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A Characterization of Basic Sequences in Banach Spaces (***)

Una caratterizzazione delle Successioni Basiche negli spazi di Banach.

SUNTO. — Una successione di uno spazio di Banach è basica se e solo se ogni sua blocco-perturbazione è *M*-basica forte.

Let (x_n) be a sequence of a Banach space B and let (f_n) be a sequence of B^* (the dual of B), we say that (x_n, f_n) is *biorthogonal* if $f_m(x_n) = \delta_{mn}$ (Kronecker indices) for every m and n. Let (x_n, f_n) be biorthogonal, we say that (x_n) is

i) *M-basis* of B if $B = [x_n]$ $(= \overline{\mathrm{span}}\,(x_n))$ and if $\{0\} = [f_n]_\perp$ $(= \{x \in B;\ f_n(x) = 0$ for every $n\})$;

ii) *strong M-basis* of B if $B = [x_n]$ and if $[f_{n_k}]_\perp = [x_{n'_k}]$ for every complementary subsequences (n_k) and (n'_k) of (n);

iii) *basis* of B if $x = \sum_{n=1}^{\infty} f_n(x) x_n$ for every x of B.

Moreover we say that (x_n) is *M-basic* (*basic*) if it is *M*-basis (basis) of $[x_n]$. If $(x_n) \subset B$, $q_0 = 0$ and (q_n) is an increasing sequence of positive integers, we say that (y_n) is

block perturbation of (x_n) if $[y_n]_{n=q_{m-1}+1}^{q_m} = [x_n]_{n=q_{m-1}+1}^{q_m}$ for every m.

Our aim is to prove the following

THEOREM I: *A sequence is basic if and only if every block perturbation is strong M-basic.*

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In § 1 we shall prove the theorem by means of a geometric characterization of the basic sequences and by a recent result of Plans.

In § 2 we shall give another direct proof of Th. I by means of the concept of unitary position.

1. - A new geometric characterization of the basic sequences

We shall prove the following

THEOREM II: *Let (x_n, f_n) be biorthogonal in B, then*

$$(x_n) \textit{ is basic} \Leftrightarrow \cap \left\{ \left[\sum_{n=1}^{m_k} f_n(x) x_n \right]_{k=1}^{\infty} ; (m_k) \subseteq (n) \right\} = [x] \textit{ for every } x \textit{ of } [x_n] .$$

In [4] Th. II is proved for the reflexive case.
In [3] it is proved that Th. II implies Th. I.

PROOF OF TH. II: Obviously it is sufficient to prove the inverse implication. Let $x \in [x_n]$, by [6] (p. 146, Th. 13.1, a), $1^o \Leftrightarrow 3^o$) it is sufficient to prove that

$$(v_m) \text{ weakly converges to } x, \text{ where } v_m = \sum_{n=1}^{m} f_n(x) x_n \text{ for every } m .$$

We affirm that (v_m) is bounded, because if $\lim_{k \to \infty} \|v_{m_k}\| = +\infty$ for a subsequence (m_k) of (m), $(v_{m_k}/\|v_{m_k}\|)$ would not have subsequence weakly convergent to $x' \neq 0$ (if $(m'_k) \subseteq (m_k)$ with $(v_{m'_k}/\|v_{m'_k}\|)$ weakly convergent to $x' \neq 0$, it would be $f_n(x') = 0$ for every n, which is impossible since (x_n) is obviously M-basic and $x' \in [x_n]$), therefore it would follow by [7] (p. 172) that

$$(1) \qquad \cap \{ [v_{m'_k}]_{k=1}^{\infty} ; (m'_k) \subseteq (m_k) \} = \{0\} ,$$

which contradicts the hypothesis.

Then (v_m) weakly converges to x, otherwise by the « third Fréchet's axiom » [1] there would exist a subsequence (v_{m_k}) without subsequences weakly convergent to x, hence without weakly convergent subsequences (since if $(m'_k) \subseteq (m_k)$ with $(v_{m'_k})$ weakly convergent to x', it would be $f_n(x - x') = 0$ for every n hence $x = x'$); therefore, since (v_{m_k}) is bounded, it would follow by [7] (p. 172) (1) again, which contradicts the hypothesis. This completes the proof of Th. II.

2. - On the concept of unitary position

Given (x_n) we say that $x \in [x_n]$ is *in unitary position as regards* (x_n) if there exists a subsequence (n_k) of (n) such that $x \in [x_{n_k}]$, $x \notin [x_{n_k}]_{k \neq k'}$ for every k'.

In [5] (see also [2] p. 121-126) it is proved, in the frame of biorthogonal sequences, that

(x_n) *is strong M-basic* $\Leftrightarrow$ *every element of* $[x_n]$ *is in unitary position as regards* (x_n).

Then in order to prove Th. I it is sufficient to prove that, if (x_n) is not basic, there exist $\bar{x} \in [x_n]$ and a block perturbation (ξ_n) of (x_n) so that $\bar{x}$ is not in unitary position as regards (ξ_n).

Firstly we prove the following

Proposition I: *Let* (v_n) *be a basic sequence of* B *and let* (u_n) *be a sequence of a finite dimensional subspace* U *of* B; *then, if* $x_n = u_n + v_n$ *for every* n, *it follows that*:

$$(x_n) \text{ is } M\text{-basic} \Leftrightarrow (x_n) \text{ is basic}.$$

Proof: Indeed suppose that (x_n) is M-basic but not basic.

By [6] (p. 58) there exists a *block sequence* (x'_n) of (x_n) (that is there exists an increasing sequence (q_n) of positive integers so that, setting $q_0 = 0$, $x'_m \in \operatorname{span}(x_n)_{n=q_{m-1}+1}^{q_m}$ for every m) such that

(2) $\quad x'_n = u'_n + v'_n$ and $\|x'_n\| = 1$ for every n, with $(u'_n) \subset U$, (v'_n) basic and $x'_{2n-1} - x'_{2n} \to 0$.

Since (u'_n) is bounded (otherwise $(v'_n/\|v'_n\|)$ would have a convergent subsequence, while it is basic), there exist $(n_k) \subseteq (n)$ and $(n'_k) \subseteq (n_k)$ so that

$$u'_{2n_k-1} \to u', \qquad u'_{2n'_k} \to u'';$$

then $u' = u'' = u$ (otherwise by (2) it would be $v'_{2n'_k-1} - v'_{2n'_k} \to u'' - u' \neq 0$, impossible since (v'_n) is basic); hence by (2) $v'_{2n'_k-1} - v'_{2n'_k} \to 0$, that is $v'_{2n'_k-1} \to 0$ and $v'_{2n'_k} \to 0$ since (v'_n) is basic; therefore by (2) $x'_{2n'_k} \to u$, impossible since (x'_n) (block sequence of (x_n)) is M-basic. This completes the proof of Proposition I.

Proposition II: *Every M-basic but not basic sequence has a block perturbation with a subsequence which is basic with brackets but not basic.*

Where (x_n) is *basic with brackets* if there exists an increasing sequence of positive integers (q_n) such that, setting $q_0 = 0$,

$$x = \sum_{m=1}^{\infty} \Big(\sum_{n=q_{m-1}+1}^{q_m} a_n x_n \Big),$$

with (a_n) unique, for every x of $[x_n]$.

Proof: Let (x_n) be M-basic and not basic.

We shall prove that there exist a sequence (y_n) of B, $1<k<\infty$ and an increasing sequence (q_n) of positive integers so that, setting $q_0=0$, for every $m>1$

$$
(3)\qquad \left|\begin{array}{l}
\text{(i)}\ (y_{2m-1}, y_{2m}) \text{ is block sequence of a block perturbation of } (x_n)_{n=q_{m-1}+1}^{q_m}; \\
\text{(ii)}\ \|y_{2m+1}+y_{2m+2}\|<1/2^{m+1} \text{ and } \|y_m\|>K2^{m+6}, \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{moreover } \|y_1+y_2\|=1; \\
\text{(iii)}\ \operatorname{dist}(y, [y_n]_{n>2m})>\|y\|/K \text{ for every } y \text{ of } [y_n]_{n=1}^{2m}.
\end{array}\right.
$$

We can suppose (x_n) in $C_{0\leftrightarrow 1}$ then, if (e_n) is the Schauder basis of $C_{0\leftrightarrow 1}$,

$$
(4)\qquad x_m=\sum_{n=1}^{\infty} c_{mn} e_n \qquad \text{for every } m.
$$

Since (x_n) is not basic, by [6] (p. 58) there exist y_1, y_2 so that (i) for $m=1$ is true moreover $\|y_1\|>2^7$ and $\|y_1+y_2\|=1$.

Fix a positive integer $p>1$.

Suppose to have $(r_n)_{n=1}^{p}$ of positive integers and $(y_n)_{n=1}^{2p}$ of B which verify (i) and (ii) of (3) for $1<m<p$; moreover for every $1<m<p$ and for every y of $[y_n]_{n=1}^{2m}$

$$
\operatorname{dist}(y, [y_n]_{n=2m+1}^{2p}+[e_n]_{n>r_m})>(1-1/2^{m+1})\|y\|
$$

where $[y_n]_{n=2m+1}^{2p}$ does not appear if $m=p$.

By (4) it is easy to see that there exist a positive integer s_p and numbers d_{pmn} so that, if

$$
x_{pm}=x_m+\sum_{n=r_p+1}^{s_p} d_{pmn} x_n \qquad \text{for } m>s_p,
$$

then $(x_{pm})_{m>s_p}\subset[e_n]_{n>r_p}$. By Propos. I $(x_{pm})_{m>s_p}$ is not basic, hence there exist q_{p+1} and (y_{2p+1}, y_{2p+2}) block sequence of $(x_{pm})_{m>s_p}$ so that

$$
\|y_{2p+1}+y_{2p+2}\|<1/2^{p+1}, \quad \|y_{2p+1}\|>2^{2p+7}, \quad (y_{2p+1}, y_{2p+1})\subset[x_n]_{n=s_p+1}^{q_{p+1}};
$$

finally there exists a positive integer r_{p+1} so that

$$
\operatorname{dist}(y, [e_n]_{n>r_{p+1}})>(1-1/2^{p+2})\|y\| \qquad \text{for every } y \text{ of } [y_n]_{n=1}^{2p+2}.
$$

So proceeding we get (y_n) as in (3) for $k=2$; which completes the proof of Propos. II.

2nd PROOF OF TH. I: We consider the particular case of (x_n) norming, that is (see [7] p. 174) there exist $1<k<\infty$ and a not decreasing sequence (l_n) of positive integers so that $\|x\|<K\operatorname{dist}(x, [x_n]_{n>l_m})$ for every x of $[x_n]_{n=1}^{m}$ and for every m. Suppose that every block perturbation of (x_n) is strong M-basic. If (x_n) is not basic, by Propos. II we have a sequence (y_n) as in (3) for (ii) and (iii), moreover with $y_m \in [x_n]_{n=q_{m-1}+1}^{q_m}$ for every m, then set

$$(5) \qquad \bar{x} = \sum_{n=1}^{\infty} (y_{2n-1} + y_{2n}) \; ; \; v_1 = y_1 \, , \; v_{n+1} = y_{2n} + y_{2n+1} \qquad \text{for every } n \geq 1 \, .$$

If $\bar{x} \notin [v_n]$ the theorem will be proved; indeed setting by (3)

$$z_1 = y_1 \quad \text{and} \quad z_n = x_n \qquad \text{for } 2 \leq n \leq q_1 \, ;$$

moreover for $m \geq 1$

$$z_{q_{2m-1}+1} = y_{2m} + y_{2m+1} = v_{m+1} \quad \text{and} \quad z_n = x_n \qquad \text{for } q_{2m-1} + 2 \leq n \leq q_{2m+1} \, ,$$

then it follows

$$x \notin [z_1 \cup (z_{q_{2m-1}+1})_{m=1}^{\infty}] \, , \quad \text{while} \quad \operatorname{dist}\left(\bar{x}, [z_1 \cup (z_{q_{2m-1}+1})_{m=1}^{m}] + [z_n]_{n=q_{2m-1}+2}^{q_{2m+1}}\right) \leq$$

$$\left\| \bar{x} - \sum_{n=1}^{m} (y_{2n-1} + y_{2n}) \right\| \leq \sum_{n=m+1}^{\infty} \| y_{2n-1} + y_{2n} \| < 1/2^m$$

for every m; that is $\bar{x}$ is not in unitary position as regards the block perturbation (z_n) of (x_n), which contradicts the hypothesis.

In order to prove that $\bar{x} \notin [v_n]$ it is sufficient to prove that

$$(6) \qquad \left\| \bar{x} - \sum_{n=1}^{m} a_n v_n \right\| < 1 \qquad \text{implies } |a_n| < 1/2^n \qquad \text{for } 1 \leq n \leq m \, ;$$

indeed, if (y_n, b_n) is biorthogonal, by (5) $b_n(\bar{x}) = 1$ for every n, while by (5)

$$b_1\left(\sum_{n=1}^{m} a_n v_n\right) = a_1 \quad \text{and} \quad b_{2k}\left(\sum_{n=1}^{m} a_n v_n\right) = b_{2k+1}\left(\sum_{n=1}^{m} a_n v_n\right) = a_{k+1} \qquad \text{for } 1 \leq k \leq m \, .$$

Fix $1 \leq p \leq m$, setting $a_{m+1} = 0$ we can suppose true the thesis of (6) for $p + 1 \leq n \leq m + 1$ and we shall prove the thesis for $n = p$. By (3) and (5) we have that

$$\left\| a_{p+1} y_{2p+1} + \sum_{n=p+2}^{m} a_n v_n \right\| \leq \left\| \sum_{n=1}^{m} a_n v_n \right\| + \left\| \sum_{n=1}^{p} a_n v_n + a_{p+1} y_{2p} \right\| \leq$$

$$\leq \left\| \sum_{n=1}^{m} a_n v_n \right\| + \left\| \sum_{n=1}^{m} a_n v_n \right\| K \, .$$

Hence by the hypothesis of (6) and by (3)

$$(7) \qquad |a_p y_{2p-1} + a_{p+1} y_{2p}| \leq \left\| \sum_{n=1}^{m} a_n v_n \right\| + \left\| \sum_{n=1}^{p-1} a_n v_n + a_p y_{2p-2} \right\| +$$

$$+ \left\| a_{p+1} y_{2p+1} + \sum_{n=p+2}^{m} a_n v_n \right\| \leq \left\| \sum_{n=1}^{m} a_n v_n \right\| (2 + 2K) \leq$$

$$\leq (\|\bar{x}\| + 1)(2 + 2K) \leq 4K(\|\bar{x}\| + 1) \, .$$

Moreover, since the thesis of (6) is true for a_{p+1}, by (3) we have that

$$(8) \qquad \begin{aligned} \|a_p y_{2p-1} + a_{p+1} y_{2p}\| &> (|a_p| - |a_{p+1}|)\,\|y_{2p-1}\| - |a_{p+1}|\,\|y_{2p-1} + y_{2p}\| > \\ &> (|a_p| - 1/2^{p+1})\,\|y_{2p-1}\| - 1/2^{2p+1} > (|a_p| - 1/2^{p+1})\,2^{2p+4}K - 1/2^{2p+1}. \end{aligned}$$

By (3) and (5) $1/2 < \|\hat{x}\| < 3/2$; hence by (7) and (8) we have that

$$|a_p| < \left(4K(\|\hat{x}\| + 1) + 1/2^{2p+1}\right)/2^{2p+4}K + 1/2^{p+1} < 1/2^{2p} + 1/2^{p+1} < 1/2^p.$$

This completes the proof.

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