Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
102° (1984), Vol. VIII, fasc. 6, pagg. 99-120

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Concrete Representations for Inverse and Distributive Exact Categories (**)

Rappresentazioni concrete per categorie inverse e categorie esatte distributive

SOMMARIO. — Per ogni categoria inversa piccola K viene esplicitamente costruita un'immersione $\widehat{\mathrm{Spc}}_{\mathsf{K}}\colon \mathsf{K} \to \mathfrak{I}$ nella categoria inversa « paradigmatica » $\mathfrak{I}$ degli *insiemi piccoli e delle biezioni parziali*; quando K è inversa idempotente (non necessariamente piccola) viene anche fornita un'immersione nella categoria inversa idempotente $\mathfrak{I}_0$ degli *insiemi piccoli e delle identità parziali*. Queste immersioni, conservando le unioni distributive finite di proiezioni, producono un'immersione esatta di ogni categoria esatta distributiva piccola (risp. esatta pre-idempotente) in $\mathfrak{I}$ (risp. in $\mathfrak{I}_0$); si noti che $\mathfrak{I}$ ed $\mathfrak{I}_0$ sono categorie esatte (sempre nel senso di Puppe [20, 19]). Questi risultati verranno usati per provare che ogni « teoria esatta » distributiva (risp. pre-idempotente) ha una categoria esatta classificante in $\mathfrak{I}$ (risp. in $\mathfrak{I}_0$); si proverà anche che varie teorie dell'algebra omologica sono pre-idempotenti: ad esempio, il complesso filtrato ed il complesso doppio.

0. - INTRODUCTION

0.1. *Summary*. We explicitly build, for every small inverse category K, an embedding $\widehat{\mathrm{Spc}}_{\mathsf{K}}\colon \mathsf{K} \to \mathfrak{I}$ into the « paradigmatic » inverse category $\mathfrak{I}$ of *small sets and partial bijections*; when K is idempotent inverse (not necessarily small) we also give an embedding in the idempotent inverse category $\mathfrak{I}_0$ of *small sets and partial identities*.

As these embeddings *preserve finite distributive unions of projections*, we derive from them an exact embedding for every *small distributive* (resp. *pre-idempotent*) exact category in $\mathfrak{I}$ (resp. in $\mathfrak{I}_0$); notice that $\mathfrak{I}$ and $\mathfrak{I}_0$ are exact categories (always in the sense of Puppe [20, 19]).

These results will be used to prove that every distributive (resp. pre-idempotent) « exact theory » has a classifying exact category which is an exact

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(**) Memoria presentata il 7 novembre 1983 da Giuseppe Scorza Dragoni, uno dei XL.

ISSN-0392-4106

subcategory of $\mathcal{I}$ (resp. $\mathcal{I}_0$); actually, various theories of interest in homological algebra are (will be shown to be) pre-idempotent: e.g. the filtered complex and the double complex.

0.2. It is easy to embed an inverse semigroup in a semigroup $\mathcal{I}(S, S)$ of partial endobijections [17, 5]; the same construction can be carried over a small inverse category. Moreover Kastl [16] proved that *every* inverse category K embeds in $\mathbf{Set}$, via the non-constructive Isbell-Freyd condition [15, 6], and derived from this the more interesting fact that K embeds in $\mathcal{I}$ (hence it is isomorphic to an inverse subcategory of $\mathcal{I}$).

0.3. However, we are interested in inverse categories *in connection with distributive exact categories* ([10, 11]; here: n. 4). As exact sequences and exact functors for distributive exact categories correspond respectively, in the context of inverse categories, to *finite distributive unions of projections* and *d-functors* (i.e. functors preserving these unions), *we need d-embeddings* $\mathsf{K} \to \mathcal{I}$. From these it is possible to derive *exact* embeddings of distributive exact categories in $\mathcal{I}$ itself.

0.4. The plan of the work is the following.

In n. 1 we recall some definitions and properties of inverse categories, as well as their transfer functor for projections, $\mathrm{Prj}_{\mathsf{K}}\colon \mathsf{K} \to \mathbf{Slt}$.

Then n. 2 builds, for every inverse category K, the *spectrum d-functor* $\mathrm{Spc}_{\mathsf{K}}\colon \mathsf{K} \to \mathcal{I}$, which is an embedding when K is transfer (i.e. the functor $\mathrm{Prj}_{\mathsf{K}}$ is faithful); n. 3 gives, for K *small*, the extended spectrum d-embedding $\widehat{\mathrm{Spc}}_{\mathsf{K}}\colon \mathsf{K} \to \mathcal{I}$.

As a consequence, n. 4 supplies exact embeddings $\mathrm{Spc}_{\mathsf{E}}\colon \mathsf{E} \to \mathcal{I}$ (E a *transfer* distributive exact category) and $\widehat{\mathrm{Spc}}_{\mathsf{E}}\colon \mathsf{E} \to \mathcal{I}$ (E a *small* distributive exact category); n. 5 treats the idempotent case: since every idempotent inverse category is transfer, we can use the results of n. 2, instead of those of n. 3, and drop the smallness hypothesis. We also remark that $\mathcal{I}$ embeds exactly in every category of modules (4.5).

Last, n. 6 sketches natural d-embeddings of semilattices and distributives lattices in lattices of parts.

0.5. The question arises whether every inverse category (with small Hom-sets) has a (possibly constructible) d-embedding in $\mathcal{I}$: this result would strengthen both Kastl's and our, and would imply that every distributive exact category (with small Hom-sets) embeds exactly in $\mathcal{I}$. As the classifying categories of exact theories are (in our formulation) necessarily small, our purposes are covered by this paper.

0.6. *Conventions.* A universe $\mathcal{U}$ is chosen once and for all. Every category C is assumed to have objects and morphisms belonging to $\mathcal{U}$. Moreover, C is *Hom-small* (or: it has small Hom-sets) whenever all the sets $\mathsf{C}(X, Y)$ belong to $\mathfrak{U}$; it is *Hom-finite* if these sets are also finite. C is small when the whole set of morphism (hence also the set of objects) belongs to $\mathfrak{U}$.

Let A be a category provided with an *involution* $a \mapsto \tilde{a}$ (i.e. an involutory contravariant endofunctor of A, identical on the objects and turning $a\colon A' \to A''$ into $\tilde{a}\colon A'' \to A'$) which is *regular* ($a = a\,\tilde{a}\,a$, for every morphism a); typically, every category of relations over an exact category is so [4, 3].

A is selfdual; the following terminology will be used. An *idempotent* of A is an endomorphism $e\colon A \to A$ such that $ee = e$; a *projection* is a symmetrical idempotent: $e = ee = \tilde{e}$. For every object A we write $\mathrm{Prj}_{\mathsf{A}}(A)$ the set of its projections. Every morphism $a \in \mathsf{A}(A', A'')$ has associated projections $e(a) = \tilde{a}a \in \mathrm{Prj}(A')$ and $i(a) = a\tilde{a} \in \mathrm{Prj}(A'')$, verifying $a = i(a)\cdot a \cdot e(a)$; every idempotent $e\colon A \to A$ is the product of the two projections: $e = e\tilde{e}\cdot\tilde{e}e = i(e)\cdot e(e)$, and conversely every product of *two* projections e_1, e_2 is idempotent (it is a projection iff e_1, e_2 commute).

A homomorphism of semilattices (resp. 1-semilattices) is assumed to preserve the product (resp. product and unit).

1. - Inverse categories and their transfer functor

Inverse categories, extending the notion of inverse semigroup, were studied in [8, 9, 10, 11, 12, 13, 16, 21, 22]. We recall here some results.

1.1. A category K is *inverse* if for every morphism $a \in \mathsf{K}(A', A'')$ there is a unique morphism $\tilde{a} \in \mathsf{K}(A'', A')$, the *generalized inverse* of a, such that:

(1) $$a = a\,\tilde{a}\,a\,; \qquad \tilde{a} = \tilde{a}\,a\,\tilde{a}\,.$$

Then the mapping $a \mapsto \tilde{a}$ supplies the unique regular involution of K. Conversely, a category provided with a regular involution is inverse iff its projections commute, iff its idempotents commute, iff every idempotent is a projection (0.6 and [8; § 1.25]).

1.2. For every object A of the inverse category K the set $\mathrm{Prj}_{\mathsf{K}}(A)$ of its projections is therefore a commutative idempotent subsemigroup of $\mathsf{K}(A, A)$, and a 1-semilattice in its own right.

We say that K is *Prj-small* (resp. *Prj-finite*) if all its projection-sets are small (resp. small and finite); usual inverse categories are Hom-small, hence also Prj-small.

1.3. An inverse category K has a *canonical order* (or *domination*) $a \sqsubset b$ (between parallel morphisms, agreeing with composition and involution), characterized by the following equivalent conditions [8]:

(1) $a = a\,\tilde{b}\,a$,

(2) $a = (a\tilde{a})\,b\,(\tilde{a}a)$,

(3) there exist projections e, f such that $a = fbe$,

(4) $a = b(\tilde{a}a)$,

(5) $a = (a\tilde{a})b$.

1.4. Every Prj-small inverse category K has a *projection functor* (or *transfer functor*) [13, § 7.3]:

$$\mathrm{Prj}_{\mathsf{K}} \colon \mathsf{K} \to \mathsf{Slt} \tag{1}$$

where Slt is the inverse category of *small* 1-*semilattices and transfer pairs*: the morphisms are the pairs $a = (a_\bullet, a^\bullet) \colon X \to Y$ such that:

$$a_\bullet \colon X \to Y \text{ and } a^\bullet \colon Y \to Y \text{ are product-preserving mappings}, \tag{2}$$

$$a^\bullet a_\bullet(x) = x a^\bullet(1), \qquad \text{for every } x \in X, \tag{3}$$

$$a_\bullet a^\bullet(y) = y a_\bullet(1), \qquad \text{for every } y \in Y, \tag{4}$$

and the composition is obvious: $(b_\bullet, b^\bullet)(a_\bullet, a^\bullet) = (b_\bullet a_\bullet, a^\bullet b^\bullet)$.

The functor (1) is defined by 1.2 on the objects, and on a morphism $a \colon A' \to A''$ by:

$$\mathrm{Prj}_{\mathsf{K}}(a) = (a_P, a^P) \colon \mathrm{Prj}(A') \to \mathrm{Prj}(A''), \tag{5}$$

$$a_P \colon \mathrm{Prj}(A') \to \mathrm{Prj}(A''); \qquad a_P(e) = a e \tilde{a}, \tag{6}$$

$$a^P \colon \mathrm{Prj}(A'') \to \mathrm{Prj}(A'); \qquad a^P(f) = \tilde{a} f a. \tag{7}$$

We also notice that the morphism $a = (a_\bullet, a^\bullet) \in \mathsf{Slt}(X, Y)$ is determined by its *covariant part* $a_\bullet$: actually, $x_0 = a^\bullet(1)$ and $a^\bullet$ can be written as:

$$x_0 = \max \{x \in X \mid x' < x \Rightarrow a_\bullet(x') < a_\bullet(x)\}, \tag{8}$$

$$a^\bullet(y) = \max \{x \in X \mid x \leqslant x_0 \;\&\; a_\bullet(x) \leqslant y\}. \tag{9}$$

Thus Slt is concrete (and coconcrete).

1.5. This functor $\mathrm{Prj}_{\mathsf{K}}$ is trivially injective on the objects, but need not be faithful; if so, we say that K is a *transfer* inverse category.

If K is Prj-finite, the projection functor takes values in the full (Hom-finite) subcategory Slt' of small finite 1-semilattices; therefore *every transfer Prj-finite inverse category is* Hom-finite.

The functorial isomorphism

$$\iota \colon 1 \to \mathrm{Prj}_{\mathsf{Slt}} \colon \mathsf{Slt} \to \mathsf{Slt}, \tag{1}$$

$$(\iota X)_\bullet x = (x \wedge -, x \wedge -); \qquad (\iota X)^\bullet e = e_\bullet(1), \tag{2}$$

proves that Slt is transfer.

1.6. In a 1-semilattice X we say that the element x is the *distributive union* of the elements x_i $(i \in I)$ [18, 11] whenever:

a) for every $y \in X$, xy is the union of the elements $x_i y$ $(i \in I)$.

The least element 0_X of X, when it exists, is the distributive union of the void family.

A homomorphism of 1-semilattices will be called a *d-homomorphism* if it preserves *finite* distributive unions.

It is easy to check that the covariant part $a_\bullet$: $X \to Y$ and the contravariant part $a^\bullet$: $Y \to X$ of a transfer pair $(a_\bullet, a^\bullet)$ always preserve (arbitrary) *distributive unions* [11; § 5.2].

1.7. We say that a family (x_i) of elements of the 1-semilattice X *d-generates* X if the only sub-1-semilattice of X, closed w.r.t. *finite* distributive unions and containing the x_i, is X itself.

One can (easily but tediously) prove that every finitely d-generated semilattice is finite. We shall derive this fact from the « concreteness theorem » of semilattices 2.6.

1.8. Every functor F: $\mathsf{K} \to \mathsf{K}'$ between Prj-small inverse categories defines homomorphisms of 1-semilattices (preserving product and unit):

$$(1) \qquad \mathrm{Prj}_F(A)\colon \mathrm{Prj}_{\mathsf{K}}(A) \to \mathrm{Prj}_{\mathsf{K}'}(FA)\,; \qquad e \mapsto Fe \qquad (A \in \mathrm{Ob}\,\mathsf{K})\,.$$

If **Sht** denotes the double category of 1-*semilattices, their homomorphisms and their transfer pairs* (with bicommutative cells), the homomorphisms (1) produce a *horizontal transformation of vertical functors* [13]:

$$(2) \qquad \mathrm{Prj}_F\colon \mathrm{Prj}_{\mathsf{K}} \to \mathrm{Prj}_{\mathsf{K}'} \cdot K\colon \mathsf{K} \to \mathbf{Sht}$$

i.e. an **Sht**-wise transformation, according to [1].

We say that F is *Prj-faithful* (resp. *Prj-full*) if all these homomorphisms (1) are injective (resp. surjective). Every faithful (resp. full) functor between inverse categories is so.

1.9. We say that F *preserves finite distributive unions* (of projections), or for short that it is a *d-functor*, when all the homomorphisms 1.8.1 are d-homomorphisms; in other words, whenever the transformation 1.8.2. takes place in **Sdt**, the double category of 1-semilattices, their d-homomorphisms and their transfer pairs.

The functors between inverse categories we are interested in *are precisely the d-functors*, because of their connection with *exact* functors between distributive exact categories (see n. 4).

Notice that every Prj-faithful, Prj-full functor preserves and reflects (arbitrary) distributive unions. In particular, the *projection functor* $\mathrm{Prj}_{\mathsf{K}}$: $\mathsf{K} \to \mathbf{Slt}$ *of every inverse category does.*

1.10. The paradigmatic inverse category is $\mathcal{I}$, the category of (small) *sets and partial bijections* [3], whose morphisms are the triples

$$(1) \qquad a = (S_0, T_0; a_0): S \to T$$

where $S_0 \subset S$, $T_0 \subset T$ and $a_0: S_0 \to T_0$ is a bijective mapping. The composition is obvious, and the generalized inverse of (1) is:

$$(2) \qquad \tilde{a} = (T_0, S_0; a_0^{-1}): T \to S\,.$$

We also consider the full (inverse) subcategory $\mathcal{I}^f$ of (small) finite sets.

2. - The spectrum representation for transfer inverse categories

We define here, for every Prj-small inverse category K, a *spectrum d-functor* $\mathrm{Spc}_{\mathsf{K}}: \mathsf{K} \to \mathcal{I}$, which is an embedding when K is transfer. The construction is based on a « standard » spectrum $\mathrm{Spc}: \mathsf{Slt} \to \mathcal{I}$, which will be extended in n. 6 to a larger category of semilattices.

2.1. First, we notice that the projection functor of $\mathcal{I}$

$$(1) \qquad \mathrm{Prj}: \mathcal{I} \to \mathsf{Slt}$$

is canonically isomorphic to the *functor of parts* (with obvious action on morphisms):

$$(2) \qquad \mathcal{S}: \mathcal{I} \to \mathsf{Slt}$$

via:

$$(3) \qquad \pi: \mathcal{S} \to \mathrm{Prj}: \mathcal{I} \to \mathsf{Slt}\,,$$

$$(4) \qquad \pi S(S_0) = (S_0, S_0; \mathrm{id}\, S_0): S \to S \qquad (\text{for } S_0 \subset S)\,.$$

We shall identify Prj with $\mathcal{S}$ via π.

2.2. We build now the *spectrum functor*:

$$(1) \qquad \mathrm{Spc}: \mathsf{Slt} \to \mathcal{I} \qquad (\mathrm{Spc}: \mathsf{Slt}^f \to \mathcal{I}^f)$$

associating to every 1-semilattice X the set $\mathrm{Spc}(X)$ of d-homomorphisms of 1-semilattices $\varphi: X \to \Omega = \{0, 1\}$.

[3] Partial bijections (also called partial injections) are used in semigroup theory to represent inverse semigroups [17, 5]. The category $\mathcal{I}$ was studied in [2, 16, 12].

On a morphism $a=(a_*, a^*)\colon X \to Y$ we set:

$$\text{(2)} \qquad \operatorname{Spc}(a) = (F_0, G_0; a_0)\colon \operatorname{Spc}(X) \to \operatorname{Spc}(Y),$$

$$\text{(3)} \qquad F_0 = \{\varphi \in \operatorname{Spc} X \mid \varphi(a^*1) = 1\}; \qquad G_0 = \{\psi \in \operatorname{Spc} Y \mid \psi(a_*1) = 1\},$$

$$\text{(4)} \qquad a_0(\varphi) = \varphi a^*; \qquad a_0^{-1}(\psi) = \psi a_*.$$

Thus, for $\varphi \in F_0$ and $x \in X$:

$$\text{(5)} \qquad a_0(\varphi)(a_*1) = \varphi(a^*a_*1) = \varphi(a^*1) = 1,$$

$$\text{(6)} \qquad (a_0^{-1}a_0\varphi)(x) = \varphi a^* a_*(x) = \varphi(x \cdot a^*(1)) = \varphi(x)\cdot\varphi(a^*(1)) = \varphi(x),$$

which, with the dual properties, prove that $a_0\colon F_0 \to G_0$ is a bijection.
It will be proved in 2.4-2.7 that Spc is a d-embedding.

2.3. Equivalently, one can consider the set $\operatorname{Spc}'(X)$ of *prime filters* $\alpha = \varphi^{-1}(1)$ of X, characterized by:

a) $\alpha \subset X$ is stable for finite intersections (hence $1 \in \alpha$),

b) if $x_1 < x_2$ in X and $x_1 \in \alpha$ then $x_2 \in \alpha$,

c) if $x = x_1 \vee x_2$ is a distributive union in X and $x \in \alpha$, then either $x_1 \in \alpha$ or $x_2 \in \alpha$; moreover $0_X \notin \alpha$ (when 0_X exists),

or also the set $\operatorname{Spc}''(X)$ of *prime ideals* $\beta = \varphi^{-1}(0)$ of X:

d) $\beta \subset X$ is stable for finite distributive unions,

e) if $x_1 < x_2$ in X and $x_2 \in \beta$ then $x_1 \in \beta$,

f) if $x = x_1 \wedge x_2$ and $x \in \beta$ then either $x_1 \in \beta$ or $x_2 \in \beta$; moreover $1 \notin \beta$.

In both cases one gets functors which are isomorphic to Spc, via:

$$\text{(1)} \qquad \operatorname{Spc}(X) \to \operatorname{Spc}'(X); \qquad \varphi \mapsto \varphi^{-1}(1),$$

$$\text{(2)} \qquad \operatorname{Spc}(X) \to \operatorname{Spc}''(X); \qquad \varphi \mapsto \varphi^{-1}(0).$$

We shall use, according to convenience, the first (Spc) or the second (Spc′) « description » of the spectrum functor.

2.4. We prove now that Spc: $\mathsf{Slt} \to \mathfrak{Z}$ preserves *finite* distributive unions.
Let X be a 1-semilattice: it is isomorphic to $\operatorname{Prj}(X)$ via 1.5.2. Therefore, if $e_0 = e_1 \vee e_2$ is a distributive union in $\operatorname{Prj}(X)$, we have $e_i = (x_i \wedge -, x_i \wedge -)$, and $x_0 = x_1 \vee x_2$ is a distributive union in X.
Now $e_i' = \operatorname{Spc}(e_i) = (F_i, F_i; \operatorname{id} F_i)$ where $F_i = \{\varphi \in \operatorname{Spc}(X) \mid \varphi(x_i) = 1\}$. Since, for every $\varphi \in \operatorname{Spc}(X)$, $\varphi(x_0) = \varphi(x_1) \vee \varphi(x_2)$ in Ω, it follows that $\varphi(x_0) = 1$

iff $\varphi(x_1) = 1$ or $\varphi(x_2) = 1$; in other words, $F_0 = F_1 \cup F_2$ in $\mathcal{P}\,\mathrm{Spc}(X)$ and $e_0' = e_1' \vee e_2'$, distributive union in $\mathrm{Prj}\,\mathrm{Spc}(X) \simeq \mathcal{P}\,\mathrm{Spc}(X)$ (2.1).

Analogously one verifies that the least element of $\mathrm{Prj}(X)$, if it exists, is turned into the empty part of $\mathrm{Spc}(X)$.

2.5. Also in order to prove that Spc is faithful (2.7), we introduce the horizontal transformation of vertical functors:

$$\eta\colon \quad V \to \mathrm{Prj}\cdot\mathrm{Spc}\colon \mathbf{Slt} \to \mathbf{Sdt}\,, \tag{1}$$

$$\eta X\colon X \to \mathrm{Prj}\cdot\mathrm{Spc}(X) \simeq \mathcal{P}(\mathrm{Spc}(X))\,, \tag{2}$$

$$\eta X(x) = \{\varphi \in \mathrm{Spc}(X) \mid \varphi(x) = 1\}\,, \tag{3}$$

where **Sdt** is defined in 1.9, and $V\colon \mathbf{Slt} \to \mathbf{Sdt}$ is the vertical inclusion; indeed η is the composition

$$V \xrightarrow{\ \iota'\ } \mathrm{Prj}_{\mathbf{Slt}} \xrightarrow{\mathrm{Prj}_{\mathrm{Spc}}} \mathrm{Prj}\cdot\mathrm{Spc}\colon \mathbf{Slt} \to \mathbf{Sdt} \tag{4}$$

of the horizontal isomorphism ι' associated to $\iota\colon 1 \to \mathrm{Prj}_{\mathbf{Slt}}\colon \mathbf{Slt} \to \mathbf{Slt}$ (1.5.1) and of the horizontal transformation $\mathrm{Prj}_{\mathrm{Spc}}$ (1.8, 1.9, 2.4).

2.6. *Theorem* (*Concrete representation for semilattices*). Every 1-semilattice X d-embeds via $\eta X\colon X \to \mathcal{P}(\mathrm{Spc}(X))$ (2.5) in a semilattice of parts (η is pointwise injective). X is finite iff it is finitely d-generated (1.7), iff $\mathrm{Spc}(X)$ is finite.

Proof. The last assertion is an obvious consequence of the first, and of the fact that if X is finitely d-generated, $\mathrm{Spc}(X)$ is finite. Therefore, assuming that $x_1 < x_2$ in X, we only need to show that $\eta X(x_1) \neq \eta X(x_2)$.

Consider the filter α_0 and the ideal β_0 of X:

$$\alpha_0 = \{x \in X \mid x \geq x_2\}\,, \quad \beta_0 = \{x \in X \mid x \leq x_1\}$$

and the set A of those filters of X which contain α_0 and do not intersect β_0. By Zorn's lemma, A has a maximal element α; since $x_2 \in \alpha$ and $x_1 \notin \alpha$, we need only to verify that α is a prime filter.

If $x \in \alpha$ and $x = x' \vee x''$ is a distributive union, we may assume (possibly exchanging x' and x'') that $y \wedge x' \nleq x_1$ for every $y \in \alpha$: otherwise, the existence of $y', y'' \in \alpha$ such that $y' \wedge x' \leq x_1$ and $y'' \wedge x'' \leq x_1$ would give

$$y' \wedge y'' \wedge x \in \alpha \quad \text{and} \quad y' \wedge y'' \wedge x = (y' \wedge y'' \wedge x') \vee (y' \wedge y'' \wedge x'') \leq x_1\,,$$

which is impossible. Therefore the filter generated by $\{y \wedge x' \mid y \in \alpha\}$ is in A and by the maximality of α, $x' \in \alpha$.

2.7. *Theorem.* The spectrum functor $\mathrm{Spc}\colon \mathsf{Slt} \to \mathfrak{I}$ is an embedding.

Proof. The functor Spc is obviously injective on the objects. The faithfulness is an easy consequence of the previous Theorem 2.6: if $a, b \in \mathsf{Slt}(X, Y)$, the commutative squares

$$\begin{tikzcd} X \arrow[r, "\eta X"] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & F(X) \arrow[d, shift right, "Fa"'] \arrow[d, shift left, "Fb"] \\ Y \arrow[r, "\eta Y"'] & F(Y) \end{tikzcd} \tag{1}$$

where $F = \mathrm{Prj}\cdot\mathrm{Spc}$ and ηY is injective, show that $Fa = Fb$ implies $a = b$.

2.8. For every Prj-small inverse category K we define the *spectrum d-functor*:

$$\mathrm{Spc}_{\mathsf{K}}\colon \mathsf{K} \to \mathfrak{I}\,; \qquad \mathrm{Spc}_{\mathsf{K}} = \mathrm{Spc}\cdot\mathrm{Prj}_{\mathsf{K}}\,. \tag{1}$$

It takes values in $\mathfrak{I}^f$ whenever K is Prj-finite, and is an *embedding* when K is *transfer*.

Notice that, for $\mathsf{K} = \mathsf{Slt}$, $\mathrm{Spc}_{\mathsf{K}} \simeq \mathrm{Spc}$ by 1.5.1.

2.9. Finally we notice that there exists a «simpler» faithful functor:

$$F\colon \mathsf{Slt} \to \mathfrak{I} \tag{1}$$

which turns every 1-semilattice X into its underlying set $F(X)$, and every transfer pair $a = (a_*, a^*)\colon X \to Y$ into the partial bijection:

$$F(a) = (X_0, Y_0; a_0)\colon F(X) \to F(Y)\,, \tag{2}$$

$$X_0 = \{x \in X \mid x < a^*(1_Y)\}\,; \qquad Y_0 = \{y \in Y \mid y < a_*(1_X)\}\,, \tag{3}$$

$$a_0(x) = a_*(x)\,; \qquad a_0^{-1}(y) = a^*(y)\,. \tag{4}$$

The faithfulness of F [3] is an easy consequence of Slt being transfer (1.6).

However this functor does *not* preserve finite distributive unions of projections, hence does *not* produce exact functors for distributive exact categories (n. 4). Actually, let $1 = x_1 \vee x_2$ be a distributive union in X; the projections $e_i = x_i \wedge -\colon X \to X$ yield a distributive union $1 = e_1 \vee e_2$ in $\mathrm{Prj}(X)$ (1.5), while

[3] It is easy to derive from F an *embedding* $F'\colon \mathsf{Slt} \to \mathfrak{I}$. For example take $F'(X) = \{(x, y) \in X^2 \mid x \leq y \text{ in } X\}$, which determines both the underlying set $F(X) = \mathrm{pr}_1 F'(X)$ (because $x \leq 1$ for every $x \in X$) and the order structure; for $a \in \mathsf{Slt}(X, Y)$, take $F'(a) = (X_0', Y_0'; a_0')\colon F'(X) \to F'(Y)$, where $X_0' = X_0 \times \{1\}$ and so on.

their F-images $f_i = F(e_i) = (X_i, X_i; 1_{X_i})\colon X \to X$, with $X_i = \{x \in X | x \leq x_i\}$ have distributive union in $\mathrm{Prj}(FX) \simeq \mathfrak{I}(FX)$:

$$f_1 \vee f_2 = (X_0, X_0; 1_{X_0})\colon X \to X,$$

with

$$X_0 = X_1 \cup X_2 = \{x \in X | x \leq x_1 \text{ or } x \leq x_2\}\,;$$

generally, $X_0 \neq X$ and $f_1 \vee f_2 \neq 1$.

3. - The extended spectrum representation for small inverse categories

Let K be a *small* inverse category, possibly non transfer: we build an *extended spectrum d-embedding* $\widehat{\mathsf{Spc}}_{\mathsf{K}}\colon \mathsf{K} \to \mathfrak{Z}$.

3.1. For every object A call $\mathrm{Var}_{\mathsf{K}}(A)$ the (small) set of *variable points* (or *variables*) of A, i.e. the morphisms of K having codomain A, provided with the multiplication:

$$x \,\Delta\, y = x\tilde{x}y \tag{1}$$

and the distinguished element $1 = 1_A\colon A \to A$.

The order relation $\sqsubset$ (1.3) on $\mathrm{Var}_{\mathsf{K}}(A)$ is determined by Δ: $x \sqsubset y$ iff $x = x \,\Delta\, y$; besides $x \in \mathrm{Var}_{\mathsf{K}}(A)$ is a projection (of A) iff $x \,\Delta\, 1 = x$.

Every morphism $a\colon A' \to A''$ of K determines a pair $\mathrm{Var}(a) = (a_V, a^V)$ of mappings:

$$a_V\colon \mathrm{Var}(A') \to \mathrm{Var}(A'')\,; \qquad a_V(x) = ax\,, \tag{2}$$

$$a^V\colon \mathrm{Var}(A'') \to \mathrm{Var}(A')\,; \qquad a^V(y) = \tilde{a}y\,. \tag{3}$$

3.2. Therefore we introduce, as a codomain for the will-be functor $\mathrm{Var}_{\mathsf{K}}$, the category Slt of *Δ-bands and transfer pairs*.

Its objects are the sets $X = (X, \Delta, 1)$ provided with a product Δ and a distinguished element 1, so that:

a) Δ is associative and idempotent ($x \,\Delta\, x = x$, for every x),

b) 1 is *a* left unit for Δ,

c) $x \,\Delta\, x' \,\Delta\, x'' = x' \,\Delta\, x \,\Delta\, x''$, for every x, x', x''.

Its morphisms (*transfer pairs of Δ-bands*) are pairs $(a_\bullet, a^\bullet)\colon X \to Y$ where:

d) $a_\bullet\colon X \to Y$ and $a^\bullet\colon Y \to X$ are Δ-homomorphisms,

e) $a_\bullet(x \,\Delta\, a^\bullet(y)) = a_\bullet(x) \,\Delta\, y$, for every $x \in X$ and $y \in Y$,

f) $a^\bullet(y \,\Delta\, a_\bullet(x)) = a^\bullet(y) \,\Delta\, x$, for every $x \in X$ and $y \in Y$.

The composition, as usual, is: $(b_\bullet, b^\bullet) \circ (a_\bullet, a^\bullet) = (b_\bullet a_\bullet, a^\bullet b^\bullet)$.

3.3. *Theorem.* $\widehat{\mathsf{Slt}}$ is an inverse category; its projection functor $\mathrm{Prj}\colon \widehat{\mathsf{Slt}} \to \mathsf{Slt}$ is isomorphic to:

(1) $P\colon \widehat{\mathsf{Slt}} \to \mathsf{Slt}$,

(2) $P(X) = X_0 = \{x_0 \in X \mid x_0 = x_0 \Delta 1\} = \{x \Delta 1 \mid x \in X\}$,

(3) $P(a_\bullet, a^\bullet) = (a_0, a^0)$; $\quad a_0(x_0) = (a_\bullet x_0) \Delta 1$; $\quad a^0(y_0) = (a^\bullet y_0) \Delta 1$,

via

(4) $\iota\colon P \to \mathrm{Prj}\colon \widehat{\mathsf{Slt}} \to \mathsf{Slt}$,

(5) $(\iota X)_\bullet(x_0) = (x_0 \Delta -, x_0 \Delta -)$; $\quad (\iota X)^\bullet(e) = e_\bullet(1)$.

Moreover, if $a = (a_\bullet, a^\bullet)\colon X \to Y$ is in $\widehat{\mathsf{Slt}}$ and $P(a) = (a_0, a^0)$ is the associated morphism in Slt:

(6) $a_\bullet(x) \Delta 1 = a_0(x \Delta 1)$, $\quad$ for every $x \in X$,

(7) $a^\bullet(y) \Delta 1 = a^0(y \Delta 1)$, $\quad$ for every $y \in Y$.

In other words, the homomorphisms of Δ-bands $\pi X\colon X \to P(X)$, $x \mapsto x \Delta 1$, form a horizontal transformation of suitable vertical functors.

Proof. $\widehat{\mathsf{Slt}}$ has an involution:

(8) $$(a_\bullet, a^\bullet)^\sim = (a^\bullet, a_\bullet)$$

which is regular, since:

(9) $a_\bullet a^\bullet a_\bullet(x) = a_\bullet(1 \Delta a^\bullet a_\bullet(x)) = a_\bullet(1) \Delta a_\bullet(x) = a_\bullet(1 \Delta x) = a_\bullet(x)$.

Moreover, for every Δ-band X we have a bijection $(\iota X)_\bullet\colon X_0 \to \mathrm{Prj}(X)$ between $P(X) = X_0$ and the set of projections of X in $\widehat{\mathsf{Slt}}$ given by the formulas (5): indeed, if $x_0 \in X_0$, $x \in X$ and $e = (e_\bullet, e^\bullet) \in \mathrm{Prj}(X)$:

(10) $x_0 \Delta 1 = x_0$,

(11) $e_\bullet(1) \Delta x = e_\bullet(1 \Delta e^\bullet(x)) = e_\bullet e^\bullet(x) = e_\bullet(x)$.

Now X_0 is a commutative subsemigroup of X, hence a 1-semilattice ($x_0 \Delta x_0' = x_0 \Delta x_0' \Delta 1 = x_0' \Delta x_0 \Delta 1 = x_0' \Delta x_0$) and $(\iota X)_\bullet$ turns the Δ-product of X_0 into the composition-product of $\mathrm{Prj}(X)$ (as $(x_0 \Delta x_0') \Delta - = x_0 \Delta (x_0' \Delta -)$): therefore the projections of X commute, and $\widehat{\mathsf{Slt}}$ is inverse.

The « will-be transformation » ι turns the mapping P into the functor Prj because, for $a = (a_\bullet, a^\bullet) \in \widehat{\mathsf{Slt}}(X, Y)$ and $x_0 \in X_0$:

$$(12) \qquad (\iota Y)^\bullet a_P (\iota X)_\bullet (x_0) = (\iota Y)^\bullet a_P (x_0 \Delta -) = (\iota Y)^\bullet (a_\bullet(x_0 \Delta a^\bullet(-)) =$$
$$= a_\bullet(x_0 \Delta a^\bullet(1)) = a_\bullet(x_0) \Delta 1 = a_0(x_0).$$

This proves at the same time that P is a functor and ι is a functorial isomorphism.

As to the last assertion, if $x \in X$:

$$(13) \qquad a_0(x \Delta 1) = a_\bullet(x \Delta 1) \Delta 1 = a_\bullet(x) \Delta a_\bullet(1) \Delta 1 = a_\bullet(1) \Delta a_\bullet(x) \Delta 1 =$$
$$= a_\bullet(1 \Delta x) \Delta 1 = a_\bullet(x) \Delta 1 .$$

3.4. Now, for every *small* inverse category K, the definitions of 3.1 (and some easy verifications) produce a functor

$$(1) \qquad \mathrm{Var}_{\mathsf{K}} \colon \mathsf{K} \to \widehat{\mathsf{Slt}}$$

which is an embedding ($a_\Gamma(1) = a$, for every morphism a).

Moreover $\mathrm{Var}_{\mathsf{K}}$ *preserves and reflects arbitrary distributive unions.* Indeed, the functorial iso ι in 3.3.4 produces, by horizontal composition with $\mathrm{Var}_{\mathsf{K}}$:

$$(2) \qquad \mathsf{K} \xrightarrow{\mathrm{Var}_{\mathsf{K}}} \mathsf{Slt} \underset{\mathrm{Prj}}{\overset{P}{\underset{\downarrow\iota}{\rightrightarrows}}} \mathsf{Slt}$$

the functorial *iso*:

$$(3) \qquad \sigma = \iota \cdot \mathrm{Var}_{\mathsf{K}} \colon P \cdot \mathrm{Var}_{\mathsf{K}} \to \mathrm{Prj} \cdot \mathrm{Var}_{\mathsf{K}} .$$

As the functors $P \cdot \mathrm{Var}_{\mathsf{K}} = \mathrm{Prj}_{\mathsf{K}}$ and Prj in (3) preserve and reflect arbitrary distributive unions (1.9), so does $\mathrm{Var}_{\mathsf{K}}$.

3.5. We want now to define the «extended spectrum» of a Δ-band X: of course it has to be larger than its projection-spectrum $\mathrm{Spc}\,\mathrm{Prj}\,(X) \simeq \mathrm{Spc}\,(X_0)$ otherwise we should get for K nothing more than the preceding spectrum functor (n. 2): $\mathrm{Spc} \cdot \mathrm{Prj} \cdot \mathrm{Var}_{\mathsf{K}} \simeq \mathrm{Spc} \cdot \mathrm{Prj}_{\mathsf{K}} = \mathrm{Spc}_{\mathsf{K}}$ (3.4).

Define the relation $x < y$ in X by:

$$(1) \qquad x < y \qquad \text{if } x = x \Delta y .$$

It is easy to see that it is an order relation, consistent with the product Δ, preserved by Δ-homomorphisms, and that:

$$(2) \qquad x \in X_0 \qquad \text{iff } x < 1 .$$

Say that z is the *distributive union* of x and y in X (or: $z = x \vee y$ is a *distributive union*) if:

a) $x, y < z$,

b) $z \Delta 1 = (x \Delta 1) \vee (y \Delta 1)$, distributive union in X_0.

By 1.6 and properties 3.3.6-7 the covariant and contravariant parts of a transfer pair of Δ-bands preserve and reflect arbitrary distributive unions.

3.6. Analogously to 2.2, define the *extended spectrum* $\widehat{\mathrm{Spc}}\,(X)$ of a Δ-band X to be the set of Δ-homomorphism $\varphi\colon X \to \Omega = \{0, 1\}$ preserving finite distributive unions and 1.

Equivalently, one can consider the set $\widehat{\mathrm{Spc}}'\,(X)$ of *prime filters* $\alpha = \varphi^{-1}(1)$ of X, characterized by:

a) $\alpha \subset X$ is a stable for Δ-products and $1 \in \alpha$,

b) if $x_1 < x_2$ in X and $x_1 \in \alpha$ then $x_2 \in \alpha$,

c) if $z = x \vee y$ is a distributive union in X and $z \in \alpha$, then either $x \in \alpha$ or $y \in \alpha$; moreover $0_X \notin \alpha$ (when 0_X exists).

Or also the set $\widehat{\mathrm{Spc}}''\,(X)$ of *prime ideals* $\beta = \varphi^{-1}(0)$ of X, having a characterization similar to 2.3*d*)-*f*).

3.7. We consider now the functor

$$\widehat{\mathrm{Spc}}\colon \mathfrak{St} \to \mathfrak{I} \tag{1}$$

associating to every Δ-band X the (small) set $\widehat{\mathrm{Spc}}\,(X)$, and to every transfer pair $a = (a_\bullet, a^\bullet)\colon X \to Y$ the partial bijection:

$$\widehat{\mathrm{Spc}}\,(a) = (F_a, G_a; a_0)\colon \widehat{\mathrm{Spc}}\,(X) \to \widehat{\mathrm{Spc}}\,(Y), \tag{2}$$

$$F_a = \{\varphi \in \widehat{\mathrm{Spc}}\,(X) | \varphi(a^\bullet(1)) = 1\}\,; \qquad G_a = \{\psi \in \widehat{\mathrm{Spc}}\,(Y) | \psi(a_\bullet(1)) = 1\}\,, \tag{3}$$

$$a_0(\varphi) = \varphi a^\bullet\,; \qquad a_0^{-1}(\psi) = \psi a_\bullet\,, \tag{4}$$

so that, e.g. for $\varphi \in F_a$ and $x \in X$:

$$\begin{aligned}(a_0^{-1} a_0(\varphi))(x) = \varphi a^\bullet a_\bullet(x) = \varphi(a^\bullet(1) \Delta x) = \\ = \varphi(a^\bullet(1)) \Delta \varphi(x) = 1 \Delta \varphi(x) = \varphi(x)\,.\end{aligned} \tag{5}$$

$\widehat{\mathrm{Spc}}$ is a *d-functor*: the proof is strictly analogous to 2.3.

3.8. Thus, every *small* inverse category K has an *extended spectrum d-embedding*:

$$\widehat{\mathrm{Spc}}_{\mathsf{K}}\colon \mathsf{K} \to \mathfrak{I}\,; \qquad \widehat{\mathrm{Spc}}_{\mathsf{K}} = \widehat{\mathrm{Spc}} \cdot \mathrm{Var}_{\mathsf{K}}\,. \tag{1}$$

When K is finite, this functor takes values in $\mathfrak{I}^f$.

3.9. Like in 2.9, we notice that $\widehat{\mathsf{Slt}}$ has a simpler faithful functor

$$(1) \qquad \hat{F}\colon \widehat{\mathsf{Slt}} \to \mathfrak{I}$$

which is an extension of F in 2.9.1, with an analogous description. Of course $\hat{F}$ is not a d-functor.

4. - Representations for distributive exact categories

A distributive exact category E canonically embeds in an inverse category $\Theta(\mathsf{E})$ [10]; together with the d-embeddings of n. 2, 3, this produces exact embeddings of E in $\mathfrak{I}$ (which is distributive exact) when E is transfer and Sub-small, or respectively small.

4.1. Let E be an exact category, in the sense of Puppe [20, 19]: it has a zero object, kernels and cokernels and every map factorizes via a conormal epi and a normal monic. We refer to [13; n. 1] for a short review of results on exact categories.

We write

$$(1) \qquad \mathrm{Sym}_{\mathsf{E}}\colon \mathsf{E} \to \mathrm{Rel}\,(\mathsf{E})$$

the *canonical symmetrization* of E, i.e. its embedding in its category of relations [4, 3, 7], provided with the usual involution (which is regular).

4.2. $\mathrm{Rel}\,(\mathsf{E})$ is *orthodox* (i.e. idempotent endomorphisms are stable under composition [8]) iff E is *distributive* (that is, has distributive lattices of subobjects) [10; § 1.10]. In such a case $\mathrm{Rel}\,(\mathsf{E})$ is provided with a *preorder* $a \sqsubset b$, characterized by properties 1.3.1-3, and with the associated congruence Φ [8].

The quotient $\Theta(\mathsf{E}) = \mathrm{Rel}\,(\mathsf{E})/\Phi$ is an *inverse* category (whose induced order coincides with the canonical one (1.3)); the composition of $\mathrm{Sym}_{\mathsf{E}}$ (4.1) with the quotient functor $Q\colon \mathrm{Rel}\,(\mathsf{E}) \to \Theta(\mathsf{E})$ gives the *canonical inverse symmetrization* (or *Θ-symmetrization*) of the distributive exact category E:

$$(1) \qquad \mathrm{Sym}^{\Theta}_{\mathsf{E}}\colon \mathsf{E} \to \Theta(\mathsf{E}) = \mathrm{Rel}\,(\mathsf{E})/\Phi$$

studied in [10]; it is still an *embedding*.

The inverse category $\Theta(\mathsf{E})$ is small iff E is so; analogously, $\Theta(\mathsf{E})$ is Prj-small iff E is *Sub-small* (i.e. well-powered). $\Theta(\mathsf{E})$ is transfer iff E is so (i.e. the transfer functor $\mathrm{Sub}_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{Mlc}$ of E [13] is faithful).

4.3. Now, let $F\colon \mathsf{E} \to \mathsf{E}'$ be a functor between (Sub-small) exact categories. F is called *exact* whenever it preserves kernels and cokernels (or, equivalently, exact sequences). It can be proved [11; § 6.2-3] that, if E and E' are *distributive*, F is exact iff:

a) F is *Θ-symmetrizable*, i.e. there exists a (necessarily unique) functor $\Theta F\colon \Theta(\mathsf{E}) \to \Theta(\mathsf{E}')$ extending F,

b) $\Theta(F)$ is a *d-functor* (1.9).

Thus *finite distributive unions and d-functors surrogate exact sequences and exact functors* for inverse categories (for further details see [11]).

4.4. An exact category E is inverse iff it is *boolean*, i.e. has boolean lattices of subobjects (see [13, § 6.4], where other characterizations are given).

For an exact inverse category E, one can assume that $\Theta(\mathsf{E}) = \mathsf{E}$, and that $\mathrm{Sym}^{\sharp}_{\mathsf{E}}\colon \mathsf{E} \to \Theta(\mathsf{E})$ is the identity [10].

4.5. It is not difficult to see that $\mathfrak{I}$ is boolean exact [12]; thus:

(1) $$\mathfrak{I} \xrightarrow{\mathrm{Sym}} \mathrm{Rel}\,(\mathfrak{I}) \xrightarrow{\ Q\ } \mathfrak{I}\,; \qquad Q\cdot\mathrm{Sym} = 1\,.$$

Moreover, notice that $\mathfrak{I}$ has an exact embedding

(2) $$F\colon \mathfrak{I} \to R\text{-}\mathsf{Mod}$$

in the category of left R-modules, where R is any non-trivial unitary ring. Indeed, for every small set S and every $a = (S_0, T_0; a_0) \in \mathfrak{I}(S, T)$, let $F(S) = R^{(S)}$ be the free R-module on S and $F(a)\colon R^{(S)} \to R^{(T)}$ the unique R-homomorphism such that:

(3) $$F(a)(x) = a_0(x)\,, \qquad \text{for } x \in S_0\,,$$

(4) $$F(a)(x) = 0\,, \qquad \text{for } x \in S - S_0\,.$$

Thus $\mathfrak{I}$ is isomorphic to its F-image $\mathfrak{I}_R$, an exact subcategory of R-Mod; *every exact embedding in* $\mathfrak{I}$ *yields an exact embedding in* R-Mod.

4.6. We are also interested in the modular expansion $\mathfrak{Z} = \mathrm{Mdl}\,(\mathfrak{I})$ [12, 14] ([*]), the (distributive, non boolean) exact category of *semitopological spaces and open-closed partial homeomorphisms*: an object is a pair (S, X) where S is a set and X a sublattice of $\mathfrak{P}S$ containing 0 and S (whose elements we call *closed* subsets of S); a morphism $u = (S_0, T_0; u_0)\colon (S, X) \to (T, Y)$ is a homeomorphism u_0 from an open subset S_0 of (S, X) onto a closed subset T_0 of (T, Y); the composition is obvious.

There are functors:

(1) $$U\colon \mathfrak{Z} \to \mathfrak{I}\,; \qquad U(S, X) = S\,,$$

(2) $$\mathfrak{Z} \xrightarrow{\mathrm{Sym}} \mathrm{Rel}\,(\mathfrak{Z}) \xrightarrow{\ Q\ } \Theta(\mathfrak{Z})$$

([*]) Since $\mathfrak{I}$ is distributive, $\mathfrak{Z}$ coincides with the distributive expansion $\mathrm{Dst}\,(\mathfrak{I})$ of $\mathfrak{I}$ [13].

and the inverse category $\Theta(\mathfrak{Z})$ can be described as the category of *semitopological spaces and partial homeomorphisms between locally closed subspaces*; the description of $\mathrm{Rel}(\mathfrak{Z})$ is more complicated (see [12]).

4.7. Our interest in $\mathfrak{Z}$ comes from the universal property of modular expansions [14]: every exact functor $F\colon \mathsf{E} \to \mathfrak{I}$, where E is an exact category, has a unique *Sub-full* lifting

$$(1) \qquad F^\sharp\colon \mathsf{E} \to \mathfrak{Z}\,; \qquad F^\sharp(A) = \big(F(A),\, \mathrm{Sub}_F(\mathrm{Sub}_{\mathsf{E}}(A))\big)$$

verifying $UF^\sharp = F$.

In particular, there is a one-to-one correspondence between the exact subcategories of $\mathfrak{I}$ and the Sub-full exact subcategories of $\mathfrak{Z}$ whose objects have different underlying sets.

4.8. *Representations for Sub-small transfer distributive exact categories.* Every *Sub-small* distributive exact category E has exact functors (deriving from the spectrum d-functors 2.8.1):

$$(1) \qquad \mathrm{Spc}_{\mathsf{E}}\colon \mathsf{E} \to \mathfrak{I}\,; \qquad \mathrm{Spc}_{\mathsf{E}} = \mathrm{Spc}_{\theta\mathsf{E}} \cdot \mathrm{Sym}^{\theta}_{\mathsf{E}}\,,$$

$$(2) \qquad \mathrm{Spc}^\sharp_{\mathsf{E}}\colon \mathsf{E} \to \mathfrak{Z}\,; \qquad U \cdot \mathrm{Spc}^\sharp_{\mathsf{E}} = \mathrm{Spc}_{\mathsf{E}}\,,$$

which are embeddings whenever E is transfer; the second is always Sub-full.

Actually, the exactness of (1) follows from 4.3: $\mathrm{Spc}_{\mathsf{E}}$ has clearly Θ-symmetrization $\mathrm{Spc}_{\theta\mathsf{E}}$, which is a d-functor by 2.8.

Remark that, when E is both exact and inverse, $\mathrm{Sym}^{\theta}_{\mathsf{E}} = 1$ (4.4) so that there is no ambiguity on the functor $\mathrm{Spc}_{\mathsf{E}}$.

Notice also that the functors (1), (2) take values in $\mathfrak{I}^f$ and $\mathfrak{Z}^f$ when E is *Sub-finite* (every object has a finite lattice of subobjects).

4.9. *Representations for small distributive exact categories.* In the same way, every *small* distributive exact category has *extended spectrum exact embeddings* (deriving from the extended spectrum d-embeddings 3.8.1):

$$(1) \qquad \widehat{\mathrm{Spc}}_{\mathsf{E}}\colon \mathsf{E} \to \mathfrak{I}\,; \qquad \widehat{\mathrm{Spc}}_{\mathsf{E}} = \widehat{\mathrm{Spc}}_{\theta\mathsf{E}} \cdot \mathrm{Sym}^{\theta}_{\mathsf{E}}\,,$$

$$(2) \qquad \widehat{\mathrm{Spc}}^\sharp_{\mathsf{E}}\colon \mathsf{E} \to \mathfrak{Z}\,; \qquad U \cdot \widehat{\mathrm{Spc}}^\sharp_{\mathsf{E}} = \widehat{\mathrm{Spc}}_{\mathsf{E}}\,.$$

These embeddings take values in $\mathfrak{I}^f$, $\mathfrak{Z}^f$ when E is finite.

4.10. *Representations of distributive RE-categories.* If A is a distributive *Prj-small* RE-category [14], we have *spectrum RE-functors* (the second being Prj-full):

$$(1) \qquad \mathrm{Spc}_{\mathsf{A}}\colon \mathsf{A} \to \mathrm{Rel}(\mathfrak{I})\,,$$

$$(2) \qquad \mathrm{Spc}^\sharp_{\mathsf{A}}\colon \mathsf{A} \to \mathrm{Rel}(\mathfrak{Z})\,,$$

given by the composition [14; § 6.8]

$$\mathsf{A} \xrightarrow{\eta \mathsf{A}} \mathrm{Fct}\,(\mathsf{A}) \xrightarrow{=} \mathrm{Rel}\,(\mathsf{E}) \xrightarrow{\mathrm{Rel}(F)} \tag{3}$$

where $\mathsf{E} = Z(\mathrm{Prp}\,(\mathrm{Fct}\,(\mathsf{A})))$ is the (distributive Sub-small) exact category associated to A, $\eta\mathsf{A}$ is full and F is respectively $\mathrm{Spc}_{\mathsf{E}}$ or $\mathrm{Spc}^f_{\mathsf{E}}$. These functors are RE-embeddings whenever A is transfer [14; § 7.3].

Analogously, every small distributive RE-category A has *extended spectrum RE-embeddings* (the second Prj-full):

$$\widehat{\mathrm{Spc}}_{\mathsf{A}} : \mathsf{A} \to \mathrm{Rel}\,(\mathfrak{I})\,, \tag{4}$$

$$\widehat{\mathrm{Spc}}^f_{\mathsf{A}} : \mathsf{A} \to \mathrm{Rel}\,(\mathfrak{J})\,. \tag{5}$$

4.11. These results will prove that every distributive exact theory has a classifying exact category which is an exact subcategory of $\mathfrak{I}$ and a Sub-full exact subcategory of $\mathfrak{J}$. Analogous results hold for RE-theories.

5. - Representations for idempotent inverse categories and pre-idempotent exact categories

5.1. We say that a category A, provided with a regular involution (or, more particularly, inverse) is idempotent (4) when all its endomorphisms are so; this happens iff A is orthodox and the inverse category A/Φ is idempotent (see 4.2 or [8]).

We say that the exact category E is *pre-idempotent* when its category of relations $\mathrm{Rel}\,(\mathsf{E})$ is idempotent (a direct characterization is given in [14; Thm. 8.8]); by the above remark this happens iff E is distributive and its canonical inverse symmetrization $\Theta(\mathsf{E})$ (4.2) is idempotent.

5.2. Here the paradigmatic case is $\mathfrak{I}_0$, the category of *small sets and common parts* (or *partial identities*): the objects are the small sets, while a morphism $L: S \to T$ is a common subset of S and T; the compositions is given by the intersection.

We identify the morphism $L: S \to T$ with the partial bijection $(L, L; 1_L)$: $S \to T$, so that $\mathfrak{I}_0$ becomes a Sub-full pre-idempotent boolean exact subcategory of $\mathfrak{I}$. Write $\mathfrak{I}^f_0 = \mathfrak{I}^f \cap \mathfrak{I}_0$ the Hom-finite pre-idempotent boolean exact category of small finite sets and common parts.

For a small set S we write $\mathfrak{I}_0\langle S\rangle$ (resp. $\mathfrak{I}^f_0\langle S\rangle$) the full subcategory of $\mathfrak{I}_0$ whose objects are the subsets (resp. finite subsets) of S; it is a pre-idempotent boolean exact subcategory of $\mathfrak{I}_0$ (resp. $\mathfrak{I}^f_0$).

(4) Probably the « good » notion for *idempotent category* is a *g-regular* category (i.e. a category in which every morphism has *some* generalized inverse [8]; called « regular » in [16]) such that every endomorphism is idempotent.

5.3. We also use $\mathfrak{J}_0 = \mathrm{Mdl}\,(\mathcal{J}_0) = \mathrm{Dst}\,(\mathcal{J}_0)$, the pre-idempotent exact category of small *semitopological spaces and open-closed subspaces* (or *open-closed partial identities*): a morphism $L\colon S \to T$ is here given by a common subspace L of S and T (*same* induced semitopology) which is open in S and closed in T; the composition is again by intersection.

We set

$$\mathfrak{J}'_0 = \mathfrak{J}' \cap \mathfrak{J}_0 .$$

5.4. *Lemma* (*). Let $F\colon \mathsf{K} \to \mathsf{K}'$ be a functor between inverse categories. If K is idempotent, F is faithful iff it is Prj-faithful.

Proof. Let F be Prj-faithful, and a, a' be parallel morphisms with $Fa = Fa'$; then $F(a\tilde{a}) = F(a'\tilde{a}')$ and $a\tilde{a} = a'\tilde{a}'$.

We can assume that $a \sqsubset a'$ (otherwise consider $a_0 = (a\tilde{a}')a = (a\tilde{a}')a'(\tilde{a}'a)$ which is dominated both by a and a', because the endomorphisms $a\tilde{a}'$ and $\tilde{a}'a$ are idempotent). Then $a = (a\tilde{a})a' = a'\tilde{a}'a' = a'$ (1.3).

5.5. *Corollary.* Every idempotent inverse category K is transfer. Moreover K is Hom-finite iff it is Prj-finite, iff every Prj-set of K is finitely d-generated (1.7).

Proof. The transfer functor $\mathrm{Prj}_{\mathsf{K}}\colon \mathsf{K} \to \mathsf{Slt}$ is faithful by 5.4. If K is Prj-finite, this functor takes values in the Hom-finite subcategory Slt'; the conclusion follows from 2.6.

5.6. *The Glueing Theorem for idempotent inverse categories.* Every idempotent inverse subcategory K of $\mathfrak{J}$ has an embedding in $\mathfrak{J}_0$, which preserves (finite or arbitrary) distributive unions when the inclusion $\mathsf{K} \to \mathfrak{J}$ does. If K is small, this embedding takes values in $\mathfrak{J}_0\langle Z\rangle$, for a suitable small set Z.

Proof. Let Z' be the (possibly non-small) disjoint union of all the sets which are objects of K, and define the equivalence relation $\sim$ in Z' by assuming that $x \in S \in \mathrm{Ob}\,\mathsf{K}$ is equivalent to $y \in T \in \mathrm{Ob}\,\mathsf{K}$ if there is some partial bijection in $\mathsf{K}(S, T)$ which turns x into y. Notice that, K being idempotent in $\mathfrak{J}$, every endomorphism of K is a partial identity: therefore $\sim$ suborders the equality on every object of K.

Set $Z'' = Z'/\!\sim$ and let Z be the disjoint union of Z'' and $\mathrm{Ob}\,\mathsf{K}$; notice that, when K is small, the sets Z', Z'', Z are so. Moreover, for every $S_0 \subset S \in \mathrm{Ob}\,\mathsf{K}$ let Z''_{S_0} be the image of $S_0 \subset Z'$ in $Z'' = Z'/\!\sim$ and $Z_S = Z''_S \cup \{S\} \subset Z$; Z''_{S_0} and Z_S are always small.

(*) This results extends, with analogous proof, to involution-preserving functors between categories with regular involution: see [13; § 7.4] for the transfer functor of such categories.

Finally we define:

(1) $\hat{Z}\colon \mathsf{K} \to \mathfrak{J}_0$,

(2) $\hat{Z}(S) = Z_S$,

(3) $\hat{Z}((S_0, T_0; a_0)\colon S \to T) = H\colon Z_S \to Z_T \quad (H = Z'_{S_0} = Z'_{T_0})$.

$\hat{Z}$ is a functor: if $a = (S_0, T_0; a_0)\colon S \to T$ and $b = (T_1, U_1; b_1)\colon T \to U$ are in K

(4) $$\hat{Z}(b) \cdot \hat{Z}(a) = Z'_{T_0} \cap Z'_{T_1} = Z'_{T_0 \cap T_1} = \hat{Z}(ba)$$

where the second equality comes from the above remark on $\sim$, applied to the object T.

$\hat{Z}$ is trivially injective on the objects and Prj-faithful; by 5.4 it is an embedding; its preserving distributive unions is again a direct consequence of the above remark on $\sim$. Finally, the last assertion has already been checked: Z is small when K is so.

5.7. *Representation Theorem for idempotent categories.*

a) Every Prj-small idempotent inverse category K has a d-embedding in $\mathfrak{J}_0$, which takes values in a suitable $\mathfrak{J}_0\langle S\rangle$ when K is small.

b) Every Sub-small pre-idempotent exact category E has an exact embedding in $\mathfrak{J}_0$ and a Sub-full exact embedding in $\mathfrak{Z}_0$.

c) Every Prj-small idempotent RE-category [14] has a RE-embedding in Rel $(\mathfrak{J}_0)$ and a Prj-full RE-embedding in Rel $(\mathfrak{Z}_0)$.

In the Prj-finite case (Sub-finite for *b*)) these embeddings take values in $\mathfrak{J}'_0$ and so on.

Proof. The assertion *a*) follows immediately from 2.8, 5.5 and 5.6; *b*) follows from *a*), 4.3 and the universal property of $\mathfrak{Z}_0 = \mathrm{Mdl}\,(\mathfrak{J}_0)$; *c*) follows from *b*), via the full RE-embedding 4.9.3. The last remark follows from 2.8.

5.8. This result will prove that the classifying exact category of every pre-idempotent exact theory is an exact subcategory of $\mathfrak{J}_0$ and a Sub-full exact subcategory of $\mathfrak{Z}_0$.

Various « homological » theories will be proved to be (pre)-idempotent.

6. - Natural representations for semilattices and distributive lattices

The possibility of embedding semilattices and distributive lattices in lattices of parts is classically known. We use here the representation 2.6 for semilattices to sketch embeddings which are natural with regard to a category **Sl** of 1-semilattices containing both **Sld** and **Slt**, or respectively to a category **Dl** of distributive 0, 1-lattices containing both **Dlh** and **Dlc**.

6.1. Call **Sl** the category of *1-semilattices and d-homomorphisms of semilattices* (possibly not preserving the unit). **Sld** is a subcategory of **Sl**.

Moreover, also the functor:

(1) $U\colon \mathbf{Slt} \to \mathbf{Sl}\,; \qquad X \mapsto X\,; \qquad (a_*, a^*) \mapsto a_*$

is an embedding, because of the last remarks in 1.4 and 1.6.

6.2. Now, the spectrum functor Spc: **Slt** $\to \mathfrak{I}$ (2.2) extends to:

(1) $\mathrm{Spc}\colon \mathbf{Sl} \to \mathbf{Sfn}^*\,,$

(2) $\mathrm{Spc}\,(f)(\psi) = \psi f$, for those $\psi \in \mathrm{Spc}\,(\mathrm{Cod}\, f)$ such that $\psi f(1) = 1\,,$

where **Sfn*** is the dual of the category of *small sets and functions* (i.e. partially defined mappings).

6.3. Analogously, the functor $\mathrm{Prj} \simeq \mathfrak{F}\colon \mathfrak{I} \to \mathbf{Blt} \subset \mathbf{Slt}$ (*) extends to a functor:

(1) $\mathfrak{F}\colon \mathbf{Sfn}^* \to \mathbf{Bl}\,,$

(2) $\mathfrak{F}(f\colon T \to S) = (\mathfrak{F}f\colon \mathfrak{F}S \to \mathfrak{F}T)\,; \qquad \mathfrak{F}f(S_0) = f^{-1}(S_0)\,,$

where **Bl** is the category of Boolean algebras and lattice homomorphisms.

6.4. The horizontal transformation η (2.5) becomes a natural transformation (where U is the inclusion functor):

(1) $\eta\colon U \to \mathfrak{F}\cdot \mathrm{Spc}\colon \mathbf{Sl} \to \mathbf{Bl}\,,$

(2) $\eta X(x) = \{\psi \in \mathrm{Spc}\,(X) | \psi(x) = 1\}\,,$

which is pointwise a 1-preserving embedding.

6.5. As to distributive lattices, call **Dl** the category of *small distributive* 0, 1-*lattices and homomorphisms of lattices*. We shall use the Macneille functor [18, 11]:

(1) $\mathbf{Dl} \xrightarrow{\mathrm{Mac}} \mathbf{Sl}$

(*) Here **Blt** is the full subcategory of **Slt** of small boolean algebras [13; § 7.3].

turning every distributive 0, 1-lattices X into the 1-semilattice $\mathrm{Mac}\,(X) = \hat{X} = X_2/\Phi$ canonically associated to the idempotent 1-semigroup:

$$X_2 = \{(x_1, x_2) \in X^2 | x_2 \leq x_1\}, \tag{2}$$

$$(x_1, x_2) \sqcap (x_1', x_2') = ((x_1 \wedge x_1') \vee x_2, (x_1 \wedge x_2') \vee x_2). \tag{3}$$

Notice that Mac sends **Dlh** into **Sld** and **Dlc** into **Slt**; moreover it can be proved that $\mathcal{F}$-Spc-Mac: **Dlc** $\to$ **Blt** is isomorphic to the composed functor:

$$\mathbf{Dlc} \xrightarrow{\mathrm{Sem}^*} \Theta(\mathbf{Dlc}) \xrightarrow{\mathrm{Ssc}} \mathbf{J} \xrightarrow{\mathrm{Prl}} \mathbf{Blt}$$

which is exact, by 4.3.

6.6. We also use the natural embedding (where U^0 is the inclusion functor):

$$\eta^0 \colon U^0 \to \mathrm{Mac} \colon \mathbf{Dl} \to \mathbf{Sl}, \tag{1}$$

$$\eta^0 X(x) = \overline{(x, 0)}. \tag{2}$$

The horizontal composition of η^0 and η in 6.6 and 6.4 gives the natural transformation (where $U' = UU^0$ is the suitable inclusion functor):

$$\eta' = \eta\eta^0 \colon U' \to \mathcal{F}\text{-Spc-Mac} \colon \mathbf{Dl} \to \mathbf{Bl}, \tag{3}$$

$$\eta' X(x) = \{\varphi \in \mathrm{Spc}\,(\mathrm{Mac}\,(X)) | \varphi\overline{(x, 0)} = 1\}. \tag{4}$$

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Direttore responsabile: Prof. A. Ballio - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
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