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On the Number of H-Sets in a Hausdorff Space (**)

ABSTRACT. — In this note some upper bounds for the number of H-sets in a Hausdorff space are given. An example shows that the number of H-sets can be greater than the number of compact sets even for H-closed spaces.

Sul numero degli H-insiemi in uno spazio di Hausdorff

RIASSUNTO. — In questa Nota sono date alcune stime per il numero degli H-insiemi in uno spazio di Hausdorff. Un esempio mostra che il numero degli H-insiemi può in effetti superare quello degli insiemi compatti anche in ispazi H-chiusi.

0. - INTRODUCTION

In [2], Burke and Hodel gave some upper bounds for the number of compact subsets of a topological space in terms of other cardinal functions. In particular they obtained the following three inequalities:

$$|\mathcal{K}(X)| \leq 2^{2^{s(X)}} \; ; \tag{0.1}$$

$$|\mathcal{K}(X)| \leq 2^{hL(X)} \; ; \tag{0.2}$$

$$|\mathcal{K}(X)| \leq 2^{s(X)\psi(X)} \; ; \tag{0.3}$$

where X is assumed to be Hausdorff and $\mathcal{K}(X)$ denotes the set of all compact subsets of X.

For any topological space a class of subsets larger than the class of compact ones consists of the so called H-sets (see [7]).

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A subset A of the topological space X is said to be a H-set if for any family $\mathfrak{U}$ of open subsets of X such that $A \subseteq \bigcup \mathfrak{U}$, there is a finite subfamily $\{U_1, \dots, U_n\} \subseteq \mathfrak{U}$ for which $A \subseteq \overline{U}_1 \cup \dots \cup \overline{U}_n$.

H-sets play an important role in the theory of H-closed spaces, in fact a Hausdorff space X is H-closed if and only if X itself is a H-set.

In this note we show that the above three inequalities remain true when $\mathcal{K}(X)$ is replaced by $\mathcal{H}(X)$, where $\mathcal{H}(X)$ denotes the set of all H-sets in X.

For any topological space X we have $|X| \leq |\mathcal{K}(X)| \leq |\mathcal{H}(X)| \leq 2^{|X|}$ and in general the middle inequality can be proper. The example in section 2 will furnish a H-closed space for which the gap between $|\mathcal{K}(X)|$ and $|\mathcal{H}(X)|$ is the largest possible, *i.e.*, $|\mathcal{K}(X)| = |X|$ and $|\mathcal{H}(X)| = 2^{|X|}$.

1. - Some definitions

For notations we follow [3] and [5]. The cardinality of a set S is denoted by $|S|$. The following cardinal functions will be used here:

— $d(X)$, density, i.e. the smallest cardinality of a dense subset of X;

— $hL(X)$, hereditary Lindelöf degree, i.e. the smallest cardinal number m such that any family $\mathfrak{U}$ of open subsets of X has a subfamily $\mathfrak{U}_0$ for which $\bigcup \mathfrak{U} = \bigcup \mathfrak{U}_0$ and $|\mathfrak{U}_0| \leq m$;

— $e(X)$, extend, i.e. the supremum of the cardinality of a discrete closed subset of X;

— $\Psi(X)$, perfect degree, i.e. the smallest cardinal number m such that any closed set can be expressed as the intersection of m open subsets of X.

A subset of a topological space is said to be regularly closed if it is the closure of an open set. The number of regularly closed subsets of X is denoted by $\varrho(X)$.

Let Y be a Hausdorff space and $X \subseteq Y$, Y is said to be a H-closed extension of X if X is dense in Y and Y is H-closed. The simple extension Y^+ corresponding to Y (see [1] and [6]) is the topological space having the same underlying set as Y, but $U \subseteq Y^+$ defined to be open if and only if $U \cap X$ is open in X and for every $p \in U \setminus X$ there exists a set V open in Y such that $p \in V$ and $V \cap X = U \cap X$.

Y^+ is again a H-closed extension of X and, moreover, $Y^+ \setminus X$ is closed discrete in Y^+.

2. - Results

We begin by showing that a Hausdorff space can actually have a number of H-sets greater than the number of compact sets.

Example: Let $I = [0, 1]$ be the unit interval of the real line. Let Y be the Alexandrov double of I, i.e. the set $I \times \{1, 2\}$ topologized in such a way that a point of the form $(x, 2)$ is isolated, while a point of the form $(x, 1)$ has as a fundamental system of neighborhoods the sets $U\times\{1\}\cup(U\setminus\{x\})\times\{2\}$, where U is an open neighborhood of x in I with respect to the Euclidean topology. Y is compact Hausdorff, hence H-closed, and the subspace $I\times\{2\}$ is dense in Y. Thus we can think of Y as a H-closed extension of $I\times\{2\}$. Let Y^+ be the simple extension corresponding to Y (see section one). We claim that $|\mathcal{K}(Y^+)|=|Y^+|=c$ and $|\mathcal{H}(Y^+)|=2^{|Y^+|}=2^c$. To prove $|\mathcal{K}(Y^+)|=c$ we consider the following facts.

(i) Let $A\subseteq I$. If x is an accumulation point of A with respect to the Euclidean topology then $(x, 1)\in\overline{A\times\{2\}}^{Y^+}$.

(ii) If A is uncountable then it has an uncountable set of accumulation points.

(i) and (ii) clearly imply:

(iii) $|\overline{A\times\{2\}}^{Y^+}\cap I\times\{1\}|>\aleph_0$ whenever $|A\times\{2\}|>\aleph_0$.

Now let $Z\in\mathcal{K}(Y^+)$. As $I\times\{1\}=Y^+\setminus I\times\{2\}$ is closed discrete in Y^+ we must have $|Z\cap I\times\{1\}|<\aleph_0$. Furthermore $|Z\cap I\times\{2\}|<\aleph_0$, since otherwise $|Z\cap I\times\{1\}|>\aleph_0$ by (iii) and the fact that Z is closed. This shows that $|Z|<\aleph_0$ and therefore $c=|Y^+|\leq|\mathcal{K}(Y^+)|\leq|Y^+|^{\aleph_0}=c^{\aleph_0}=c$.

To prove that $|\mathcal{H}(Y^+)|=2^c$ we recall that a regularly closed subset of a H-closed space is a H-set.

Every subset of $I\times\{2\}$ is open in Y^+ and, moreover, for any two $A_1, A_2\subseteq I\times\{2\}$ such that $A_1\neq A_2$ we have $\overline{A_1}^{Y^+}\neq\overline{A_2}^{Y^+}$. This clearly implies $\varrho(Y^+)\geq 2^c$ and therefore $2^c=2^{|Y^+|}\geq|\mathcal{H}(Y^+)|\geq 2^c$.

Theorem 2.1: *If X is a Hausdorff space than $|\mathcal{H}(X)|\leq 2^{\varrho(X)}$.*

Proof: Let $H\in\mathcal{H}(X)$ and let $p\in X\setminus H$. For any $x\in H$ choose an open neighborhood U_x of x such that $p\notin\overline{U}_x$. The family $\{U_x\}_{x\in H}$ is an open cover of H and so there exists a finite subset $\{x_1, \ldots, x_n\}\subseteq H$ such that

$$H\subseteq\overline{U_{x_1}\cup\ldots\cup U_{x_n}}.$$

$\overline{U_{x_1}\cup\ldots\cup U_{x_n}}$ is a regularly closed subset and $p\notin\overline{U_{x_1}\cup\ldots\cup U_{x_n}}$. This shows that H is the intersection of a family of regularly closed subsets of X and thus $|\mathcal{H}(X)|\leq 2^{\varrho(X)}$. q.e.d.

Recalling that for any topological space X we have $\varrho(X)\leq 2^{d(X)}$, we obtain

Theorem 2.2: *If X is a Hausdorff space then $|\mathcal{H}(X)|\leq 2^{2^{d(X)}}$.*

We need two elementary lemmas to prove our next result

Lemma 2.1: *If X is a Hausdorff space then for any $x\in X$ there is a family of open neighborhoods $\mathfrak{U}$ of x, closed under finite intersection, such that $|\mathfrak{U}|\leq|X|$ and* $\bigcap_{U\in\mathfrak{U}}\overline{U}=\{x\}$.

LEMMA 2.2: *Let X be a topological space. If H is a H-set in X and $(U_\alpha)_{\alpha\in A}$ is a collection of open subsets of X such that $H\cap\left(\bigcap_{\alpha\in A}\overline{U}_\alpha\right)=\emptyset$ then there exist $\alpha_1,\dots,\alpha_n$ for which $H\cap U_{\alpha_1}\cap\dots\cap U_{\alpha_n}=\emptyset$.*

THEOREM 2.3: *If X is a Hausdorff space then $|\mathcal{K}(X)|\leq 2^{bL(X)}$.*

PROOF: Let $m=bL(X)$. By formula (0.2) in the introduction we have $|X|\leq 2^m$ and by Lemma 2.1 for any $x\in X$ there is a family of open neighborhoods $\mathfrak{U}_x$ of x, closed under finite intersection, such that $|\mathfrak{U}_x|\leq 2^m$ and $\bigcap_{U\in\mathfrak{U}_x}\overline{U}=\{x\}$. Let $\mathfrak{U}=\bigcup_{x\in X}\mathfrak{U}_x$ and let $\mathfrak{G}$ be the set of all unions of $\leq m$ members of $\mathfrak{U}$. Clearly $|\mathfrak{G}|\leq 2^m$. Let $H\in\mathcal{K}(X)$. By Lemma 2.2 for any $x\in X\setminus H$ there exists a neighborhood $U_x\in\mathfrak{U}_x$ such that $U_x\subseteq X\setminus H$ and so $X\setminus H=$ $=\bigcup_{x\in X\setminus H}U_x$. Since $bL(X)=m$ we have $X\setminus H\in\mathfrak{G}$. This establishes a one to one map from $\mathcal{K}(X)$ into $\mathfrak{G}$ and so $|\mathcal{K}(X)|\leq 2^m$. q.e.d.

THEOREM 2.4: *If X is a Hausdorff space then $|\mathcal{K}(X)|\leq 2^{e(X)\psi(X)}$.*

PROOF: Using Lemmas 2.1 and 2.2 as in the preceding theorem, the proof can proceed as in [4, Th. 9.3], thus we omit the details.

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