteristics and in the same way for η).

We divide T_0 into characteristic rectangles as in fig. 2.1: $\chi(\xi, \eta)$ can be obtained as a solution of a sequence of Darboux problems. It maintains the monotonicity property of the data with respect to ξ on ξ-characteristics, and so for η. As $A(P)>0$, constructing χ, we could meet the first line of influence in T_0 if these exists some rectangle R_i

$$(2.4)\qquad \begin{cases} R_i=[\xi_i, \xi_j]\times[\eta_i, \eta_j] \text{ with} \\ \chi(\xi_i, \eta)\updownarrow \quad \text{in } [\eta_i, \eta_j] \qquad \text{and} \qquad \chi(\xi, \eta_i)\updownarrow \quad \text{in } [\xi_i, \xi_j]. \end{cases}$$

If both data are >0, strictly decreasing, and $\chi(\xi_j, \eta_j)<0$, there exists $\xi', \xi_i< <\xi'<\xi_j$, such that $\chi(\xi', \eta_j)=0$, and $\eta', \eta_i<\eta'<\eta_j$, such that $\chi(\xi_j, \eta')=0$ (fig. 2.2).

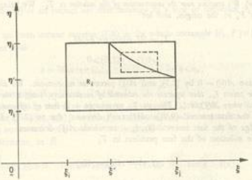

Fig. 2.2

By applying Dini's theorem (in monotonicity hypothesis) into the rectangle $[\xi', \xi_j]\times[\eta', \eta_j]$, we observe that $\chi(\xi, \eta)=0$ implicitly defines a line

$$\eta=\eta(\xi) \quad (\xi=\xi(\eta)), \quad \eta\in C^0(\xi', \xi_j), \quad \eta(\xi)\downarrow \text{ in } (\xi', \xi_j).$$

Such arc will be an *impact arc* $\in\gamma_0$.

Let instead be $\chi(\xi, \eta_i)\downarrow$ and $\chi(\xi_i, \eta)$ constant: then, if $\chi(\bar{\xi}, \eta_i)=0$, $\chi(\xi, \eta)$ changes sign in R_i on the characteristic segment $\xi=\bar{\xi}$, that will belong to γ_0.

Hence by construction, γ_0 is constitued by a finite number of arcs and segments in T_0. Furthermore in T_0 only one line of influence can be found (see [2]).

By constructing the solution above the first line of influence, we calculate first y in R_i beyond the impact arc:

$$(2.5)\qquad y(\xi,\eta)=z(\xi,\eta(\xi))+z(\xi(\eta),\eta)+z(\xi(\eta),\eta(\xi))-z(\xi(\eta),\eta(\xi)).$$

Being $\xi(\eta)$, $\eta(\xi)$ decreasing, we have

$$(2.6)\qquad y(\xi_i,\eta)\uparrow \quad \text{in } [\eta',\eta_i]\,, \qquad y(\xi,\eta_i)\uparrow \quad \text{in } [\xi',\xi_i]\,.$$

So we obtain y on a piecewise characteristic line, partially coinciding and in part lying above γ_0, and $y(\xi,\eta)$ is piecewise monotonic on this line. It can be so extended to the whole of T_0 by elementary problems, maintaining (1.11), (1.12).

b) We consider now the construction of the solution in T_1. We bring for simplicity P_0 to the origin, and let

$$T_1=\{(\xi,\eta)|0<\eta<\xi<a\}\,,$$

$$(2.7)\qquad A(\xi)=y(\xi,0)>0\,.$$

We have $A(0)=0$ by (2.3), and $A(\xi)$ piecewise monotonic. *We divide* $[0,a]$ *with* m *points* ξ_i, *that separate the intervals of monotonicity, being* n *the (separate) intervals where* $A(\xi)$ *in* $\downarrow$. *Then, in* T_1, *at most* $(n+1)$ *lines of influence lie.*

In the first interval $(0,\xi_1)$, $A(\xi)$ can't decrease, due to (2.7). Let ξ_k the left edge of the first interval (ξ_k,ξ_{k+1}) in which $A(\xi)$ decreases.

The solution of the free problem in T_1

$$(2.8)\qquad z(\xi,\eta)=A(\xi)-A(\eta)$$

satisfies the unilateral condition in $0<\eta<\xi<\xi_k$, and is $\not\uparrow$ (not increasing) with respect to η and $\not\downarrow$ with respect to ξ. Hence it is there $z(\xi,\eta)=y(\xi,\eta)$.

Let $R_k=[\xi_k,\xi_{k+1}]\times[0,\xi_k]$; in R_k is

$$(2.9)\qquad \left|\begin{array}{ll} z(\xi,\eta)=A(\xi)+z(\xi_k,\eta)-A(\xi_k)\,, & \\ z(\xi,\eta)\downarrow & \text{with respect to } \xi\,,\ \forall\eta\,, \\ z(\xi,\eta)\not\uparrow & \text{with respect to } \eta\,,\ \forall\xi\,, \end{array}\right.$$

$z(\xi,\eta)$ does not satisfy (1.6) in R_k.

The segment $[0,\xi_k]$ can be divided by $(k-1)$ points so that at any subinterval, $z(\xi_k,\eta)$ is either $\downarrow$ or constant.

Let

$$\eta_2 = \sup\{\eta \in [0, \xi_k] \mid y(\xi_k, \eta) > 0\}. \tag{2.10}$$

Obviously $y(\xi_k, \eta_2) = 0$, $0 < \eta_2 < \xi_k$.

By (2.10) η_2 is the right extremity of a segment $[\eta_1, \eta_2]$ with $y(\xi_k, \eta)\downarrow$ in $[\eta_1, \eta_2]$. Moreover we have, by (2.9),

$$z(\xi, \eta_2) < 0, \qquad \xi_k < \xi < \xi_{k+1}, \tag{2.11}$$

$$z(\xi, \eta_1) > 0, \qquad \xi_k < \xi < \xi' < \xi_{k+1}. \tag{2.12}$$

Because of the monotonicity with respect to ξ and η, $z(\xi, \eta) = 0$ implicitly defines in $R' = [\xi_k, \xi'] \times [\eta_1, \eta_2]$ a line

$\eta = \eta(\xi)$ with $\eta(\xi)$ continuous and $\downarrow$ in $[\xi_k, \xi']$.

$\eta = \eta(\xi)$ is an impact arc that belongs to a line of influence γ_1.

The free solution instead satisfies (1.6) in the whole rectangle $[\xi_k, \xi'] \times [0, \eta_1]$.

We set now

$$\bar{\eta} = \operatorname{Max}\{\eta \mid z(\xi_{k+1}, \eta) > 0\}, \qquad 0 < \bar{\eta} < \eta_2, \tag{2.13}$$

and the rectangle $\bar{R} = [\xi_k, \xi_{k+1}] \times [\bar{\eta}, \eta_2]$.

We observe, by the same arguments as at point a), that $z(\xi, \eta) = 0$ defines in $\bar{R}$ a line $\xi = \xi(\eta)$, $\xi(\eta)\uparrow$ in $[\bar{\eta}, \eta_2]$, constituted by a finite number of space-like arcs connected by η-characteristic segments.

Furthermore, in $\bar{R}$:

$$z(\xi, \eta) > 0 \quad \text{for } \xi < \xi(\eta), \qquad z(\xi, \eta) < 0 \quad \text{for } \xi > \xi(\eta).$$

Then $\xi = \xi(\eta)$ is part of the line of influence γ_1 (fig. 2.3).

We calculate y on the segment $\eta = \eta_2$, $\xi \in [\xi_k, \xi_{k+1}]$; we have (see fig. 2.3)

$$y(P) = y(P_0), \qquad y(\xi, \eta_2)\uparrow \quad \text{in } [\xi_k, \xi_{k+1}]. \tag{2.14}$$

but, due to (2.10), $y(\xi, \eta_2) = y(\xi, \xi_k)$ in $[\xi_k, \xi_{k+1}]$, and consequently also the solution of the Goursat problem in $\xi_k < \eta < \xi < \xi_{k+1}$ is > 0.

We observe that the already known results on lines of influence guarantee that no impact point can lie in the characteristic curved triangle above a line of influence, but nothing was said on the existence of an impact point in $\xi_k < \eta < \xi < \xi_{k+1}$.

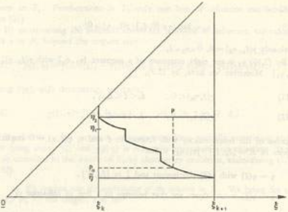

Fig. 2.3

Let us conclude the construction of γ_1:

The segment $\xi=\xi_k$, $\eta_1<\eta<\xi_k$ and the line $\xi=\xi(\eta)$ belong to γ_1; if $z(\xi,\bar\eta)$ is >0 in $\xi_{k+1}<\xi<a$, then γ_1 is completed in T_1 by the segment $\eta=\bar\eta$, $\xi>\xi_{k+1}$.

On the contrary, if $z(\xi,\bar\eta)$ assumes also negative values, γ_1 will contain other impact arcs but *always in correspondence to intervals* $\xi_i<\xi<\xi_{i+1}$ where $A(\xi)\downarrow$ (fig. 2.4).

Furthermore we observe, with an argument similar to (2.14), that $y(\xi,\bar\eta)$, $\xi>\xi_{k+1}$, is still piecewise monotonic. These intervals are not necessarily as many as the ones in $A(\xi)$ for $\xi>\xi_{k+1}$; nevertheless the decreasing intervals of $y(\xi,\bar\eta)$ are, *at most*, as many as the ones of $A(\xi)\downarrow$, and each contained in a decreasing interval of $A(\xi)$.

Now we can construct the possible second line of influence. We calculate, by means of a Darboux problem

(2.15) $$y(\xi,\xi_k)=y(\xi_{k+1},\xi_k)+y(\xi,\bar\eta)>0 \qquad \text{in } \xi>\xi_{k+1}\,.$$

Let

(2.16) $$y(\xi,\xi_k)=A_1(\xi)\,, \qquad \xi_k<\xi\,;$$

If $A(\xi)$ has n decreasing intervals, $A_1(\xi)$ will be $\downarrow$ at most in $(n-1)$ intervals. In correspondence of the first interval in which $A_1(\xi)\downarrow$, we shall construct a second line of influence γ_2 and so on.

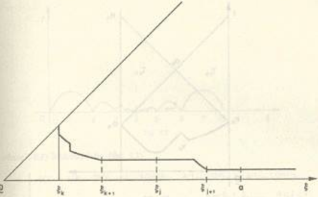

Fig. 2.4

Moreover we observe that γ_0 constructed in *a*), if it exists, either intersects T_1 before γ_1 (possibly at $(0, 0)$), or connects with γ_1. (On the contrary two lines of influence should intersect in Z, what is absurd).

Finally, repeating at most n times the construction procedure, we obtain al most $(n + 1)$ lines of influence in T_1.

c) *Let* $B(\eta) = y(\xi, \eta)|_{\Gamma_2\Gamma_3}$ (see fig. 2.1). $B(\eta)$ *is piecewise monotonic and let* m *be the number of decreasing intervals of* $B(\eta)$. *Then in* T_2 *can lie at most* $(m + 1)$ *lines of influence.*

The monotonicity properties of the solution of the Goursat problem can be applied as in *b*). In fact if the datum $B(\eta)$ is monotonic and the datum on $x = l$ is constant, then z is monotonic with respect to η with the same direction, and with the opposite direction with respect to ξ. So we can prove the property as in *b*).

d) *Let* $(\xi, \eta) \in \mathring{Z}$. *Then* $y(\xi, \eta)$ *is obtained in a unique way by extending the solution beyond a finite number of lines* γ_n *of influence.*

If only a finite number of γ_n exists, the statement is proved. On the contrary we suppose that γ_n are infinite and let $P_n = (\xi_n, \xi_n)$ be the intersection of γ_n with $x = 0$, Q_n the intersection with $x = l$, G_n, H_n and the sets T_{0n}, T_{1n}, T_{2n} as in fig. 2.5.

$\{\xi_n\}$ is a infinite increasing sequence: if $\{\xi_n\} \to +\infty$, we can meet the point (ξ, η) in a finite number of steps. On the contrary let $\xi_n \to \bar{\xi} < +\infty$: than $\forall \varepsilon > 0$, $\exists \bar{n}$, such that, $\xi_{\bar{n}} + \varepsilon > \bar{\xi}$.

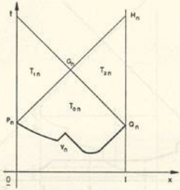

Fig. 2.5

Let us extend the solution beyond γ_k; in T_{0k} no lines of influence are found; y is piecewise monotonic in P_kG_k. Consequently, at most a finite number of lines of influence are found in T_k. If $\overline{P_kG_k} > \varepsilon$, there exist an infinity of n such that $\xi_n > \bar{\xi}$, against the hypothesis.

If $\overline{P_kG_k} < \varepsilon$, we extend y on G_kH_k (which is possible by constructing a *finite* number of lines of influence in T_{2k}). In this way y is piecewise monotonic on P_kH_k, which is again absurd.

Finally it is known that the property (1.11) of the datum on σ_0 is maintained by extending the solution beyond a line of influence.

3. - Example of infinite impact arcs with cluster point at finite

We consider the problem in the halfstrip Z introduced in (2.2) with $l = 8\sqrt{2}$,

$$y_{t\eta} = J, \tag{3.1}$$

y satisfying (1.5), with $y(8\sqrt{2}, t) = 8$, (1.6), (1.7), (1.9) and (2.3) with

$$\sigma_0 = \{(\xi, \eta) | \eta = -8, 0 < \xi < 8 \text{ or } \xi = 0, -8 < \eta < 0\},$$

and the data on σ_0

$$\begin{cases} y(\xi, -8) = A(\xi) + 8\,, & 0 < \xi < 8\,, \\ y(0, \eta) \quad = -\eta\,, & -8 < \eta < 0\,, \end{cases} \tag{3.3}$$

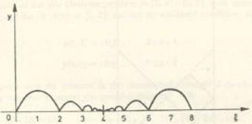

Fig. 3.1

where $A(\xi)$ is defined by (fig. 3.1)

$$(3.4)\quad \left|\begin{array}{ll} A(\xi)=1/2^n-2^n(\xi-(4-3/2^n))^2\,, & 4(1-1/2^n)<\xi<4(1-1/2^{n+1})\,, \\ & n=0,1,2,\ldots\,,\quad 0<\xi<4\,, \\ A(4)=0\,, & \\ A(\xi)=A(8-\xi)\,,\quad 4<\xi<8\,. & \end{array}\right.$$

We observe that $A(\xi)$ is *absolutely continuous*, but *does not satisfy* (1.12); in fact $\xi=4$ is a *cluster point of infinite oscillations.*

The solution y coincides in R_0 (see fig. 3.2) with the free solution:

$$(3.5)\qquad y(\xi,\eta)=z(\xi,\eta)=A(\xi)-\eta>0 \qquad \text{in } R_0\,.$$

We can note that this problem in Z corresponds to the most usual one with initial condition for $t=0$:

$$(3.6)\quad \left|\begin{array}{l} y(x,0)=y(\xi,-\xi)=A(\xi)+\xi=A(x/\sqrt{2})+x/\sqrt{2}\,,\qquad 0<x<8\sqrt{2}\,, \\ y_t(x,0)=(z_\xi(\xi,-\xi)+z_\eta(\xi,-\xi))/\sqrt{2}=(A'(\xi)-1)/\sqrt{2}= \\ \qquad\qquad =(A'(x/\sqrt{2})-1)/\sqrt{2}\,. \end{array}\right.$$

Data for problems in T_1 and T_2 are respectively:

$$(3.7)\qquad y(\xi,0)=A(\xi)>0\,,\qquad 0<\xi<8\,,$$

$$(3.8)\qquad y(8,\eta)=-\eta>0\,,\qquad -8<\eta<0\,.$$

By (3.8) we immediatly have

$$(3.9)\qquad y(\xi,\eta)=-\eta+8+(\xi-16)=(\xi-\eta)-8>0 \qquad \text{in } T_2\,.$$

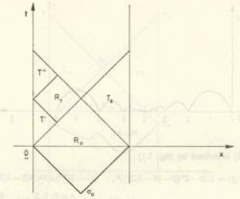

Fig. 3.2

We divide the T_1 triangle into three domains

(3.10) $$T' = \{0<\eta<\xi<4\}, \ R_1 = [4, 8] \times [0, 4], \ T'' = \{4<\eta<\xi<8\}.$$

and we construct y in T'. Let us consider firstly the Goursat problem for $0<\xi<2$; we have

(3.11) $$A(\xi) = 1-(\xi-1)^2, \qquad 0<\xi<2.$$

(3.12) $$z(\xi, \eta) = A(\xi) - A(\eta) = (\eta-\xi)(\xi+\eta-2).$$

Consequently $\eta = 2-\xi$, $1<\xi<2$ is the equation of an impact arc. The first line of influence is then

(3.13) $$\gamma_1 = \{\eta = 2-\xi \mid 1<\xi<2\} \cup \{\eta = 0 \mid \xi>2\}.$$

We calculate, by means of (2.14)

$$y(\xi, 1) = z(1, 2-\xi) = (\xi-1)^2, \qquad 1<\xi<2,$$

$$y(2, \eta) = A(2-\eta) = A(\eta), \qquad 0<\eta<1,$$

amd then, by a Goursat problem

$$y(2, \eta) = 1-(\eta-1)^2 = A(\eta), \qquad 1<\eta<2.$$

The solution of the free Darboux problem in $[2,4]\times[0,2]$, with datum $A(\xi)$ on $[2,4]$ and datum $A(\eta)$ on $[0,2]$, satisfies the unilateral condition. In detail we have

$$y(\xi,2)=A(\xi)\,,\qquad 2<\xi<4\,, \tag{3.14}$$

$$y(4,\eta)=A(\eta)\,,\qquad 0<\eta<2\,. \tag{3.15}$$

We construct now the solution in the characteristic triangle $2<\eta<\xi<4$: we can carry out (for example) the transformation:

$$\xi'=2(\xi-2)\,,\qquad \eta'=2(\eta-2)\,,\qquad Y(\xi',\eta')=2y(\xi',\eta')\,. \tag{3.16}$$

The differential equation $Y_{\xi'\eta'}=f$ remains unchanged, as the datum vanishes on $\xi'=\eta'$; moreover we have, by (3.4)

$$\begin{cases} y(\xi',0)=A(\xi)=A(2(\xi-2))/2=A(\xi')/2\,,\\ Y(\xi',0)=A(\xi')\,,\qquad 0<\xi'<4\,. \end{cases} \tag{3.17}$$

Then we repeat the previous argument and, coming back to coordinates ξ and η, we obtain the impact arc

$$\eta=5-\xi\,,\qquad 2+1/2<\xi<3\,, \tag{3.18}$$

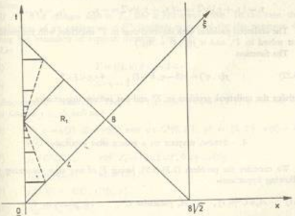

Fig. 3.3

ed

$$y(4, \eta) = A(\eta) , \qquad 2<\eta<3 .$$

By repeating the procedure infinite times, we obtain infinite impact segments (with edges respectively on the straight lines $\eta = \xi$ and $\eta = 2(\xi - 2)$), of equations

$$\xi + \eta = 2(4 - 3/2^n) , \qquad 4 - 3/2^n < \xi < 4(1 - 1/2^{n+1}) . \tag{3.19}$$

Furthermore we obtain

$$y(4, \eta) = A(\eta) , \qquad 0<\eta<4 . \tag{3.20}$$

Point $(4, 4)$ is therefore an impact arcs cluster point. Due to continuity, it must be $y(4, 4) = 0$.

The solution in R_1 (see (3.10)), with data $A(\xi)$ on $\eta = 0$ and $A(\eta)$ on $\xi = 4$, is calculated by means of a Darboux problem and is

$$y(\xi, \eta) = z(\xi, \eta) = A(\xi) + A(\eta) > 0 . \tag{3.21}$$

Now we consider the triangle T'': Datum is $y(\xi, 4) = A(\xi)$, $4<\xi<8$, and has infinite oscillations in any neighborhood of $\xi = 4$.

We obtain a (a priori non unique) solution by an indirect method. Let $\xi' = 8 - \eta$, $\eta' = 8 - \xi$; this is the same as inverting the time: in fact is

$$y_t = (y_\xi + y_\eta)/\sqrt{2} = -(y_{\xi'} + y_{\eta'})/\sqrt{2} = -y_{t'} , \qquad y_{\xi\eta} = y_{\xi'\eta'} .$$

The unilateral problem with inverted time in T'' coincides with the problem just solved in T', and is $y(\xi', 4) = A(\xi')$.

The function

$$y(\xi', \eta') = y(8 - \eta, 8 - \xi) , \qquad 4<\eta<\xi<8 \tag{3.22}$$

satisfies the unilateral problem in T'' and has infinite impact arcs.

4. - String subject to a force sign changing

We consider the problem (1.2)-(1.9), being F of any sign, assuming the following hypotheses

$$A_0 \in C^0[0, l] , \qquad A_0 \text{ piecewise } C^1 , \qquad A_1 \text{ piecewise } C^0 . \tag{4.1}$$

We accept also (differently from (1.3)) that A_0 could vanish in a (some) point other than the origin. Moreover we require that:

(4.2) Let $E = \{x \in [0, l] \mid A_0(x) = 0\}$: is finite union of intervals and isolated points;

(4.3) $A_1(x) > 0$ a.e. on E.

Furthermore, we shall require that $F \in C^0(Z)$ and satisfies the $\mathcal{F}$ property (see [6], page 188). Otherwise, if $F = f + \sum_1^n \alpha_k \delta_{x_k}$ (as in the case of a more general obstacle with corners [7]), will be $f \in C^0(Z)$, f satisfying the $\mathcal{F}$ property, or $f(x) \in L^1(0, l)$, $f < 0$ a.e. on a set that could be rapresented as a *finite* union of intervals.

Finally, the admissible function class to which the solution belongs, will be the one described in [6], § 2. Under the assumption made on the initial data (4.1), (4.2), (4.3), the first line of influence γ_0 is constituted by a finite number of space-like arcs and characteristic segments; it can also contain a finite number of points belonging to the x-axis.

The techniques shown in [1], [6], and [7] in any case allow the extension beyond γ_0 to the domain $Z \cap \{x > t\}$.

The only thing left to do is to extend the solution in a T_1 characteristic triangle above $P_0 = (\xi_0, \xi_0)$, intersection of γ_0 with the t-axis (see fig. 1.1).

Let $A(\xi) = y(\xi, \xi_0)$, $\xi > \xi_0$, be the trace of the solution on the ξ-characteristic outcoming from P_0: due to (1.12), $A(\xi)$ is either strictly increasing or $= 0$ on the whole interval (ξ_0, ξ_1). If $F > 0$, the Goursat problem solution in T_1 is always > 0 at least in $\xi_0 < \eta < \xi < \xi_1$, and so satisfies the unilateral problem.

If $F < 0$ or changes sign in T_1, this is not always true. In this case, the introduction of an analogous of Π^* problem is needed for the pourpose of finding the boundary of support domains (see [6]).

Let

(4.4) $$T = \{(\xi, \eta) \mid 0 < \eta < \xi < a\},$$

(4.5) $$\Delta : \eta = \xi .$$

Adjusted Π^* problem: *Find* $T' = \{(\xi, \eta) \mid 0 < \eta < \xi < \xi'\} \subseteq T$, *a time-like arc* Γ, *a function* $y_1(\xi, \eta)$ *such that*

(4.6) Γ: $\eta = \varphi(\xi)$ $(\xi = \gamma(\eta))$ *with* $\varphi \in C^0[0, \xi']$, $\varphi \uparrow$ *in* $[0, \xi']$, $\varphi(0) = 0$, $\varphi(\xi) < \xi$, $0 < \xi < \xi'$.

(4.7) $y_1 \in C^1(Z_1)$, *with* $Z_1 = \{0 < \xi < \xi', 0 < \eta < \varphi(\xi)\}$.

(4.8) $y_{1\xi\eta} = f(\xi, \eta)$ *in* $\mathcal{D}'(\mathring{Z})$.

(4.9) $y_1(\xi, 0) = A(\xi) \in C^1[0, a]$,

(4.10) $y_1|_\Gamma = y_{1\xi}|_\Gamma = 0$.

(4.11) $y_1 > 0$ *in* Z_1.

In order to solve the Adjusted Π^* problem, we introduce the function

$$G(\xi, \eta) = A'(\xi) + \int_0^\eta f(\xi, \beta)\, d\beta \,. \tag{4.12}$$

and we observe that a necessary condition for Π^* to have a solution is

$$A'(0) = 0 \,. \tag{4.13}$$

Regarding some sufficient condition to solve the Adjusted Π^* problem, we refer to theorems in [7], § 3 and § 4, relevant to the existence and uniqueness of the solution of the Π_1^* problem in R^-.

In fact we observe that the solution of the Adjusted Π^* problem *depends only on* $A(\xi)$ *and* f *and does not depend on the Goursat datum on* $\xi = \eta$. So, identically, the mentioned theorems (only observing that in [6], [7] a (—) signum on f force has been stressed for convenience) can be used.

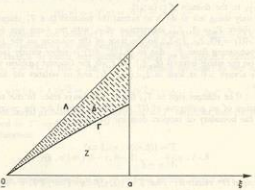

Fig. 4.1

We state finally the following extension laws:

a) Let $A(\xi) = 0$, $f < 0$ a.e. *in* T. *Then the solution of the support problem is* $y(\xi, \eta) = 0$ *in* T.

We observe that in this case $z(\xi, \eta) < 0$ in $\hat{T}$.

b) Let $A(\xi) > 0$, *and one only solution* Γ, y_1 *of the adjusted* Π^* *problem exists in* T. *Let moreover* Δ *be the domain in* fig. 4.1, *being* $f(\xi, \eta) < 0$ a.e. *in* Δ. *We* *have then*

$$y(\xi,\eta)=\begin{cases} y_1(\xi,\eta) & \text{in } Z_1\,, \\ 0 & \text{in } \Delta\,. \end{cases}$$

We can verify, by means of a simple calculation, that, under these assumptions, the free problem solution does not satisfy the unilateral condition in Δ.

REFERENCES

[1] L. Amerio - G. Prouse, *Study of the motion of a string vibrating against an obstacle*, Rend. Mat., Ser. 6, 8, n. 2 (1975), 563-585.

[2] L. Amerio, *Continuous solutions of the problem of a string vibrating against an obstacle*, Rend. Sem. Mat. Univ. Padova, 59 (1978), 67-96.

[3] C. Citrini, *Controesempi all'unicità del moto di una corda in presenza di una parete*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Natur., Ser. VIII, 67, n. 3-4 (1979), 179-185.

[4] M. Schatzman, *A hyperbolic problem of second order with unilateral constraints: the vibrating string with a concave obstacle*, J. Math. Anal. Appl., 73 (1980), 138-191.

[5] R. Burridge - J. Kappraff - C. Morshedi, *The sitar string, a vibrating string with a one-sided inelastic constraint*, SIAM J. Appl. Math., 42 (6) (1982), 1231-1251.

[6] L. Amerio, *Studio del moto di una corda vibrante contro una parete di forma qualsiasi, sotto l'azione di una forza esterna arbitraria; domini di appoggio: un problema unilaterale di frontiera libera*, Rend. Accad. Naz. XL, VIII (10) (1984), 185-246.

[7] C. Marchionna, *Studio del moto di una corda vibrante contro una parete con spigoli: domini e segmenti d'appoggio*, Rend. Accad. Naz. XL, IX (13) (1985), 275-312.