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On the structure of linearly compact rings and their dualities (***)

Sulla struttura degli anelli linearmente compatti e loro dualità (***)

RIASSUNTO. — Nel presente lavoro si studiano gli anelli linearmente compatti (l.c.) apportando contributi nelle seguenti tre direzioni: un teorema di rappresentazione di un anello l.c. come anello di endomorfismi di un modulo ad esso canonicamente associato, e tale modulo fornisce utili informazioni sulla struttura dell'anello; una teoria della dualità atta a caratterizzare gli anelli l.c.; l'esistenza di un anello cobasico e di un anello basico (entrambi l.c.) per ogni anello l.c. Si danno diverse applicazioni dei risultati ottenuti.

0. - INTRODUCTION

0.1. In the present work we study linearly compact (l.c.) rings giving contributions in the following three directions: a theorem of representation of any l.c. ring as the endomorphism ring of a module canonically associated to the ring; a duality theorem characterizing the l.c. rings; the existence of a pair of l.c. rings (cobasic ring and basic ring) canonically associated to a given l.c. ring.

First we give some definitions and notations. For unexplained terms see Section 1.

0.2. All rings in consideration have a nonzero identity and all modules are unital. Let R be a ring. We denote by R-Mod (Mod-R) the category of left (right) R-modules. Module morphisms will be written on the opposite side to that of scalars.

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Ring and module topologies are linear and Hausdorff. From now on (R, τ) denotes a left linearly topologized (l.t.) ring R with topology τ and R_τ-LT denotes the category of left l.t. modules over the topological ring (R, τ). We denote by $\mathcal{F}_\tau$ the filter of open left ideals of (R, τ) and by $\mathcal{C}_\tau$ the class of τ-*torsion* modules:

$$\mathcal{C}_\tau = \{M \in R\text{-Mod} : \forall x \in M, \operatorname{Ann}_R(x) \in \mathcal{F}_\tau\}.$$

Every module $M \in \mathcal{C}_\tau$ has an injective hull in $\mathcal{C}_\tau$ which will be denoted by $E_\tau(M)$.

Let $(V_\gamma)_{\gamma \in \Gamma}$ be a system of representatives of the simple non isomorphic modules belonging to $\mathcal{C}_\tau$ and let, for every $\gamma \in \Gamma$, $D_\gamma = \operatorname{End}_R(V_\gamma)$. Then V_γ is a right vector space over the division ring D_γ. Let r_γ be the dimension of V_γ over D_γ.

The module ${}_RW = E_\tau\left(\bigoplus_{\gamma \in \Gamma} V_\gamma\right)$ is the *minimal injective cogenerator* of $\mathcal{C}_\tau$. The ring $A = \operatorname{End}({}_RW)$ is endowed with the finite topology σ, so that (A, σ) is right l.t. The meaning of $\mathcal{F}_\sigma$, $\mathcal{C}_\sigma$, E_σ and LT-A_σ is clear. Recall that a module $M \in R_\tau$-LT is l.c. if every family of closed cosets of M with the finite intersection property has non empty intersection.

0.3. The paper is divided in eight sections. Section 1 contains preliminary notions on l.c. rings.

In Section 2 we give the representation theorem. First we prove that for every left l.t. ring (R, τ), $\operatorname{Soc}({}_RW) = \operatorname{Soc}(W_A)$ and $\operatorname{Soc}(W_A)$ is essential in W_A. Moreover $\operatorname{Soc}(W_A) = \bigoplus_{\gamma \in \Gamma} D_\gamma^{(r_\gamma)}$ and, denoting by $J(A)$ the Jacobson radical of A, $A/J(A) \cong \prod_{\gamma \in \Gamma} D_\gamma$. Then we prove the equivalence of the following three conditions: (*a*) (R, τ) is l.c.; (*b*) (R, τ) and (A, σ) are both l.c.; (*c*) $R \cong \operatorname{End}(W_A)$ and W_A is an injective cogenerator, with essential socle, of $\mathcal{C}_\sigma$. In this case $W_A = E_\sigma\left(\bigoplus_{\gamma \in \Gamma} D_\gamma^{(r_\gamma)}\right)$.

Denoting by τ_* the Leptin topology of (R, τ), the representation theorem for a l.c. ring (R, τ) is contained in the following topological isomorphism: $(R, \tau_*) \cong \operatorname{End}(W_A)$ where the endomorphism ring has the finite topology. In particular, if (R, τ) is strictly linearly compact (s.l.c.), i.e. for every $I \in \mathcal{F}_\tau$ R/I is artinian, then $W_A = \bigoplus_{\gamma \in \Gamma} [E_\sigma(D_\gamma)]^{(r_\gamma)}$.

0.4. In Section 3 we give some applications of the representation theorem, obtaining a very short and natural proof of the classical Leptin's theorem on the structure of semiprimitive l.c. rings and of the theorem of Zelinsky stating that every commutative l.c. ring is a topological product of local l.c. rings.

0.5. Section 4 is dedicated to the duality. Let (R, τ) be a left l.c. ring, let ${}_RK$ be an injective cogenerator of $\mathcal{C}_\tau$ with essential socle, $B = \operatorname{End}({}_RK)$ and let B have the finite topology β. ${}_RK$ and K_B are supposed to be discrete, so that ${}_R K \in R_\tau\text{-}LT$ and $K_\beta \in LT\text{-}B_\beta$. Consider the functor $D_1\colon R_\tau\text{-}LT \to LT\text{-}B_\beta$ which associates to every $M \in R_\tau\text{-}LT$ the right A-module of continuous morphisms of M in ${}_R K$, $\mathrm{Chom}_R(M, {}_R K)$, provided with the finite topology and denote by M^* the topological module obtained in this way. Clearly $M^* \in LT\text{-}B_\beta$. The functor $D_2\colon LT\text{-}B_\beta \to R_\tau\text{-}LT$ is defined in an analogous way. Denote by $\mathfrak{B}({}_R K)$ ($\mathfrak{B}(K_B)$) the subcategories of $R_\tau\text{-}LT$ ($LT\text{-}B_\beta$) consisting of modules M whose topology coincides with the weak topology given by $\mathrm{Chom}(M, K)$. Now the duality theorem can be stated as follows. The following conditions are equivalent: (*a*) (R, τ) is l.c.; (*b*) (R, τ) and (B, β) are both l.c.; (*c*) $R \cong \mathrm{End}(K_B)$ and K_B is an injective cogenerator of $\mathfrak{C}_\beta$ with essential socle; (*d*) for every $M \in \mathfrak{B}({}_R K)$ ($M \in \mathfrak{B}(K_B)$) the canonical morphism $\omega_M\colon M \to M^{**}$ is a topological isomorphism; (*e*) for every $M \in R_\tau\text{-}LT$ ($M \in LT\text{-}B_\beta$) the canonical morphism ω_M is a continuous isomorphism. Moreover, if (*e*) holds then the topology of M is equivalent to the weak topology of ω_M.

The above duality is equivalent to that considered by Ánh [1]; however our approach—inspired by [11] and [12]—is completely different.

0.6. Section 5 is devoted to various applications of duality. The most significant one is a characterization of the s.l.c. rings: (R, τ) is s.l.c. iff D_1 induces a duality between $\mathfrak{C}_\tau$ and the subcategory $NLC_\beta\text{-}B_\beta$ of $LT\text{-}B_\beta$ consisting of l.c. topologically noetherian modules endowed with their Leptin topology. If (R, τ) is absolutely linearly compact (a.l.c.)—i.e., for every $I \in \mathcal{F}_\tau$, R/I has finite length—then (B, β) is a.l.c. too. Moreover (D_1, D_2) induces a duality between $\mathfrak{C}_\tau$ and $ALC\text{-}B_\beta$ and between $\mathfrak{C}_\beta$ and $R_\tau\text{-}ALC$. This is a well known result of Gabriel [6].

Section 5 ends with a comparison between the duality D_1 and the Oberst duality for the class $\mathfrak{C}_\tau$ of a s.l.c. ring.

0.7. In Section 6 an example of a s.l.c. not a.l.c. ring is given. Besides this example solves Problem 2 in [13].

0.8. In Section 7 we introduce the notion of cobasic ring and of basic ring associated to a l.c. ring (R, τ).

By the representation theorem,

$$(R, \tau_*) \cong \mathrm{End}(W_A),\ \sigma = \sigma_*,\ W_A = E_\sigma\Big(\bigoplus_\gamma D_\gamma^{(\nu_\gamma)}\Big) \text{ and } A/J(A) = \prod_{\gamma \in \Gamma} D_\gamma .$$

Then, by definition, (A, σ) is the *cobasic ring* of (R, τ) and $\nu_\Gamma = (\nu_\gamma)_{\gamma \in \Gamma}$ is the *grade* of (R, τ). The main result of this section is the following. Let (A, σ) be a right l.c. ring with $\sigma = \sigma_*$ and $A/J(A) = \prod_{\gamma \in \Gamma} D_\gamma$ with D_γ division rings and let $\nu_\Gamma = (\nu_\gamma)_{\gamma \in \Gamma}$ be an arbitrary Γ-tuple of non zero cardinal numbers. Then there exists a left l.c. ring (R, τ), with $\tau = \tau_*$, unique up to topological isomorphisms, having (A, σ) as cobasic ring and ν_Γ as grade.

The basic ring of (R, τ) is the cobasic ring of (A, σ). It has the form eRe where e is an idempotent in R.

0.9. In Section 8 we apply the preceding results to the study of primary l.c. rings, precising some results of Leptin and Ánh.

Remark: Some of the above results have been announced at the Conference on Topology held at L'Aquila, March 1983. See [3].

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1. - Preliminaries

Let (R, τ) be a left l.t. Hausdorff ring. Let $M \in R_\tau\text{-}LT$. A *topological submodule* of M is a submodule of M endowed with the relative topology. Writing $(M, \varepsilon) \in R_\tau\text{-}LT$ we mean that the module M, endowed with the topology ε, is an object of $R_\tau\text{-}LT$. If $L, M \in R_\tau\text{-}LT$, $\mathrm{Chom}_R(L, M)$ is the group of continuous morphisms of L in M.

Let R be a ring. For every $M \in R$-Mod we denote by $E({}_RM)$—or also by $E(M)$—the injective envelope of ${}_RM$ in R-Mod.

Let R, B be two rings and let ${}_RK_B$ be a bimodule. We say that ${}_RK_B$ is *faithfully balanced* if the canonical morphisms $R \to \mathrm{End}(K_B)$, $B \to \mathrm{End}({}_RK)$ are ring isomorphisms.

1.1. Let R be a ring and $L, K \in R$-Mod. For every subset F of L set

$$O(F) = \{\zeta \in \mathrm{Hom}_R(L, K) : F\zeta = 0\}$$

$O(F)$ is a subgroup of $\mathrm{Hom}_R(L, K)$. Observe that $O(F) = O(RF)$, where RF is the submodule of L generated by F.

The subgroups $O(F)$, where F is a finite subset of L—or also a finitely generated (f.g.) submodule of L—can be assumed as a basis of neighbourhoods of 0 for a group topology on $\mathrm{Hom}_R(L, K)$, called the *finite topology*. Such topology is complete and Hausdorff. If $L = K$ then the $O(F)$'s are right ideals of $\mathrm{End}({}_RK)$. $\mathrm{End}({}_RK)$, with the finite topology, is a right l.t. ring complete and Hausdorff.

Let ${}_RK \in R$-Mod. The *K-topology* of R is the ring topology on R obtained by taking as a basis of neighbourhoods of 0 the left ideals of the form $\mathrm{Ann}_R(F)$ where F is a finite subset of K. R, endowed with the K-topology, is Hausdorff iff $\mathrm{Ann}_R(K) = 0$, i.e. ${}_RK$ is faithful. Set $B = \mathrm{End}({}_RK)$. Then, considering K as a right B-module, the finite topology and the K-topology of B coincide.

1.2. From now on (R, τ) denotes a fixed, but arbitrary, left l.t. Hausdorff ring. Let $\mathcal{F}_\tau$ be the filter of open left ideals of (R, τ) and let $\mathfrak{T}_\tau$ be the class of τ-torsion modules. $\mathfrak{T}_\tau$ is a *closed* subcategory of R-Mod, since $\mathfrak{T}_\tau$ is closed under submodules, epimorphic images and arbitrary direct sums. For every $M \in R$-Mod denote by $t_\tau(M)$ the τ-*torsion submodule* of M. t_τ is a left exact preradical in R-Mod. Note that, for every left ideal I of R:

$$I \in \mathcal{F} \Leftrightarrow R/I \in \mathfrak{T}_\tau .$$

Denote by $\hat{R}$ the Hausdorff completion of (R, τ). The class $\mathfrak{T}_\tau$ has the following properties:

a) *A module* $M \in R$-Mod, *endowed with the discrete topology, is an object of* R_τ-LT *iff* $M \in \mathfrak{T}_\tau$.

b) *Every module belonging to* $\mathfrak{T}_\tau$ *is, in a natural way, a left* $\hat{R}$-*module and every* R-*linear morphism in* $\mathfrak{T}_\tau$ *is* $\hat{R}$-*linear*.

Let $M \in \mathfrak{T}_\tau$. Put $E_\tau(M) = t_\tau(E(M))$. Then $E_\tau(M)$ is the injective hull of M in $\mathfrak{T}_\tau$. Recall that $\mathfrak{T}_\tau$ *is a Grothendieck category*.

Let $(V_\gamma)_{\gamma \in \Gamma}$ be a fixed system of representatives of the isomorphism classes of simple modules belonging to $\mathfrak{T}_\tau$. Then:

$${}_RU = \bigoplus_{\gamma \in \Gamma} E_\tau(V_\gamma)$$

is the minimal cogenerator of $\mathfrak{T}_\tau$, while

$${}_RW = E_\tau\Big(\bigoplus_{\gamma \in \Gamma} V_\gamma\Big) \simeq E_\tau\Big(\bigoplus_{\gamma \in \Gamma} E_\tau(V_\gamma)\Big)$$

is the minimal injective cogenerator of $\mathfrak{T}_\tau$.

Note that ${}_RU$ and ${}_RW$, with the discrete topology, belong to $\mathfrak{T}_\tau$. Observe also that ${}_RU$ and ${}_RW$ are both faithful left R-modules since (R, τ) is Hausdorff. Let $A = \mathrm{End}\,({}_RW)$ and denote by σ the finite topology of A. (A, σ) is a right l.t. ring. The symbols $\mathcal{F}_\sigma$, $\mathfrak{T}_\sigma$, LT-A_σ are now clear.

The notations just established will be of current use and, in general, their meaning will not be recalled.

1.3. Let ${}_RK \in R$-Mod and denote by N the set of positive integers.

We shall say that ${}_RK$ is *strongly quasi-injective* (s.q.i.) if for every submodule $B < {}_RK$, for every morphism $f\colon {}_RB \to {}_RK$ and for every $x \in K \setminus B$, f extends to an endomorphism g of ${}_RK$ such that $xg \neq 0$.

It is clear that, if ${}_RK$ is s.q.i., then ${}_RK$ is quasi-injective (q.i.).

${}_RK$ is said to be a *self-cogenerator* if for every $n \in \mathrm{N}$, for every submodule $B <{}_RK^n$ and for every $x \in K^n \setminus B$, there exists a morphism $f\colon {}_RK^n \to {}_RK$ such that $Bf = 0$ and $xf \neq 0$.

It was proved by S. Bazzoli [2] that *${}_R K$ is s.q.i. iff ${}_R K$ is a q.i. selfcogenerator.*

Let ${}_R K \in R$-Mod. Suppose that ${}_R K$ is faithful and denote by w the K-topology of R. Then

a) *${}_R K$ is q.i. (s.q.i.) iff ${}_R K$ is injective (an injective cogenerator) in $\mathfrak{C}_w$.*

b) *Let (R, τ) be a left l.t. ring. If ${}_R K$ is an injective object (cogenerator) in $\mathfrak{C}_\tau$, then ${}_R K$ is q.i. (s.q.i.).*

1.4. Let (R, τ) be a left l.t. ring, ${}_R K \in \mathfrak{C}_\tau$ and assume ${}_R K$ discrete, so that ${}_R K \in R_\tau$-LT. K is said to be a *cogenerator* of R_τ-LT if for every $M \in R_\tau$-LT and for every $x \in M$, $x \neq 0$, there exists a continuous morphism $f \colon M \to K$ such that $xf \neq 0$. If K is a cogenerator of R_τ-LT, then for every closed submodule H of M and for every $x \in M \setminus H$ there exists a continuous morphism $f \colon M \to K$ such that $Hf = 0$, $xf \neq 0$.

K is said to be an *injective* object of R_τ-LT if for every $M \in R_\tau$-LT and for every topological submodule H of M, every continuous morphism of H in K can be extended to a continuous morphism of M in K. Observe that, for every discrete module $K \in R_\tau$-LT, every continuous morphism of H in K can be extended to a continuous morphism of an appropriate open submodule of M containing H, in K. Therefore, K is an injective object of R_τ-LT iff for every $I \in \mathcal{F}_\tau$ every continuous morphism of I in K extends to a morphism of R in K.

Finally it is easy to prove that for every ${}_R K \in \mathfrak{C}_\tau$:

${}_R K$ is a (an injective) cogenerator in R_τ-LT iff ${}_R K$ is a (an injective) cogenerator in $\mathfrak{C}_\tau$.

1.5. A module $M \in R$-Mod is said to be *finitely cogenerated* (f.c.) if the socle Soc(M) of M is f.g. and essential in M. A submodule H of a module M is said to be *cofinite* if M/H is f.c.

Let (R, τ) be a left l.t. ring and let ε and ε_1 be two topologies on M, such that $(M, \varepsilon), (M, \varepsilon_1) \in R_\tau$-$LT$. The topologies ε and ε_1 are said to be *equivalent* if they have the same closed submodules.

The *Leptin topology* ε_* of (M, ε) is the topology having as a basis of neighbourhoods of 0 in M the cofinite ε-open submodules in M. Clearly $(M, \varepsilon_*) \in R_\tau$-$LT$, $\varepsilon_* \subseteq \varepsilon$, ε_* and ε are equivalent. The Leptin topology τ_* of (R, τ) coincides with the ${}_R U$-topology, hence τ_* is a ring topology.

If (M, ε) is l.c. then (M, ε_*) is l.c. and the topology ε_* is minimal in the set of topologies ε_1 on M such that $(M, \varepsilon_1) \in R_\tau$-$LT$. If (M, ε) is s.l.c. then $\varepsilon = \varepsilon_*$.

The abbreviation l.c.d. means «linearly compact in the discrete topology».

Let $(M, \varepsilon) \in R_\tau$-$LT$ be l.c. It is known that among all topologies equivalent to ε there exists a finest one which will be denoted by ε^*. The existence of ε^* was established by various authors ([19], [1], [10]).

Following [19] the topology ε^* has as a basis of neighbourhoods of 0 in M the closed submodules H of (M, ε) such that M/H is l.c.d. (M, ε^*) is l.c. and $\varepsilon_* \subseteq \varepsilon \subseteq \varepsilon^*$.

If (R, τ) is a left l.c. ring, then τ^* is a ring topology.

The following lemma will be very useful in the sequel.

1.6. LEMMA (cf. [9], Lemma 6): *Let (R, τ) be a left l.t. ring and let ${}_RK$ be a cogenerator of $\mathcal{C}_\tau$. Then the K-topology of R is equivalent to τ and coarser than τ.*

2. - REPRESENTATION OF LINEARLY COMPACT RINGS

2.1. LEMMA: *Let R be a ring and let ${}_RK$ be a module with essential socle such that R, endowed with the K-topology τ, is l.c. Then $\tau = \tau_*$.*

PROOF: Evidently $\tau \supseteq \tau_*$. Conversely, let $I \in \mathcal{F}_\tau$. Then $I \supseteq \bigcap_{i=1}^{n} \mathrm{Ann}_R(x_i)$ with $x_i \in K$. Then it is sufficient to prove that for every $x \in K$, $x \neq 0$, $Rx \cong R/\mathrm{Ann}_R(x)$ is f.c. Indeed, since $\mathrm{Ann}_R(x) \in \mathcal{F}_\tau$, Rx is l.c.d., hence $\mathrm{Soc}(Rx)$ is f.g. On the other hand the socle is essential by assumption, hence Rx is f.c.

2.2. LEMMA: *Let (R, τ) be a left l.c. ring, ${}_RW$ and ${}_RU$ respectively the minimal injective cogenerator and the minimal cogenerator of $\mathcal{C}_\tau$, ${}_RU < {}_RW$. Then for every $x \in {}_RW$ there exists $y \in {}_RU$ such that $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(y)$.*

PROOF: Consider the inclusions

$$ {}_RU = \bigoplus_{\gamma \in \Gamma} E_\tau(V_\gamma) < {}_RW < \prod_{\gamma \in \Gamma} E_\tau(V_\gamma) \, . $$

Let $x \in {}_RW$, $x \neq 0$, $x = (x_\gamma)_{\gamma \in \Gamma}$ with $x_\gamma \in E_\tau(V_\gamma)$. Rx is l.c.d., hence $\mathrm{Soc}(Rx)$ is f.g. and essential in Rx $(Rx < {}_RW)$. Thus Rx is f.c. Setting $I = \mathrm{Ann}_R(x)$ we have $I = \bigcap_{\gamma \in \Gamma} \mathrm{Ann}_R(x_\gamma)$. By Proposition 8 in [9] there exists a finite subset F of Γ such that $\bigcap_{\gamma \in F} \mathrm{Ann}_R(x_\gamma) < I$. Then $\bigcap_{\gamma \in F} \mathrm{Ann}_R(x_\gamma) = I$. Let $y \in {}_RU$ be such that $y_\gamma = x_\gamma$ for $\gamma \in F$ and $y_\gamma = 0$ for $\gamma \notin F$. Then $\mathrm{Ann}_R(x) = \mathrm{Ann}_R(y)$.

2.3. REMARK: The preceding lemmata yield that for every left l.c. ring (R, τ) the U-topology and the W-topology of R coincide both with the Leptin topology τ_*.

2.4. Let (R, τ) be a left l.t. ring, ${}_RK \in \mathcal{C}_\tau$ and $B = \mathrm{End}({}_RK)$. Endow B with the K-topology β and the bimodule ${}_RK_B$ with the discrete topology. For $M \in R_\tau\text{-}LT$ we denote by M^* the group $\mathrm{Chom}_R(M, {}_RK)$ endowed with the finite topology. Then $M^* \in LT\text{-}B_\beta$ being a topological submodule of the module $K_B^M \in LT\text{-}B_\beta$.

Let $M^{**} = \mathrm{Chom}_B(M^*, K_B)$ endowed with the finite topology. Then $M^{**} \in R_\tau\text{-}LT$. Remark that M^{**} is a topological submodule of the module $\mathrm{Hom}_B(M^*, K_B)$ endowed with the finite topology.

For every $x \in M$ denote by $\tilde{x}$: $M_B^* \to K_B$ the morphism defined as follows:

$$\tilde{x}(\zeta) = (x)\zeta \qquad (\zeta \in M^*).$$

Clearly $\tilde{x} \in M^{**}$ since coincides with the restriction to M^* of the canonical projection π_x: $K_B^M \to K_B$. Finally, denote by ω_M: $M \to M^{**}$ the *canonical morphism* given by $(x)\omega_M = \tilde{x}$. Since $M^{**} < \mathrm{Hom}_B(M^*, K_B)$, $\omega = \omega_M$ can be considered also as a morphism of M in $\mathrm{Hom}_B(M^*, K_B)$.

2.5 Theorem: *Let (R, τ) be a left l.t. ring, ${}_RK \in \mathcal{C}_\tau$, $B = \mathrm{End}({}_RK)$ and let B have the K-topology β. Let $M \in R_\tau\text{-}LT$ and let $\mathrm{Hom}_B(M, K_B)$ have the finite topology. In the notations of 2.4, the following hold:*

a) The canonical morphism ω: $M \to \mathrm{Hom}_B(M^*, K_B)$ is continuous and $\mathrm{Im}(\omega) < < M^{**} < \mathrm{Hom}_B(M^*, K_B)$.

b) ω is open on the image iff the quotient topology of $M/\mathrm{Ker}(\omega)$ coincides with the weak topology induced by $\mathrm{Chom}_B(M, {}_RK)$.

c) If ${}_RK$ is a faithful selfcogenerator, then:

1) *for every f.g. submodule L of M_B^* and for every algebraic morphism f: $L \to K_B$, there exists $x \in M$ such that $\tilde{x}_{|L} = f$, hence f can be extended to a morphism of M_B^* in K_B.*

2) *$M\omega$ is dense in $\mathrm{Hom}_B(M^*, K_B)$.*

3) *$\mathrm{End}(K_B)$ coincides with the Hausdorff completion of R with respect to the K-topology.*

d) (Ánh [1]) *Suppose that ${}_RK$ is a cogenerator of $\mathcal{C}_\tau$. Then M is l.c. iff the following two conditions hold:*

i) *the morphism ω: $M \to \mathrm{Hom}_B(M^*, B)$ is an algebraic isomorphism;*

ii) *for every submodule L of M^*, every algebraic morphism of L in K_B extends to a morphism of M^* in K_B.*

Proof: Statements *a*), *b*), *c*) can be proved by arguments similar to those used in the density theorems in [11] and [17]. *d*) suppose that ${}_RK$ is a cogenerator of $\mathcal{C}_\tau$, hence of $R_\tau\text{-}LT$. Then the necessity of condition i) follows from *c*), 2). The necessity of ii) and the sufficiency of both i) and ii) can be proved by the method used in the proof of Lemma 4 of [14] (see [1], Theorem 2.8).

2.6. A module $M \in R\text{-Mod}$ will be called *weakly quasi-injective* (w.q.i.) if for every $n \in \mathbb{N}$ and for every f.g. submodule H of M^n, every morphism of H in M extends to a morphism of M^n in M.

Obviously every q.i. module is w.q.i. If (R, τ) is a left l.t. ring, then ${}_RU$ is w.q.i., but not q.i. in general.

2.7. Let (R, τ) be a left l.t. ring, $\mathfrak{F}$ the set of maximal open left ideals of (R, τ), ${}_RK$ a fixed cogenerator of $\mathfrak{C}_\tau$, $B = \mathrm{End}\,({}_RK)$. It is clear that

a) $\mathrm{Soc}\,({}_RK) = \sum_{P \in \mathfrak{F}} \mathrm{Ann}_K\,(P)$.

Both $\mathrm{Soc}\,({}_RK)$ and $\mathrm{Ann}_K(P)$ are right B-modules, being fully invariant submodules of ${}_RK$.

b) *Suppose that ${}_RK$ is w.q.i. Then, for every $P \in \mathfrak{F}$, $\mathrm{Ann}_K\,(P)$ is a simple submodule of K_B, and every simple submodule of K_B has the form $\mathrm{Ann}_K\,(P)$ with $P \in \mathfrak{F}$. Consequently:*

$$\mathrm{Soc}\,({}_RK) = \mathrm{Soc}\,(K_B)\,.$$

Proof: Let $x, y \in \mathrm{Ann}_K\,(P)$, $x \neq 0 \neq y$. Then $\mathrm{Ann}_R\,(x) = P = \mathrm{Ann}_R\,(y)$ since P is open in R and ${}_RK$ is a cogenerator of R_τ-LT. Hence there exists a morphism $f\colon Rx \to Ry$ such that $xf = y$. f extends to an endomorphism b of ${}_RK$, so that $y = xb$, which yields that $\mathrm{Ann}_K\,(P)$ is a simple submodule of K_B.

Conversely, let S_B be a simple submodule of K_B, $x \in S_B$, $x \neq 0$. There exists $P \in \mathfrak{F}$ such that $\mathrm{Ann}_R\,(x) < P$. Let

$$y \in \mathrm{Ann}_K\,(P)\,, \qquad y \neq 0\,.$$

Since $\mathrm{Ann}_R\,(y) = P$ there exists a surjective morphism $f\colon Rx \twoheadrightarrow Ry$ such that $xf = y$. This yields $y = xb$ with $b \in B$, hence $\mathrm{Ann}_K\,(P) < S_B$. Since S_B is simple, this gives $\mathrm{Ann}_K\,(P) = S_B$.

Let $(V_\gamma)_{\gamma \in \Gamma}$ be, as usual, a system of representatives of the simple non isomorphic modules belonging to $\mathfrak{C}_\tau$. For a fixed $\gamma \in \Gamma$ let $P \in \mathfrak{F}$ be such that $V_\gamma \simeq R/P$ and set $V_\gamma^* = \mathrm{Hom}_R\,(V_\gamma, {}_RK) \simeq \mathrm{Hom}_R\,(R/P, {}_RK)$. For $e = 1 + P \in R/P$ the correspondence $f \mapsto f(e)$, when $f \in \mathrm{Hom}_R\,(R/P, {}_RK)$, defines an isomorphism of the right B-modules V_γ^* and $\mathrm{Ann}_K\,(P)$. Therefore V_γ^* is isomorphic to a simple submodule of K_B.

Denote by $\Sigma(V_\gamma)$ the isotypic component of $\mathrm{Soc}\,({}_RK)$ with respect to V_γ and by $\Sigma(V_\gamma^*)$ the isotypic component of $\mathrm{Soc}\,(K_B)$ with respect to V_γ^*. Set $D_\gamma = \mathrm{End}_R\,(V_\gamma)$.

2.8. Theorem: *Let (R, τ) be a left l.t. ring, ${}_RK \in \mathfrak{C}_\tau$ a w.q.i. cogenerator of $\mathfrak{C}_\tau$, $B = \mathrm{End}\,({}_RK)$. Then*

a) $\mathrm{Soc}\,({}_RK) = \mathrm{Soc}\,(K_B)$.

b) $\mathrm{Soc}\,(K_B)$ *is essential in* K_B.

c) *For every* $\gamma \in \Gamma$, $\Sigma(V_\gamma) = \Sigma(V_\gamma^*)$.

d) If ${}_R W$ is the minimal injective cogenerator of $\mathcal{C}_\tau$ and $A = \operatorname{End}({}_R W)$, then $V_\gamma^* = \operatorname{Hom}_R(V_\gamma, {}_R W) \simeq D_\gamma$ hence

$$\operatorname{Soc}(W_A) = \bigoplus_{\gamma \in \Gamma} D_\gamma^{(\nu_\gamma)}$$

where ν_γ is the dimension of V_γ as a right vector space over the division ring D_γ.

Proof: *a)* has already been proved.

b) Since $\operatorname{Soc}(K_B)$ is the intersection of all essential submodules of K_B, it is enough to prove that $\operatorname{Soc}({}_R K)$ as a B-submodule of K_B is essential in K_B. Let $x \in K_B$, $x \neq 0$, and let $P \in \mathcal{F}$ such that $\operatorname{Ann}_R(x) < P$; then ${}_R K$ contains a simple submodule V isomorphic to R/P. There exists a morphism $f: Rx \to V$ such that $xf \neq 0$. Then there exists $b \in B$ such that $xb \neq 0$.

c) Let $\gamma \in \Gamma$, $\mathcal{F}_\gamma = \{P \in \mathcal{F} \colon R/P \simeq V_\gamma\}$. It is clear that

$$\Sigma(V_\gamma) = \sum_{P \in \mathcal{F}_\gamma} \operatorname{Ann}_K(P).$$

This yields $\Sigma(V_\gamma) < \Sigma(V_\gamma^*)$. Now let $x \in K$ be such that $xB \simeq V_\gamma^*$. Fix $P \in \mathcal{F}_\gamma$ and $y \in \operatorname{Ann}_K(P) < \Sigma(V_\gamma)$, $y \neq 0$. Then $yB = \operatorname{Ann}_K(P) \simeq V_\gamma^*$, since $\operatorname{Ann}_K(P)$ is a simple submodule of K_B. So there exists a B-linear morphism $f: yB \to xB$ such that $f(y) = x$. By Theorem 2.5 *c)*, 3) there exists $r \in R$ such that $ry = x$. Since $ry \in \Sigma(V_\gamma)$, $x \in \Sigma(V_\gamma)$; therefore $\Sigma(V_\gamma^*) < \Sigma(V_\gamma)$.

d) is now obvious.

2.9. Proposition: *Let (R, τ) be a left l.t. ring, let ${}_R K$ be an injective cogenerator of $\mathcal{C}_\tau$ with essential socle, $B = \operatorname{End}({}_R K)$ and let $J(B)$ be the Jacobson radical of B. Then*

$$\operatorname{Ann}_B(\operatorname{Soc}({}_R K)) = J(B)$$

and

$$\operatorname{End}_R(\operatorname{Soc}({}_R K)) \simeq B/J(B)$$

canonically. In particular $\operatorname{End}_R(\operatorname{Soc}({}_R W)) \simeq A/J(A) \simeq \prod_{\gamma \in \Gamma} D_\gamma$.

Proof: Let us consider the exact sequence

$$0 \to \operatorname{Soc}({}_R K) \to {}_R K \to {}_R K/\operatorname{Soc}({}_R K) \to 0.$$

Applying $\operatorname{Hom}_R(-, {}_R K)$ we obtain the exact sequence

$$0 \to \operatorname{Ann}_B(\operatorname{Soc}({}_R K)) \to B \to \operatorname{End}_R(\operatorname{Soc}({}_R K)) \to 0.$$

By assumption $\mathrm{Ann}_B(\mathrm{Soc}({}_RK))$ coincides with the ideal J of B consisting of the endomorphisms of ${}_RK$ having essential kernels. Since ${}_RK$ is q.i. it is well known that $J = J(B)$ (see for example [5]).

The last statement follows from the equality

$$\mathrm{Soc}({}_RW) = \bigoplus_{\gamma \in \Gamma} V_\gamma .$$

The following theorem is a slight generalization of a result due to C. Menini ([9], Main Theorem and Theorem 10).

2.10. THEOREM: *Let (R, τ) be a left l.t. ring and let ${}_RK$ be a cogenerator of $\mathcal{C}_\tau$, $B = \mathrm{End}({}_RK)$, β the K-topology of B. Then the following conditions are equivalent:*

(*a*) *(R, τ) is l.c.*

(*b*) *The bimodule ${}_RK_B$ is faithfully balanced and K_B is q.i.*

Moreover if (*a*) *holds and ${}_RK$ is w.q.i., then the following two conditions are equivalent:*

1) *$\mathrm{Soc}({}_RK)$ is essential in ${}_RK$;*

2) *K_B is s.q.i.*

Finally, if (*a*) *and* 2) *hold, then the following two conditions are equivalent:*

(i) *(B, β) is right l.c.*

(ii) *${}_RK$ is s.q.i.*

PROOF: (*a*) $\Leftrightarrow$ (*b*). Follows from Theorem 2.5 *d*).

Suppose that (*a*) holds and ${}_RK$ is w.q.i.

1) $\Rightarrow$ 2). It is enough to prove that if L is a submodule of K_B and $x \in K_B \setminus L$, then $\mathrm{Ann}_R(L) \nless \mathrm{Ann}_R(x)$. Assume that $\mathrm{Ann}_R(L) < \mathrm{Ann}_R(x)$, i.e.

$$\bigcap_{y \in L} \mathrm{Ann}_R(y) < \mathrm{Ann}_R(x).$$

$\mathrm{Soc}(Rx)$ is essential in Rx and Rx is l.c.d., hence $\mathrm{Soc}(Rx)$ is f.g., so $Rx \cong R/\mathrm{Ann}_R(x)$ is f.c. Applying to the module ${}_RR$ a classical linear compactness argument ([14], Lemma 2 and [9], Lemma 8), we get a finite subset $\{y_1, \ldots, y_n\}$ of L, such that

$$\bigcap_{i=1}^{n} \mathrm{Ann}_R(y_i) < \mathrm{Ann}_R(x). \tag{1}$$

Set $u = (y_1, \ldots, y_n) \in K^n$ and define the morphism $f\colon Ru \to Rx$ setting $(ru)f = rx$ $(r \in R)$. f is correctly defined because of (1). Since ${}_RK$ is w.q.i. there exists a morphism $g\colon {}_RK^n \to {}_RK$ which extends f. Hence there exist elements $b_1, \dots, b_n \in B$ such that $ng = \sum_{i=1}^{n} y_i b_i = x$; this yields $x \in L$. A contradiction.

2) $\Rightarrow$ 1). Since β is the K-topology of B, K_B is an injective cogenerator of $\mathcal{C}_\beta$ (see 1.3 a)). To get the conclusion it suffices to apply Theorem 2.8 to K_B.

Now suppose that conditions (a) and 2) hold.

(i) $\Rightarrow$ (ii). Since (a) $\Rightarrow$ (b), ${}_RK$ is q.i., so w.q.i. By virtue of Theorem 2.8, Soc (K_B) is essential in K_B. The implication 1) $\Rightarrow$ 2) yields ${}_RK$ s.q.i.

(ii) $\Rightarrow$ (i). Since K_B is s.q.i. and β is the K-topology of B, K_B is an injective cogenerator of $\mathcal{C}_\beta$. Now the equivalence (a) $\Rightarrow$ (b) implies that (B, β) is l.c.

2.11. Remark: In the notations of 2.10 suppose that ${}_RK$ is an injective cogenerator of $\mathcal{C}_\tau$ with essential socle and suppose that the bimodule ${}_RK_B$ is faithfully balanced. Then if (R, τ) is l.c., (B, β) is l.c. too. The converse is not true in general as the following example shows.

Let p be a prime number, let $\mathbf{Z}(p^\infty)$ be the Prüfer p-group, $\mathbf{Z}_p$ the ring of rationals whose denominator is prime to p, J_p the ring of p-adic integers. Consider the trivial extensions

$$R = \mathbf{Z}(p^\infty) \oplus 1 \cdot \mathbf{Z}_p, \qquad \overline{R} = \mathbf{Z}(p^\infty) \oplus 1 \cdot J_p,$$

and let $R \hookrightarrow \overline{R}$ be the canonical embedding.

R is a valutation ring with a simple and essential socle S. Hence the only linear Hausdorff topology on R is the discrete one and $E({}_RS) = E({}_RR)$. It is easy to see that $E({}_RS) = {}_R\overline{R}$, so that ${}_R\overline{R}$ is the minimal injective cogenerator of R-Mod. Then, for $A = \mathrm{End}_R(\overline{R})$, the bimodule ${}_R\overline{R}_A$ is faithfully balanced. R is not l.c.d., but A is l.c. in its finite topology.

Indeed, since $\overline{R}$ is commutative, $\overline{R}$ is in an obvious way a subring of A. Let F be a finite subset with n elements of $\overline{R}$, $O(F)$ is an $\overline{R}$-submodule of A such that $A/O(F)$ is isomorphic to a $\overline{R}$-submodule of $\overline{R}^n$. Since $\overline{R}$ is l.c.d., $A/O(F)$ is a l.c.d. $\overline{R}$-module. Consequently $A = \varprojlim A/O(F)$ is a l.c. $\overline{R}$-module, hence A_A is l.c. in the finite topology.

The following corollary makes applications of 2.10 easier.

2.12. Corollary: *In the notations of* 2.10 *suppose that ${}_RK$ is an injective cogenerator of $\mathcal{C}_\tau$. Then the following conditions are equivalent:*

(a) *(R, τ) is l.c. and* Soc $({}_RK)$ *is essential in ${}_RK$.*

(b) *(R, τ) and (B, β) are both l.c. and* Soc $({}_RK)$ *is essential in ${}_RK$.*

(c) *The bimodule ${}_RK_B$ is faithfully balanced and ${}_RK$ is an injective cogenerator of $\mathcal{C}_\beta$.*

If these conditions hold, then Soc $({}_RK)$ = Soc (K_B) and *Soc (K_B) is essential in K_B.*

PROOF: $(a) \Rightarrow (c)$. Follows from 2.10 using 1.3 *b*).

$(c) \Rightarrow (b)$. Follows immediately from 2.10.

$(b) \Rightarrow (a)$. Obvious.

The last statement follows from Theorem 2.8.

In the notations of 2.12 suppose (R, τ) l.c. and $\mathrm{Soc}\,({}_RK)$ essential in ${}_RK$, so that (B, β) is l.c., all f.g. submodules of K_B are l.c.d. and ${}_RK_B$ is faithfully balanced. Then by Theorem 1.6 of [10] the finest topology τ^* in the equivalence class of τ (see 1.5) has as a basis of neighbourhoods of 0 in R the annihilators of l.c.d. submodules of K_B.

Let (R, τ) be a left l.t. ring: (R, τ) is said to be *topologically noetherian* (*artinian*) if for every $I \in \mathcal{F}$, R/I is a noetherian (artinian) module.

2.13. LEMMA: *Let ${}_RK_B$ be a faithfully balanced R-B bimodule and let τ and β be the K-topologies of R and B respectively. Suppose that K_B is a selfcogenerator. If (R, τ) is topologically artinian (noetherian), then (B, β) is topologically noetherian (artinian).*

The easy proof of this Lemma is similar to that of Proposition 2.3 of [13].

2.14. REPRESENTATION THEOREM FOR l.c. RINGS: *Let (R, τ) be a left l.c. ring, ${}_RW = E_\tau\big(\bigoplus_{\gamma \in \Gamma} V_\gamma\big)$ the minimal injective cogenerator of $\mathcal{C}_\tau$, $A = \mathrm{End}\,({}_RW)$ and let A be endowed with the W-topology σ. Let $D_\gamma = \mathrm{End}_R\,(V_\gamma)$ and let r_γ be the dimension of V_γ as a right vector space over D_γ. Then*

a) The bimodule ${}_RW_A$ is faithfully balanced, $\mathrm{Soc}\,({}_RW) = \mathrm{Soc}\,(W_A)$, W_A is s.q.i. with essential socle and (A, σ) is right l.c. with $\sigma = \sigma_*$.

b) (R, τ_*) is topologically isomorphic in a natural way to the ring $\mathrm{End}\,(W_A)$ endowed with the W-topology.

c) $W_A = E_\sigma\big(\bigoplus_{\gamma \in \Gamma} D_\gamma^{(r_\gamma)}\big)$.

d) $A/J(A) \cong \prod_{\gamma \in \Gamma} D_\gamma$ and the right A-modules D_γ are a system of representatives of the simple non isomorphic modules belonging to $\mathcal{C}_\sigma$.

e) If (R, τ) is s.l.c. then

$$W_A = \bigoplus_{\gamma \in \Gamma} E_\sigma(D_\gamma^{(r_\gamma)}) = \bigoplus_{\gamma \in \Gamma} (E_\sigma(D_\gamma))^{(r_\gamma)}.$$

PROOF: *a*) follows from Corollary 2.12 and Lemma 2.1.

b) According to *a*) the canonical morphism of R in $\mathrm{End}\,(W_A)$ is a ring isomorphicm. Endowing these rings with the respective W-topologies this isomorphism is topological. By Lemma 2.1 the W-topology of R coincides with the Leptin topology τ_*.

c) By virtue of Theorem 2.8 $\mathrm{Soc}\,(W_A) = \bigoplus_{\gamma\in\Gamma} D_\gamma^{(\gamma)}$ and the inclusions

$$\mathrm{Soc}\,(W_A) < W_A < E_\sigma(\mathrm{Soc}\,(W_A))$$

are essential.

Since σ is the W-topology of A and W_A is s.q.i., W_A is an injective object of $\mathcal{C}_\sigma$, hence $W_A = E_\sigma(\mathrm{Soc}\,(W_A))$.

d) By Proposition 2.9 $A/J(A) \simeq \prod_{\gamma\in\Gamma} D_\gamma$. Since $\sigma = \sigma_*$ and W_A is s.q.i., W_A is a cogenerator of $\mathcal{C}_\sigma$. The conclusion follows by the structure of W_A.

e) If (R, τ) is s.l.c., then (A, σ) is topologically noetherian according to 2.13. In this case, applying classical methods (see for example [1]), it can be proved that in $\mathcal{C}_\sigma$ direct sums of injective modules are injective.

3. - Some applications

3.1. Theorems of Leptin and Zelinsky on semiprimitive l.c. rings. Let (R, τ) be a left l.c. ring, let $J(R)$ be its Jacobson radical and let ${}_RW$ be the minimal injective cogenerator of $\mathcal{C}_\tau$. According to 2.14, 2.8 and 2.9

$$\mathrm{Ann}_R\,(\mathrm{Soc}\,(W_A)) = J(R) = \mathrm{Ann}_R\,(\mathrm{Soc}\,({}_RW)) .$$

Suppose now that (R, τ) is semiprimitive, i.e. $J(R) = 0$. Then $W_A = \mathrm{Soc}\,(W_A)$ and ${}_RW = \mathrm{Soc}\,({}_RW)$. By virtue of 2.9 $J(A) = 0$, so that $A = \prod_{\gamma\in\Gamma} D_\gamma$. This gives the canonical isomorphisms:

$$(R, \tau_*) \simeq \mathrm{End}_A\,(\mathrm{Soc}\,(W_A)) \simeq \mathrm{End}_A\Big(\bigoplus_{\gamma\in\Gamma} V_\gamma\Big) = \prod_{\gamma\in\Gamma} \mathrm{End}_{D_\gamma}(V_\gamma)$$

which are topological when the endomorphism rings are provided with their finite topology and the product on the right-hand side is provided with the product topology. We get in this way the classical Leptin theorem on the structure of semiprimitive l.c. rings.

Now we add some remarks concerning such a ring (R, τ).

a) $\tau = \tau_*$.

Proof: Since $\tau \supseteq \tau_*$ it is enough to prove that every $I \in \mathcal{F}_\tau$ is open in (R, τ_*). Since I is τ-closed and τ is equivalent to τ_*, I is also τ_*-closed, hence closed in the W-topology of R. Setting $L_A = \mathrm{Ann}_W(I)$ we have $I = \mathrm{Ann}_A\,(L)$. Applying $\mathrm{Hom}_A(-, W_A)$ to the exact sequence

$$0 \to L_A \to W_A \to W_A/L_A \to 0$$

we get the exact sequence

$$0 \to I \to R \to \mathrm{Hom}_A(L_A, W_A) \to 0 .$$

In this way we establish the following isomorphism of left R-modules:

$$R/I \simeq \mathrm{Hom}_A(L_A, W_A) .$$

Since $W_A = \mathrm{Soc}(W_A)$ is semisimple, it follows that $L_A = \bigoplus_{\lambda \in \Lambda} L_\lambda$ where L_λ are simple submodules of W_A. Then

$$R/I \simeq \mathrm{Hom}_A(L_A, W_A) \simeq \prod_{\lambda \in \Lambda} \mathrm{Hom}_A(L_\lambda, W_A) .$$

Since R/I is l.c.d. and every $\mathrm{Hom}_A(L_\lambda, W_A)$ is non-zero, it follows that Λ is finite. For every $\lambda \in \Lambda$, L_λ is isomorphic to some D_γ. Now $\mathrm{Hom}_A(D_\gamma, W_A) \simeq \simeq V_\gamma$, so every $\mathrm{Hom}_A(L_\lambda, W_A)$ is a simple R-module. Since Λ is finite, this means that R/I is f.c., hence I is τ_*-open.

b) (R, τ) is s.l.c. since every ring $\mathrm{End}_{D_\gamma}(V_\gamma)$ is s.l.c. in the finite topology. Moreover $A = \prod_{\gamma \in \Gamma} D_\gamma$ is semiprimitive and (A, σ) is s.l.c.

c) The ring (R, τ) determines uniquely the cardinality of Γ and the vector spaces V_γ over the division rings D_γ up to semilinear isomorphism (more detailed proof of this fact can be found in [17]).

d) If (R, τ) is left primitive, then Γ has a single element since a primitive ring is not decomposable in a direct product of rings.

e) If (R, τ) has a basis of neighbourhoods of 0 consisting of open two sided ideals, then, for every $\gamma \in \Gamma$, V_γ is finite dimensional over D_γ (see [20]). Indeed, let V be a right vector space over the division ring D and let $(R, \tau) = = \mathrm{End}(V_D)$ have the finite topology. By *a*), τ is the only l.c. topology on R and every non zero two sided ideal is dense in (R, τ). This implies the finiteness of dimension of V over D, since R cannot have non trivial two sided ideals.

3.2. Lemma: *Let (R, τ) be a left l.c. ring and suppose $R = \prod_{\lambda \in \Lambda} R_\lambda$, where the R_λ's are rings. Then*

a) *For every $\lambda \in \Lambda$, the relative topology of R_λ coincides with the quotient topology on $R/\prod_{\mu \neq \lambda} R_\mu \simeq R_\lambda$, and τ_λ is l.c.*

b) *τ coincides with the product topology of the τ_λ's.*

c) *$\tau = \tau_*$ iff $\tau_\lambda = (\tau_\lambda)_*$ for every $\lambda \in \Lambda$.*

PROOF: *a*) Denote by e_λ the identity of R_λ and by $\pi_\lambda\colon R \to R_\lambda$ the canonical projection. For every $I \in \mathfrak{F}_\tau$ we have: $\pi_\lambda(I) = e_\lambda I = I \cap R_\lambda$. Now it is obvious that τ_λ is l.c.

b) Let τ' be the product topology of the topologies τ_λ. It is clear that $\tau \supseteq \tau'$ and that τ and τ' are equivalent. Let $I \in \mathfrak{F}_\tau$. I is τ'-closed, hence I coincides with the product of its projections I_λ. Now R/I is l.c.d. and $R/I \cong \prod_{\lambda\in\Lambda} R_\lambda/I_\lambda$. This implies $I_\lambda = R_\lambda$ for almost all λ. Then I is τ'-open.

c) is obvious.

3.3. PROPOSITION: *Let (R,τ) be a left l.c. ring with $\tau = \tau_*$. Set $A_\gamma = \mathrm{End}_R(E_\tau(V_\gamma))$, $R_\gamma = \mathrm{End}_A(E_\sigma(D_\gamma^{(\nu_\gamma)}))$ and endow these rings with their finite topology. Then the following conditions are equivalent.*

(*a*) *For every $\gamma \neq \gamma'$, $\mathrm{Hom}_R(E_\tau(V_\gamma), E_\tau(V_{\gamma'})) = 0$.*

(*b*) *For every $\gamma \neq \gamma'$, $\mathrm{Hom}_A(E_\sigma(D_\gamma^{(\nu_\gamma)}), E_\sigma(D_{\gamma'}^{(\nu_{\gamma'})})) = 0$.*

(*c*) *(A,σ) is topologically isomorphic to the topological product of the A_γ's.*

(*d*) *(R,τ) is topologically isomorphic to the topological product of the R_γ's.*

If these conditions hold, then ${}_RU = {}_RW$ and $W_A = \bigoplus_{\gamma\in\Gamma} E_\sigma(D_\gamma^{(\nu_\gamma)})$.

PROOF: Observe that (*a*) implies that $\bigoplus_{\gamma\in\Gamma} E_\tau(V_\gamma)$ is fully invariant in $\prod_{\gamma\in\Gamma} E(V_\gamma)$. Then $\bigoplus_{\gamma\in\Gamma} E_\tau(V_\gamma)$ is q.i. so that, since $\tau = \tau_*$, $\bigoplus_{\gamma\in\Gamma} E_\tau(V_\gamma)$ is injective in $\mathcal{C}_\tau$. Thus, ${}_RU = {}_RW$.

(*a*) $\Rightarrow$ (*c*) is now obvious.

(*c*) $\Rightarrow$ (*b*). Let $\varepsilon_\gamma \in A$ be the endomorphism whose γ-component is the identity of A_γ and the other components are zero. For every $\gamma \in \Gamma$ consider $E_\gamma = E_\sigma(D_\gamma^{(\nu_\gamma)})$ as a right A-module by the canonical projection $A \to A_\gamma$. Let $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$, $x \in E_\gamma$, $f \in \mathrm{Hom}_A(E_\gamma, E_{\gamma'})$. Then:

$$f(x) = f(x\varepsilon_\gamma) = f(x)\,\varepsilon_\gamma = 0\,.$$

(*b*) $\Rightarrow$ (*d*). Argue as in (*a*) $\Rightarrow$ (*c*).

(*d*) $\Rightarrow$ (*a*). Argue as in (*c*) $\Rightarrow$ (*b*).

3.4. ZELINSKY'S THEOREM ON THE STRUCTURE OF COMMUTATIVE l.c. RINGS: Let (R,τ) be a commutative l.c. ring. Since $R \cong \mathrm{End}(W_A)$ and $W_A \cong E_\sigma\big(\bigoplus_\gamma D_\gamma^{(\nu_\gamma)}\big)$ is q.i., it is obvious that $\nu_\gamma = 1$ for every $\gamma \in \Gamma$. So we have

$$W_A \cong E_\sigma\Big(\bigoplus_{\gamma\in\Gamma} D_\gamma\Big) \cong E_\sigma\Big(\bigoplus_{\gamma\in\Gamma} E_\sigma(D_\gamma)\Big).$$

Let $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$. The ring $\operatorname{End}_A(E_\sigma(D_\gamma) \oplus E_\sigma(D_{\gamma'}))$ is commutative since every such endomorphism extends to an endomorphism of W_A. Then every endomorphic image of $E_\sigma(D_\gamma) \oplus E_\sigma(D_{\gamma'})$ is fully invariant, so that

$$\operatorname{Hom}_A(E_\sigma(D_\gamma), E_\sigma(D_{\gamma'})) = 0 .$$

By Proposition 3.3, (R, τ_*) is topologically isomorphic to the topological product of the rings $R_\gamma = \operatorname{End}_A(E_\sigma(D_\gamma))$. By Lemma 3.2, (R, τ) is topologically isomorphic to the topological product of the rings R_γ endowed with the relative topology. Each $\operatorname{End}_A(E_\sigma(D_\gamma))$ is local since $E_\sigma(D_\gamma)$ is an indecomposable injective in $\mathcal{C}_\sigma$.

We obtain in this way the well known Zelinsky's theorem [20]: any commutative l.c. ring is a topological product of local l.c. rings.

By Proposition 3.3, $W_A = \bigoplus_\gamma E_\sigma(D_\gamma)$ is the minimal cogenerator of $\mathcal{C}_\sigma$.

Finally applying 3.2 once more, we deduce that (A, σ) is the topological product of the rings A_γ, which are local too.

3.5. Remark: Let (R, τ) be a commutative l.c. ring. We do not know if $A = \operatorname{End}({}_RW)$ is commutative. An affirmative answer to this question would imply that every commutative local l.c.d. ring has a Morita duality (see [4]): this is the well known Zelinsky-Müller conjecture.

4. - Duality

In a recent paper [1], Ánh introduced the notion of *topological Morita duality* (briefly TMD) and proved that a l.t. ring is l.c. iff it admits a TMD.

Let (R, τ) be a left l.t. ring and let (B, β) be a right l.t. ring. We say that (R, τ) admits a TMD with (B, β) if there exists a faithfully balanced bimodule ${}_RK_B$ such that ${}_RK$ and ${}_BK$ are injective cogenerators of $\mathcal{C}_\tau$ and $\mathcal{C}_\beta$ respectively.

Observe that in such a case, according to Theorem 2.8, $\operatorname{Soc}({}_RK) = \operatorname{Soc}(K_B)$ and this bimodule is essential in ${}_RK$ and K_B. Introducing in B the K-topology β', which is equivalent to β by Lemma 1.6, it follows from Corollary 2.12 that (B, β'), and consequently (B, β), is l.c. On the same way (R, τ) is l.c. (then, by Lemma 2.1 the K-topologies of R and B coincide with the respective Leptin topologies). It is proved in this way that if (R, τ) admits a TMD then (R, τ) is l.c. Conversely, if (R, τ) is l.c., then using the bimodule ${}_RW_A$ we get a TMD by means of Theorem 2.14.

In this section we study a duality for l.t. rings using methods from [11] and [12], obtaining in this way a characterization of l.c. rings which sharpens that of Ánh.

4.1. In this section (R, τ) denotes a left l.t. ring, ${}_RK$ denotes a fixed cogenerator of $\mathcal{C}_\tau$, $B = \operatorname{End}({}_RK)$. The ring B is always endowed with the K-topology β. Denoting by τ' the K-topology of $R(\tau' \subseteq \tau)$, $\operatorname{End}(K_B)$ coincides with the Hausdorff completion $\bar{R}$ of (R, τ'), according to Theorem 2.5, so that the bimodule ${}_{\bar{R}}K_B$ is faithfully balanced. By Lemma 1.6 the topologies τ and τ' are equivalent.

The modules ${}_RK$ and K_B are supposed always to be endowed with the discrete topology, so that ${}_RK \in R_\tau$-LT and $K_B \in LT$-B_β. In particular ${}_RK$ is a cogenerator of R_τ-LT.

Some of the following results can be deduced from [12]; however it is more convenient to obtain them directly in our particular case.

Let $M \in R_\tau$-LT ($M \in LT$-B_β). A *character* of M is a continuous morphism of M in ${}_RK$ (in K_B). We say that a module $M \in R_\tau$-LT is ${}_RK$*-completely regular* if M is topologically isomorphic to a topological submodule of a module of the form ${}_RK^X$ where X is a non empty set. Denote by $\mathfrak{B}({}_RK)$ the full subcategory of R_τ-LT consisting of all ${}_RK$-completely regular modules. The subcategory $\mathfrak{B}(K_B)$ of LT-B_β is defined in an analogous way.

The following assertion is obvious.

a) Let $M \in R_\tau$-LT ($M \in LT$-B_β). Then $M \in \mathfrak{B}({}_RK)$ ($M \in \mathfrak{B}(K_B)$) iff the topology of M coincides with the weak topology of the characters of M. Consequently for every $M \in \mathfrak{B}({}_RK)$ ($M \in \mathfrak{B}(K_B)$) the characters of M separate the points of M.

Let $M \in R_\tau$-LT ($M \in LT$-B_β). The *module of characters*, or the *dual* of M, is the module M^* defined in 2.4. For every $x \in M$ the morphism $\tilde{x}$ defined in 2.4 is a character of M^*. The *canonical* morphism $\omega_M : M \to M^{**}$ is defined as in 2.4. M is said to be *reflexive* if ω_M is a topological isomorphism. Observe that $M^* \in \mathfrak{B}(K_B)$ ($M^* \in \mathfrak{B}({}_RK)$). Therefore if M is reflexive then M belongs to $\mathfrak{B}({}_RK)$ ($\mathfrak{B}(K_B)$).

4.2. Lemma:

a) If $M \in R_\tau$-LT ($M \in LT$-B_β), ω_M is continuous.

b) If $M \in \mathfrak{B}({}_RK)$ ($M \in \mathfrak{B}(K_B)$) ω_M is a topological embedding.

c) If $M \in R_\tau$-LT, ω_M is injective.

Proof: *a*) and *b*) follow from Theorem 2.5. *c*) is consequence of the fact that ${}_RK$ is a cogenerator of R_τ-LT.

4.3. Remark: Let $M \in R_\tau$-LT and let w be the weak topology of the characters of M. Then $(M, w) \in R_\tau$-LT and $M^* = (M, w)^*$. The same holds when $M \in LT$-B_β and the characters of M separate the points of M.

The following statement are clear.

a) $(R, \tau')^*$ is topologically isomorphic to K_B, hence $(R, \tau')^{**}$ is topologically isomorphic to $\bar{R} = \mathrm{End}\,(K_B)$.

b) $(B, \beta)^*$ is topologically isomorphic to ${}_RK$, hence $(B, \beta)^{**}$ is topologically isomorphic to (B, β).

The following key lemma is due to C. Menini.

4.5. Lemma: *Suppose that every $M \in \mathfrak{B}({}_RK)$ is reflexive. Let $x, y \in M$ be such that $y \notin Rx$. Then there exists a character $\zeta \in M^*$ such that $x\zeta = 0$, $y\zeta \neq 0$.*

Proof: Suppose the conclusion false; then for every $\zeta \in M^*$, $x\zeta = 0$ yields $y\zeta = 0$. Consider the module M^{N} endowed with the product topology. Then $M^{\mathrm{N}} \in \mathfrak{B}({}_RK)$. Let $\bar{x} = (x_n)_{n \in \mathrm{N}}$ with $x_n = x$ for every $n \in \mathrm{N}$ and consider the submodule

$$H = R\bar{x} + M^{(\mathrm{N})}$$

of M^{N}. $H \in \mathfrak{B}({}_RK)$ and H is dense in M^{N} since $H > M^{(\mathrm{N})}$. Hence the restriction to H of the characters of M^{N} gives an algebraic isomorphism of $(M^{\mathrm{N}})^*$ on H^*. By Proposition 2.4 of [16] $H^* = \mathrm{Chom}_R(H, {}_RK) \simeq \mathrm{Chom}_R(M^{\mathrm{N}}, {}_RK) \simeq$ $\simeq (M^*)^{(\mathrm{N})}$ algebraically.

Observe that every element $t = (t_n)_{n \in \mathrm{N}}$ of M^{N} can be interpreted as a morphism $f_t \colon (M^*)^{(\mathrm{N})} \to K_R$, not necessarily continuous, setting $f_t(\zeta) = \sum_{n \in \mathrm{N}} t_n \zeta_n$, where $\zeta = (\zeta_n)_{n \in \mathrm{N}}$ with $\zeta_n \in M^*$ and almost all ζ_n zero. Since H is reflexive the characters of H^* are exactly these of the form f_t with $t \in H$. In particular $f_{\bar{x}} \in H^{**}$.

Let $\bar{y} = (y_n)_{n \in \mathrm{N}}$ with $y_n = y$ for every n. We prove that

$$\operatorname{Ker}(f_{\bar{y}}) > \operatorname{Ker}(f_{\bar{x}}) \tag{1}$$

which would imply $f_{\bar{y}} \in H^{**}$. Indeed, let $\zeta = (\zeta_n)_{n \in \mathrm{N}}$ be an element of $(M^*)^{(\mathrm{N})}$ with $\zeta \in \operatorname{Ker}(f_{\bar{x}})$. Then:

$$0 = f_{\bar{x}}(\zeta) = \sum_{n \in \mathrm{N}} x\zeta_n = x \sum_{n \in \mathrm{N}} \zeta_n .$$

Since $\sum_n \zeta_n$ is a character of M, it follows that $y \sum_n \zeta_n = 0$, by assumption. Then $f_{\bar{y}}(\zeta) = y \sum_n \zeta_n = 0$. This proves (1), hence $f_{\bar{y}} \in H^{**}$ and consequently $\bar{y} \in H$. Then $\bar{y} = r\bar{x} + z$ with $r \in R$ and $z \in M^{(\mathrm{N})}$. Let $n \in \mathrm{N}$ such that $z_n = 0$. Then $y = rx \in Rx$, a contradiction.

4.6. Theorem: *The following conditions are equivalent:*

(*a*) $R = \mathrm{End}(K_R)$ *and the modules* ${}_RK$ *and* K_R *are both q.i.*

(*b*) *For every* $M \in \mathfrak{B}({}_RK)$ *and for every* $L \in \mathfrak{B}(K_R)$ *the canonical morphisms* ω_M *and* ω_L *are topological isomorphisms.*

Proof: (*a*) $\Rightarrow$ (*b*). Let $M \in \mathfrak{B}({}_RK)$. By Lemma 4.2 ω_M is a topological embedding. Let us prove that ω_M is surjective. Consider M^* as a topological submodule of K_R^M and let α be a character of M^*. By Proposition 3.9 of [11]

α extends to a character $\bar{\alpha}$ of K_B^M. By Corollary 2.4 of [16] $(K_B^M)^*$ can be identified with the left R-module generated by the projections $\pi_x \colon K_B^M \to K_B$, $x \in M$. Hence there exist a finite subset F of M and elements $r_x \in R$ such that $\bar{\alpha} = \sum_{x \in F} r_x \pi_x$. Since $\pi_x | M^* = \tilde{x}$, it follows that $\alpha = \sum_{x \in F} r_x \tilde{x} = \widetilde{\sum_{x \in F} r_x x}$, which proves that ω_M is surjective. If $L \in \mathcal{B}(K_B)$ an analogous argument can be used since ${}_RK$ is q.i. and $\operatorname{End}({}_RK) = B$.

$(b) \Rightarrow (a)$. By Lemma 4.4 $(R, \tau') \cong (R, \tau')^{**} \cong \operatorname{End}(K_B)$, hence $R \cong$ $\cong \operatorname{End}(K_B)$ canonically.

To finish the proof we must show that K_B is q.i. Let $L < K_B$: then L provided with the discrete topology belongs to $\mathcal{B}(K_B)$. Let $i \colon L \hookrightarrow K_B$ be the inclusion and $f \in \operatorname{Hom}_B(L, K_B)$. Then $f \in Ri$ will imply the conclusion. Assume that $f \notin Ri$. f and i are elements of $L^* \in \mathcal{B}({}_RK)$. By the previous lemma there exists $\zeta \in L^{**}$ such that $i\zeta = 0$ and $f\zeta \neq 0$. Then $\zeta = \tilde{x}$ with $x \in L$ since L is reflexive. Now

$$i\zeta = i\tilde{x} = i(x) = 0 \; ; \qquad f\zeta = f\tilde{x} = f(x) \neq 0 \; ;$$

a contradiction.

4.7. Let $D_1 \colon R_\tau\text{-}LT \to LT\text{-}B_\delta$ be the contravariant functor which associates to every $M \in R_\tau\text{-}LT$ its dual M^* and to every continuous morphism $f \colon L \to M$ in $R_\tau\text{-}LT$ its transposed $f^* \colon M^* \to L^*$. It is easy to verify that f^* is continuous. The functor $D_2 \colon LT\text{-}B_\delta \to R_\tau\text{-}LT$ is defined analogously. Denote by D_K the couple (D_1, D_2). Let ${}_R\mathcal{M}$ and $\mathcal{M}_B$ be full subcategories of $R_\tau\text{-}LT$ and $LT\text{-}B_\delta$ respectively. We say that D_K *induces a duality* between ${}_R\mathcal{M}$ and $\mathcal{M}_B$ if $D_1({}_R\mathcal{M}) = \mathcal{M}_B$, $D_2(\mathcal{M}_B) = {}_R\mathcal{M}$ and all modules in $\mathcal{M}_B$ and ${}_R\mathcal{M}$ are reflexive.

4.8. Denote by $\mathcal{C}({}_RK)$ the full subcategory of $\mathcal{B}({}_RK)$ consisting of all modules topologically isomorphic to closed submodules of topological products of the form ${}_RK^X$. The modules belonging to $\mathcal{C}({}_RK)$ will be called ${}_RK$-*compact*.

4.9. Let $M \in R\text{-Mod}$ such that $\operatorname{Hom}_R(M, {}_RK)$ separates the points of M. Denote by χ_M the weak topology of M with respect to $\operatorname{Hom}_R(M, {}_RK)$. Clearly $(M, \chi_M) \in \mathcal{B}({}_RK)$. A module $(M, \epsilon) \in R_\tau\text{-}LT$ is called ${}_RK$-*discrete* if $\epsilon = \chi_M$. Denote by $\mathcal{D}({}_RK)$ the full subcategory of $\mathcal{B}({}_RK)$ consisting of ${}_RK$-discrete modules. A module $M \in \mathcal{B}({}_RK)$ is ${}_RK$-discrete iff

$$\operatorname{Chom}_R(M, {}_RK) = \operatorname{Hom}_R(M, {}_RK) .$$

Every algebraic morphism between two ${}_RK$-discrete modules is continuous. If $L, M \in \mathcal{D}({}_RK)$, then $\operatorname{Chom}_R(L, M) = \operatorname{Hom}_R(L, M)$.

The categories $\mathcal{C}(K_B)$ and $\mathcal{D}(K_B)$ are defined analogously.

The following theorem is an improvement of Theorem 6.6 of [12] and characterizes l.c. rings by means of D_K.

4.10. THEOREM: *Let (R, τ) be a left l.t. ring, ${}_RK$ an injective cogenerator of $\mathcal{C}_\tau$ with essential socle, $B = \operatorname{End}({}_RK)$ and let B have the K-topology β. The following conditions are equivalent:*

(*a*) *(R, τ) is l.c.*

(*b*) *(R, τ) and (B, β) are both l.c.*

(*c*) *$R \cong \operatorname{End}(K_B)$ and K_B is q.i.*

(*d*) *D_K induces a duality between $\mathfrak{B}({}_RK)$ and $\mathfrak{B}(K_B)$.*

(*e*) *K_B is q.i. and D_K induces a duality between $\mathfrak{D}({}_RK)$ and $\mathfrak{C}(K_B)$.*

(*f*) *For every $M \in R_\tau\text{-}LT$ ($M \in LT\text{-}B_\beta$) the canonical morphism ω_M is a continuous isomorphism.*

(*g*) *(R, τ) admits a TMD with (B, β) induced by the bimodule ${}_RK_B$.*

If these conditions are satisfied, then:

1) *For every $M \in R_\tau\text{-}LT$ ($M \in LT\text{-}B_\beta$) the topology of M is equivalent to the weak topology of ω_M.*

2) *D_K induces a duality between $\mathfrak{C}({}_RK)$ and $\mathfrak{D}(K_B)$.*

PROOF: $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (g)$ by Theorem 2.10 and Corollary 2.12.

$(c) \Leftrightarrow (d)$ by Theorem 4.6.

$(g) \Leftrightarrow (e)$. Let $M \in \mathfrak{D}({}_RK)$. Since $\operatorname{Chom}_R(M, {}_RK) = \operatorname{Hom}_R(M, {}_RK)$, it follows that $M^* \in \mathfrak{C}(K_B)$. Now let $M \in \mathfrak{C}(K_B)$, set $\overline{M} = \operatorname{Chom}_B(M, K_B)$. To get the conclusion it is enough to prove that every algebraic morphism ${}_R\overline{M} \to {}_RK$ is continuous when $\overline{M}_R$ is endowed with the finite topology. Consider ${}_R\overline{M}$ as an abstract module; by virtue of Theorem 4.7 from [11] the canonical morphism $M \to \operatorname{Hom}_R({}_R\overline{M}, {}_RK)$ is an isomorphism, since K_B is s.q.i. This proves that every algebraic morphism ${}_R\overline{M} \to {}_RK$ has the form $\tilde{x}$ with $x \in M$. Finally $\tilde{x}$ is continuous when ${}_R\overline{M}$ is endowed with the finite topology (see also [13], Lemma 4.11 and Theorem 4.12).

$(e) \Rightarrow (d)$. The bimodule ${}_RK_B$ is faithfully balanced and the modules ${}_RK$ and K_B are both q.i. Now apply Theorem 4.8.

$(d) \Rightarrow (f)$ and $(d) \Rightarrow 1)$. Let $(M, \varepsilon) \in R_\tau\text{-}LT$. Since ${}_RK$ is a cogenerator of $R_\tau\text{-}LT$ the characters of (M, ε) separate the points of M. Denote by ε_1 the weak topology of the characters of (M, ε). Then $(M, \varepsilon_1) \in \mathfrak{B}({}_RK)$ and it is clear that $(M, \varepsilon)^* = (M, \varepsilon_1)^*$. Hence $(M, \varepsilon)^{**} = (M, \varepsilon_1)^{**}$. The canonical morphism $\omega_1\colon (M, \varepsilon_1) \to (M, \varepsilon)^{**}$ is a topological isomorphism, so the canonical morphism $\omega\colon (M, \varepsilon) \to (M, \varepsilon)^{**}$ is a continuous isomorphism. Let

$L \in LT\text{-}B_\beta$ Since $(d) \Leftrightarrow (g)$ K_B is a cogenerator of $LT\text{-}B_\beta$ and the above argument applies. This proves (f). Let us prove 1).

Let $(M, \varepsilon) \in R_\tau\text{-}LT$. Since ω_1 above is a topological isomorphism and $(M, \varepsilon_1) \in \mathfrak{B}({}_RK)$, the weak topology of ω coincides with ε_1. Now, $\varepsilon_1 \subseteq \varepsilon$, hence every ε_1-closed submodule is ε-closed. On the other hand, since ${}_RK$ is a cogenerator of $R_\tau\text{-}LT$, every ε-open submodule of M is intersection of ε_1-open submodules; therefore every ε-closed submodule of M is intersection of ε_1-open submodules. This proves that every ε-closed submodule of M is also ε_1-closed. For $L \in LT\text{-}B_\beta$ we use analogous arguments.

$(f) \Rightarrow (d)$. Follows from Lemma 4.2 $b)$ and $c)$.

4.11. Consider the torsion class $\mathfrak{T}_\tau < R$-Mod. Denote again by $\mathfrak{T}_\tau$ the subclass of $R_\tau\text{-}LT$ consisting of modules belonging to $\mathfrak{T}_\tau$ with the discrete topology. By $\overline{\mathfrak{T}}_\tau$ we denote the subclass of $\mathfrak{B}({}_RK)$ consisting of the modules (M, ε) such that $M \in \mathfrak{T}_\tau$ and $\varepsilon = \chi_M$. Clearly $\mathfrak{T}_\tau \subseteq \mathfrak{D}({}_RK)$. A module $M \in \mathfrak{D}({}_RK)$ belongs to $\mathfrak{T}_\tau$ iff its support belongs to $\mathfrak{T}_\tau$. ${}_RK$ with the discrete topology is in $\overline{\mathfrak{T}}_\tau$. Analogous notations and conventions hold for the class $\mathfrak{T}_\beta$.

Let $M \in \mathfrak{T}_\tau$. Since every submodule of M is closed in (M, χ_M), χ_M is equivalent to the discrete topology of M. The same holds for $\mathfrak{T}_\beta$ if K_B is a cogenerator of $\mathfrak{T}_\beta$.

5. - Application of the duality

5.1. Throughout this section (R, τ) denotes a left l.c. ring, ${}_RK$ denotes a fixed injective cogenerator with essential socle of $\mathfrak{T}_\tau$, $B = \mathrm{End}\,({}_RK)$ and B will be always endowed with the K-topology β. Then (B, β) is a right l.c. ring, K_B is an injective cogenerator of $\mathfrak{T}_\beta$ with essential socle, ${}_RK_B$ is faithfully balanced and $\beta = \beta_*$.

The functors D_1 and D_2 and $D_K = (D_1, D_2)$ have the same meaning as in the preceding section.

5.2. We denote by $R_\tau\text{-}LC$ the (full) subcategory of $R_\tau\text{-}LT$ consisting of l.c. modules and by $R_\tau\text{-}SLC$ the (full) subcategory of $R_\tau\text{-}LC$ consisting of s.l.c. modules. A modules $M \in R_\tau\text{-}LT$ is said to be *pre-linearly compact* (briefly p.l.c.) if for every open submodule H of M the quotient M/H is l.c.d. Denote by $R_\tau\text{-}PLC$ the subcategory of $R_\tau\text{-}LT$ consisting of p.l.c. modules. $R_\tau\text{-}LC_*$ denotes the subcategory of $R_\tau\text{-}LC$ consisting of l.c. modules endowed with the Leptin topology. $R_\tau\text{-}PLC_*$ is defined in an analogous way.

Let $\mathcal{N}_\tau$ be the subcategory of $\mathfrak{T}_\tau$ consisting of submodules of f.g. modules belonging to $\mathfrak{T}_\tau$. $\mathcal{N}_\tau$ is closed with respect to submodules, quotients and finite direct sums. Since (R, τ) is l.c., every $N \in \mathcal{N}_\tau$ is l.c.d. Let $R_\tau\text{-}FLC$ be the subcategory of $R_\tau\text{-}LC$ consisting of the modules M such that for every open submodule H of M $M/H \in \mathcal{N}_\tau$. The meaning of $R_\tau\text{-}FLC_*$ is clear. We adopt analogous notations concerning (B, β).

5.3. Lemma: *Let $M \in R$-Mod and suppose that every f.g. submodule of M is artinian. Then M is l.c.d. iff M is artinian.*

Proof: See Lemma 1.4 of [13].

5.4. Lemma: *Let $M \in \mathcal{N}_\tau$, $N \in \mathcal{N}_\beta$ be both f.g. Then:*

a) $M^* \in \mathfrak{C}_\beta$.

b) $N^* \in \mathcal{N}_\tau$.

c) If $\tau = \tau_*$, then $M^* \in \mathcal{N}_\beta$.

Proof: Since M and N are f.g., the duals M^* and N^* are discrete.

a) Let $\zeta \in M^*$. Then $M\zeta$ is f.g., so

$$M\zeta = \sum_{i=1}^{n} Rx_i, \qquad x_i \in K.$$

Let

$$I = \bigcap_{i=1}^{n} \mathrm{Ann}_B(x_i)\,;$$

then $(M\zeta)I = 0$. This yields $\zeta I = 0$, hence $\mathrm{Ann}_B(\zeta) \in \mathcal{F}_\beta$.

b) We can suppose, without loss of generality, that N is cyclic, i.e. $N = xB$, $x \in N$. Since β is the K-topology of B, there exists a f.g. submodule F of ${}_BK$ such that $\mathrm{Ann}_B(x) \supseteq \mathrm{Ann}_B(F)$. Thus there exists a natural surjective homomorphism $B/\mathrm{Ann}_B(F) \to B/\mathrm{Ann}_B(x) \to 0$, hence there exists an injection

$$0 \to (xB)^* \to (B/\mathrm{Ann}_B(F))^* = \mathrm{Ann}_K \mathrm{Ann}_B(F) = F.$$

Since $F \in \mathfrak{C}_\tau$, $(xB)^* \in \mathcal{N}_\tau$.

c) If $\tau = \tau_*$, then τ_* coincides with the K-topology (see Lemma 2.1) and we argue as in *b*).

5.5. Lemma: *Let $M \in R_\tau$-FLC, $N \in$ FLC-B_β. Then $M^* \in \mathfrak{C}_\beta$ and $N^* \in \mathfrak{C}_\tau$.*

Proof: Let ζ be a character of M. $M\zeta \cong M/\mathrm{Ker}(\zeta)$ is a submodule of ${}_RK$ and a submodule of a f.g. module belonging to $\mathfrak{C}_\tau$. Since ${}_RK$ is an injective in $\mathfrak{C}_\tau$, $M\zeta$ is a submodule of a f.g. submodule of ${}_RK$. Hence $M\zeta < \sum_{i=1}^{n} Rx_i$, $x_i \in {}_RK$. Let

$$I = \bigcap_{i=1}^{n} \mathrm{Ann}_B(x_i).$$

Then $I \in \mathcal{F}_\beta$ and $\zeta I = 0$, hence $\mathrm{Chom}_B(M, {}_RK) \in \mathfrak{C}_\beta$. Let ε be the topology of M and let ε_1 be the weak topology of the characters of (M, ε). Then $M^* = (M, \varepsilon_1)^*$. Now $(M, \varepsilon_1) \in \mathfrak{C}({}_R K)$, hence by Theorem 4.10 $(M, \varepsilon_1)^* \in \mathfrak{D}(K_\beta)$. Therefore $M^* \in \mathfrak{T}_\beta$. For $M \in FLC\text{-}B_\beta$ one argues analogously.

5.6. Proposition: *Let $(M, \varepsilon) \in R_\tau\text{-}PLC$. Then the weak topology of the characters of (M, ε) coincides with the Leptin topology ε_*. If (M, ε) is l.c., then $(M, \varepsilon_*) \in \mathfrak{C}({}_R K)$. An analogous result holds for $PLC\text{-}B_\beta$.*

Proof: Let ε_1 be the weak topology of the characters of (M, ε). Since ${}_R K$ is a cogenerator of $R_\tau\text{-}LT$, ε_1 is Hausdorff and $(M, \varepsilon_1) \in \mathfrak{B}({}_R K)$. A subbasis of neighbourhoods of 0 for (M, ε_1) is given by the submodules $\operatorname{Ker}(\zeta)$, $\zeta \in (M, \varepsilon)^*$. Since $M/\operatorname{Ker}(\zeta) \simeq M\zeta < {}_R K$ is l.c.d. and has essential socle, $M/\operatorname{Ker}(\zeta)$ is f.c. This yields $\varepsilon_1 = \varepsilon_*$.

If (M, ε) is l.c., then (M, ε_*) is l.c., hence complete, so $(M, \varepsilon_*) \in \mathfrak{C}({}_R K)$.

Denote by $R_\tau\text{-}PSLC$ the subcategory of $R_\tau\text{-}PLC$ consisting of the modules whose quotients with respect to open submodules are artinian. $PSLC\text{-}B_\beta$ is defined analogously.

5.7. Proposition:

a) $R_\tau\text{-}SLC \subseteq R_\tau\text{-}LC_* \in \mathfrak{C}({}_R K)$.

b) If (R, τ) is s.l.c., then $R_\tau\text{-}PLC = R_\tau\text{-}PSLC$, so $R_\tau\text{-}LC = R_\tau\text{-}SLC$. Analogous inclusions hold for (B, β).

Proof: *a)* Follows by 5.6 since a s.l.c. module has always the Leptin topology.

b) Let $M \in R_\tau\text{-}PLC$ and let L be an open submodule of M. Then $M/L \in \mathfrak{T}_\tau$ and M/L is l.c.d. Since (R, τ) is s.l.c., every f.g. module in $\mathfrak{T}_\tau$ is artinian. By Lemma 5.3 M/L is artinian.

5.8. Let β^* be the finest topology on B equivalent to β (see 1.5 and 2.12). A basis of neighbourhoods of 0 in (B, β^*) is given by $\{\operatorname{Ann}_B(L)\}$ where L is a l.c.d. submodule of ${}_R K$. Observe that every $I \in \mathfrak{F}_{\beta^*}$ has the form $I = \operatorname{Ann}_B(X)$ where X is a l.c.d. submodule of ${}_R K$. In fact $I \supseteq \operatorname{Ann}_B(L)$ with a l.c.d. submodule of ${}_R K$, hence

$$\operatorname{Ann}_K(I) < \operatorname{Ann}_K \operatorname{Ann}_B(L) = L\,,$$

so $X = \operatorname{Ann}_K(I)$ is l.c.d. Since I is closed in (B, β), it follows that

$$I = \operatorname{Ann}_B \operatorname{Ann}_K(I) = \operatorname{Ann}_B(X)\,.$$

By the definition given in 1.5 a right ideal I of B belongs to $\mathfrak{F}_{\beta^*}$ iff I is closed in (B, β) and B/I is l.c.d.

Let $\mathcal{C}(\beta^*)$ be the subcategory of $\mathfrak{B}(K_B)$ defined as follows:

$$\mathcal{C}(\beta^*) = \{M \in \mathfrak{B}(K_B) : \forall x \in M, \operatorname{Ann}_B(x) \in \mathcal{F}_{\beta^*}\}.$$

$\mathcal{C}(\tau^*)$ is defined in the same way.

5.9. Remark: If (R, τ) is s.l.c. then every l.c.d. submodule of ${}_BK$ is artinian according to Lemma 5.3. Hence (B, β^*) is topologically noetherian.

5.10. Proposition:

a) A module $M \in \mathfrak{B}(K_B)$ belongs to $\mathcal{C}(\beta^*)$ iff the f.g. submodules of M are l.c.d.

b) D_K induces a duality between R_τ-PLC_* and $\mathcal{C}(\beta^*)$.

Analogous results hold for (R, τ).

Proof: *a)* Let $M \in \mathcal{C}(\beta^*)$, $x \in M$, $x \neq 0$. Then $\operatorname{Ann}_B(x) \in \mathcal{F}_{\beta^*}$, hence $xB \simeq B/\operatorname{Ann}_B(x)$ is l.c.d.

Conversely, let $M \in \mathfrak{B}(K_B)$ and for every $x \in M$ let xB be l.c.d. Then $B/\operatorname{Ann}_B(x) \simeq xB$ is l.c.d. Since $\operatorname{Ann}_B(x)$ is closed in (B, β) this yields $\operatorname{Ann}_B(x) \in \mathcal{F}_{\beta^*}$.

b) Let $M \in R_\tau$-PLC_*, $\zeta \in M^*$. Then $M\zeta$ is a l.c.d. submodule of ${}_BK$, hence $I = \operatorname{Ann}_B(M\zeta) \in \mathcal{F}_{\beta^*}$. Since $\zeta I = 0$, then $M^* \in \mathcal{C}(\beta^*)$.

Let F be a f.g. module belonging to $\mathcal{C}(\beta^*)$. There exists a surjective morphism

$$\bigoplus_{i=1}^{m} B/I_i \to F \to 0 \qquad \text{with } I_i \in \mathcal{F}_{\beta^*}.$$

Applying $\operatorname{Hom}_B(-, K_B)$ we obtain the injection

$$0 \to \operatorname{Hom}_B(F, K_B) \to \bigoplus_{i=1}^{m} \operatorname{Hom}_B(B/I_i, K_B).$$

Since F is f.g., F^* is discrete, so it can be identified with a submodule of $\bigoplus_{i=1}^{m} \operatorname{Hom}(B/I_i, K_B)$. Since B/I_i is cyclic, $\operatorname{Hom}_B(B/I_i, K_B) \simeq \operatorname{Ann}_K(I_i)$. For every $j = 1, 2, \dots, m$ there exists a l.c.d. submodule L_j of ${}_BK$ such that $I_j = \operatorname{Ann}_B(L_j)$; consequently $\operatorname{Hom}_B(B/I_j, K_B) \simeq L_j$. Hence $F^* < \bigoplus_{j=1}^{m} L_j$, so F^* is l.c.d.

Now let $M \in \mathcal{C}(\beta^*)$ and let F be a f.g. submodule of M. Since $M^*/O(F)$ is topologically isomorphic to F^* which is l.c.d., it follows that $M^* \in R_\tau$-PLC_*.

This gives rise to the following duality theorem.

5.11. Theorem: *Let* (R, τ) *be a left l.c. ring,* ${}_RK$ *an injective cogenerator of* $\mathcal{C}_\tau$ *with essential socle,* $B = \mathrm{End}\,({}_RK)$ *and let* B *have the* K*-topology* β*. Then:*

a) D_K *induces a duality between* $R_\tau\text{-}LC_*$ *and* $\mathcal{C}(\beta^*) \cap \mathfrak{D}(K_B)$.

b) D_K *induces a duality between* $\mathcal{C}(\tau^*) \cap \mathfrak{D}({}_RK)$ *and* $LC_*\text{-}B_\beta$.

Proof: *a*) By Proposition 5.6 and 5.7

$$R_\tau\text{-}LC_* = (R_\tau\text{-}PLC_*) \cap \mathcal{C}({}_RK)\,.$$

By Theorem 4.10 D_K induces a duality between $\mathcal{C}({}_RK)$ and $\mathfrak{D}(K_B)$. The conclusion follows from Proposition 5.10.

b) Proceed as in *a*).

5.12. Corollary: *In the conditions of the preceding theorem, the following conditions are equivalent:*

(*a*) $R_\tau\text{-}LC_* = \mathcal{C}({}_RK)$.

(*b*) ${}_RK$ *is* l.c.d.

(*c*) K_B *is an injective cogenerator of* Mod-B.

(*d*) B_β *is l.c.d.*

Proof: (*a*) $\Rightarrow$ (*b*). Follows from the fact that ${}_RK$ endowed with the discrete topology belongs to $\mathcal{C}({}_RK)$.

(*b*) $\Rightarrow$ (*a*). Now $\mathcal{C}({}_RK) \subseteq R_\tau\text{-}LC_*$; the other inclusion was verified in 5.7.

(*b*) $\Leftrightarrow$ (*c*) and (*b*) $\Rightarrow$ (*d*) follow from Proposition 1.3 of [18].

(*d*) $\Rightarrow$ (*b*). Suppose that B_β is l.c.d. and let d be the discrete topology of B_B. Clearly $d = \beta^*$, so there exists a l.c.d. submodule L of ${}_RK$ such that $\mathrm{Ann}_B\,(L) = 0$. Since ${}_RK$ is s.q.i. and $B = \mathrm{End}\,({}_RK)$, it follows that $L = {}_RK$, hence ${}_RK$ is l.c.d.

5.13. Remark: In the conditions of 5.11, let the equivalent conditions from 5.12 hold. Then $\mathcal{C}(\beta^*) = LT\text{-}B_\beta$ so that D_K induces a duality between $R_\tau\text{-}LC_*$ and $\mathfrak{D}(K_B)$. Since K_B is a cogenerator of Mod-B, $\mathfrak{D}(K_B)$ is equivalent to Mod-B by means of the functor D which associates to every $M \in \mathfrak{D}(K_B)$ the underlying abstract module. Then the functor $D \circ D_1$ is a duality between $R_\tau\text{-}LC_*$ and Mod-B whose inverse is given by the functor which associates to every $M \in$ Mod-B the module $\mathrm{Hom}_B\,(M, K_B)$ endowed with the finite topology.

5.14. Theorem: *In the conditions of Theorem 5.11:*

a) D_K *induces a duality between* $R_\tau\text{-}FLC_*$ *and* $\mathcal{C}_\beta$.

b) *If* $\tau = \tau_*$*, then* D_K *induces a duality between* $\mathcal{C}_\tau$ *and* $FLC_*\text{-}B_\beta$.

Proof: *a*) Let $M \in R_\tau\text{-}FLC_*$. By Lemma 5.5 $M^* \in \mathfrak{C}_\beta$. Let $M \in \mathfrak{C}_\beta$. Then $M^* \simeq \varinjlim M^*/O(F)$, where F runs over the f.g. submodules of M, the isomorphism being topological. On the other hand $M^*/O(F) \simeq F^*$ as discrete modules. Following Lemma 5.4 $F^* \in \mathcal{N}_\tau$, hence $M^* \in R_\tau\text{-}FLC_*$.

b) If $\tau = \tau_*$ one argues as above.

The duality D_K enables us to characterize the s.l. rings (R, τ) by means of an explicit description of the duals of the modules belonging to $\mathfrak{C}_\tau$.

Let (R, τ) be l.c.; denote by $R_\tau\text{-}NLC$ the subcategory of $R_\tau\text{-}LC$ consisting of those l.c. modules which are topologically noetherian, i.e. the modules $M \in R_\tau\text{-}LC$ such that for every open submodule H of M M/H is noetherian. Equivalently, $M \in R_\tau\text{-}LC$ and M satisfies the A.C.C. for open submodules. The meaning of $R_\tau\text{-}NLC_*$ is clear. Analogous notations will be adopted for (B, β).

5.15. Theorem: *Under the hypotheses of Theorem 5.11 the following conditions are equivalent:*

(*a*) (R, τ) *is s.l.c.*

(*b*) $\tau = \tau_*$ *and* (B, β) *is topologically noetherian.*

(*c*) *For every f.g. module* $F \in \mathfrak{C}_\tau$ *there exist* $n \in \mathbb{N}$ *and an injection of* F *in* ${}_RK^n$.

(*d*) *For every left ideal* $I \in \mathfrak{F}_\tau$ *there exists a finite subset* F *of* K *such that* $I = \mathrm{Ann}_R(F)$.

(*e*) *For every* $M \in \mathfrak{C}_\tau$, M^* *is l.c. and its discrete quotients are f.g.*

(*f*) D_K *induces a duality between* $\mathfrak{C}_\tau$ *and* $NLC_*\text{-}B_\beta$.

Proof: $(a) \Leftrightarrow (b)$. Follows from Propositions 2.1 and 2.13.

$(a) \Rightarrow (c)$. Since (R, τ) is s.l.c., F is artinian, hence f.c. So there exists an injection $F \hookrightarrow \bigoplus_{i=1}^{n} E_\tau(V_i)$ where V_i are simple modules belonging to $\mathfrak{C}_\tau$. Therefore there exists an injection $N \hookrightarrow {}_RK^n$.

$(c) \Rightarrow (d)$. R/I can be embedded in ${}_RK^n$, hence $I = \bigcap_{i=1}^{n} \mathrm{Ann}_R(x_i)$, where $(x_1, \ldots, x_n)$ is the image of $1 + I$ in ${}_RK^n$.

$(d) \Rightarrow (a)$. It is well known that a module M is artinian iff every quotient of M is f.c.. So it suffices to prove that for every $I \in \mathfrak{F}_\tau$, R/I is f.c. There exist $x_1, \ldots, x_n \in K$ such that $I = \bigcap_{i=1}^{n} \mathrm{Ann}_R(x_i)$. Then $R/I \hookrightarrow \bigoplus_{i=1}^{n} Rx_i < {}_RK^n$. This implies that $\mathrm{Soc}(R/I)$ is essential in R/I. Since R/I is l.c.d., $\mathrm{Soc}(R/I)$ is f.g.. Therefore R/I is f.c.

$(a) \Rightarrow (f)$. Since (R, τ) is s.l.c., $\tau = \tau_*$: Then by Theorem 5.14, *b*) D_K induces a duality between $\mathfrak{C}_\tau$ and $FLC_*\text{-}B_\beta$. Since $(a) \Leftrightarrow (b)$, (B, β) is topologically noetherian, hence every module belonging to $\mathcal{N}_\beta$ is noetherian, so that $FLC_*\text{-}B_\beta = NLC_*\text{-}B_\beta$.

$(f) \Rightarrow (e)$ is obvious.

$(e) \Rightarrow (d)$. Let $I \in \mathcal{F}_\tau$, then $R/I \in \mathfrak{T}_\tau$. $(R/I)^* = \mathrm{Hom}_R(R/I, {}_RK)$ with the discrete topology since R/I is f.g.. On the other hand $\mathrm{Hom}_R(R/I, {}_RK) \cong \cong \mathrm{Ann}_K(I)$ is f.g. by assumption. Now $I = \mathrm{Ann}_R \mathrm{Ann}_K(I)$ and $\mathrm{Ann}_K(I) = = \sum_{i=1}^{n} x_i B$ with $x_i \in K$. This yields $I = \mathrm{Ann}_R(F)$ with $F = \{x_1, \ldots, x_n\}$.

5.16. Remark: Under the hypotheses of Theorem 5.11 it may happen that (B, β) is topologically noetherian and (R, τ) is not s.l.c. Consider in fact the ring J_p with the discrete topology. Then $\mathbb{Z}(p^\infty)$ is the minimal injective cogenerator of J_p-Mod and J_p is l.c. and not s.l.c. In this case (B, β) coincides with the ring J_p endowed with the p-adic topology, so (B, β) is topologically noetherian.

5.17. Corollary: *Under the hypotheses of Theorem* 5.11 *suppose that* $\tau = \tau_*$. *Then the following conditions are equivalent*:

(*a*) (R, τ) *is topologically noetherian.*

(*b*) (B, β) *is s.l.c.*

(*c*) D_K *induces a duality between* $R_\tau\text{-}NLC$ *and* $\mathfrak{T}_\beta$.

5.18. Let (R, τ) be a left l.c. ring. A module $M \in R_\tau\text{-}LC$ is said to be *absolutely linearly compact* (a.l.c.) if M is topologically artinian and topologically noetherian at the same time. This means that for every open submodule H of M, M/H has finite length. In particular such a module is s.l.c. and has the Leptin topology. Denote by $R_\tau\text{-}ALC$ the subcategory of $R_\tau\text{-}LC$ consisting of a.l.c. modules. The category $ALC\text{-}B_\beta$ is defined analogously.

Suppose now that (R, τ) is a.l.c. Then $R_\tau\text{-}LC = R_\tau\text{-}LC_* = R_\tau\text{-}SLC$ since a.l.c. implies s.l.c. Moreover, every f.g. module of $\mathfrak{T}_\tau$ has finite length, hence $R_\tau\text{-}ALC = R_\tau\text{-}FLC = R_\tau\text{-}NLC$. In general $R_\tau\text{-}ALC \subsetneq R_\tau\text{-}SLC$: take for example the ring J_p endowed with the p-adic topology; then J_p is a.l.c. and $\mathbb{Z}(p^\infty)$ is s.l.c., but not a.l.c. in the discrete topology.

The following result is due essentially to P. Gabriel (cf. [6], pg. 395) and follows easily from Theorem 5.15 and Corollary 5.17.

5.19. Theorem: *Under the hypotheses of Theorem* 5.15 *the following conditions are equivalent*:

(*a*) (R, τ) *is left a.l.c.*

(*b*) $\tau = \tau_*$ *and* (B, β) *is right a.l.c.*

(*c*) D_K *induces a duality between* $\mathfrak{T}_\tau$ *and* $ALC\text{-}B_\beta$.

(*d*) $\tau = \tau_*$ *and* D_K *induces a duality between* $R_\tau\text{-}ALC$ *and* $\mathfrak{T}_\beta$.

PROOF: It is enough to prove $(b) \Leftrightarrow (d)$.

$(b) \Rightarrow (d)$. Corollary 5.17 says that D_K induces a duality between R_τ-NLC_* and $\mathcal{C}_\beta$. Since (B, β) is a.l.c., (R, τ) is a.l.c. too, hence R_τ-$NLC_* = R_\tau$-ALC.

$(d) \Rightarrow (b)$. Since $K_R \in \mathcal{C}_\beta$, $K_R^* = (R, \tau_*) \in R_\tau$-$ALC$. By $\tau = \tau_*$, (R, τ) is a.l.c., hence (B, β) is a.l.c.

5.20. REMARK: It may be shown that if (R, τ) is a left a.l.c. ring with Jacobson radical J then $\bigcap_{n \in \mathbb{N}} J^n = 0$. (See [6]).

5.21. *Comparison between the duality D_K and the duality of Oberst for the class $\mathcal{C}_\tau$ of a s.l.c. ring.*

Let (R, τ) be a left s.l.c. ring. ${}_RK_B$, (B, β) etc., have the usual meaning. Since $\mathcal{C}_\tau$ is a Grothendieck category, we can study the dual category of $\mathcal{C}_\tau$ in the sense of Oberst [15], using the couple $(\mathcal{N}_\tau, {}_RK)$ and taking in account Theorem 5.15.

For every $M \in \mathcal{C}_\tau$, the Oberst-dual of M is the right B-module $M^\# = \mathrm{Chom}_R(M, K)$ endowed with the topology $\varepsilon^\#$ having as a basis of neighbourhoods of O the submodules $O(N)$ with $N < M$ and $N \in \mathcal{C}_\tau$. If ε is the topology of M^*, then $\varepsilon \subseteq \varepsilon^\#$. It is not difficult to prove the following:

a) The topology $\beta^\#$ of B, as the Oberst-dual of ${}_RK$, coincides with β, so that $M^\# \in LT$-B_β for every $M \in \mathcal{C}_\tau$.

b) For every $M \in \mathcal{C}_\tau$, $M^\#$ is l.c.d. so that the topologies ε and $\varepsilon^\#$ are equivalent.

By our Theorem 5.15 and by Theorem 5.4 of [15], the assignement $M \mapsto M^\#$ defines a duality between $\mathcal{C}_\tau$ and the subcategory $STC(B)$ of LT-B_β consisting of all strict complete and topologically coherent modules (cf. [15], pg. 486). In our case $STC(B)$ coincides with the subcategory $SLNC$-B_β consisting of the modules $L \in NLC$-B_β which are strict, i.e.: if H is a closed submodule of L and $L/H \in \mathcal{N}_\beta$, then H is open in L. Consequently the categories NLC_*-B_β and $SLNC$-B_β are equivalent. If $(L, \varepsilon) \in SLNC$-$B_\beta$, in general $\varepsilon \neq \varepsilon^*$.

6. - EXAMPLE OF A STRICTLY LINEARLY COMPACT RING WHOSE DISCRETE FACTOR MODULES ARE OF INFINITE LENGTH

The existence of s.l.c. rings which are not a.l.c. was proved by Leptin (see [8], pg. 298). Nevertheless we think that the following example has some interest.

6.1. Let R be a commutative, noetherian local ring with maximal ideal $\mathcal{M}$. Suppose that R is non artinian, equicharacteristic and complete in its $\mathcal{M}$-adic topology. $K = R/\mathcal{M}$ is the maximal subfield of R and $R = K \oplus \mathcal{M}$ so that every $r \in R$ may be written in a unique way in the form

(1) $$r = r_1 + r_2 \qquad (r_1 \in K, r_2 \in \mathcal{M}).$$

Let H be the injective envelope in R-Mod of the unique simple R-module $R/\mathcal{M}$ and consider the bimodule ${}_R H_K$. We extend the action of K over H to an action of R by means of the canonic morphism $R \to K$. Then H gets a structure of right R-module. We have $H\mathcal{M} = 0$ and ${}_R H_R$ is a bimodule. For every $x \in H$, $x \neq 0$, there exists $n \in \mathbf{N}$ such that $\mathrm{Ann}_R(x) \supset \mathcal{M}^n$.

6.2. Let $\overline{R} = H \oplus 1R$ be the trivial extension of H by R. $\overline{R}$ is the ring consisting of the couples (x, r), $x \in H$, $r \in R$, where the addition is defined pointwise while the multiplication is given by

$$(x, r)(y, s) = (ry + xs, rs).$$

Observe that, according to (1), $xr = xr_1$. The identity of $\overline{R}$ is $(0, 1)$ and $\overline{R}$ is not commutative.

Let ${}_R N \leq {}_R H$ and let I be a proper ideal of R. Then

a) $N \oplus I$ is a proper left ideal of $\overline{R}$.

In fact let $(x, r) \in N \oplus I$, $(y, s) \in \overline{R}$. Then

$$(y, s)(x, r) = (sx + yr, sr) = (sy, sr)$$

since $yr = 0$ being $I \leq \mathcal{M}$.

Let N and I be as above. Put

$$\overline{N} = \{(x, 0) : x \in N\}, \qquad \overline{I} = \{(0, r) : r \in I\}.$$

Then $N \oplus I$ coincides with the internal direct sum $\overline{N} \oplus \overline{I}$. The following statement is obvious.

b) Let L be a left ideal of $\overline{R}$ consisting of elements whose first (second) component is zero. Then L is of the form $\overline{I}$ with I an ideal of R ($\overline{N}$ with ${}_R N \leq {}_R H$).

$\overline{R}$ is a local ring with maximal ideal $H \oplus \mathcal{M}$, since every element belonging to $\overline{R} \setminus (H \oplus \mathcal{M})$ is a unit in $\overline{R}$. Thus $J(\overline{R}) = H \oplus \mathcal{M}$ and $\overline{R}/J(\overline{R}) \simeq K$. $\overline{H}$ is a two sided ideal of $\overline{R}$ and $\overline{R}/\overline{H} \simeq R$.

Note that for every $(x, r) \in J(\overline{R})$ and for every $(y, s) \in \overline{R}$ it is: $(x, r)(y, s) = (ry, rs)$.

6.3. *For every $n \in \mathbf{N}$, the cyclic left $\overline{R}$-module $\overline{R}/\overline{\mathcal{M}}^n$ is artinian not noetherian.*

Indeed $\overline{R}/J(\overline{R})$ is a left $\overline{R}$-module of finite length. Moreover

$$J(\overline{R})/\overline{\mathcal{M}}^n = \frac{H \oplus \overline{\mathcal{M}}}{\mathcal{M}} \simeq H \oplus \overline{\mathcal{M}}/\overline{\mathcal{M}}^n.$$

Now, by a) and b) of 6.2, ${}_{\overline{R}}H$ is artinian not noetherian, while $\overline{\mathcal{M}}/\overline{\mathcal{M}}^n$ has finite length since, for every $n \in \mathrm{N}$, ${}_{\overline{R}}\overline{\mathcal{M}}^n$ is f.g.

6.4. Endow $\overline{R}$ with the $\overline{\mathcal{M}}$-adic topology τ, by taking as a basis of neighbourhoods of zero the left ideals $\overline{\mathcal{M}}^n$, $n \in \mathrm{N}$. Note that τ is a ring topology. In fact let $n \in \mathrm{N}$, $(x, r) \in \overline{R}$. There exists $p > n$ such that $\mathcal{M}^p < \mathrm{Ann}_R(x)$. Then Then $\overline{\mathcal{M}}^p(x, r) \leqslant \overline{\mathcal{M}}^n$. $(\overline{R}, \tau)$ is Hausdorff since $\bigcap_{n \in \mathrm{N}} \overline{\mathcal{M}}^n = 0$. $(\overline{R}, \tau)$ is complete since τ coincides with the product topology of the discrete topology on H by the $\mathcal{M}$-adic topology on R.

Thus $(\overline{R}, \tau)$ is a complete left l.t. ring and for every $I \in \mathcal{F}$, $\overline{R}/I$ is artinian. It follows that $(\overline{R}, \tau)$ is s.l.c. $(\overline{R}, \tau)$ is not a.l.c., by 6.3.

6.5. Remark: Let ω be the ordinal of N. Then:

$$J(\overline{R})^{\omega} = \bigcap_{n \in \mathrm{N}} J(\overline{R})^n = H \neq 0,$$

$$J(\overline{R})^{\omega+1} = 0.$$

In fact $J(\overline{R})^{\omega} J(\overline{R}) = H(H \oplus \overline{\mathcal{M}}) = 0$.

We now illustrate briefly the structure of ${}_{\overline{R}}W$ and that of $A = \mathrm{End}\,({}_{\overline{R}}W)$.

6.6. The module ${}_{\overline{R}}W$. Put ${}_{\overline{R}}W = E_{\tau}(V)$, where $V \cong \overline{R}/J(\overline{R})$ is the unique simple left $\overline{R}$-module. $V \cong R/\mathcal{M}$ and H is the injective hull of V in R-Mod.

Consider the R-module $\mathrm{Hom}_R(\overline{R}, H)$ and endow $\mathrm{Hom}_R(\overline{R}, H)$ with a structure of left $\overline{R}$-module in the following way. Let $\zeta \in \overline{R}$, $f \in \mathrm{Hom}_R(\overline{R}, H)$. Put

$$(\zeta f)(\mu) = f(\mu\zeta) \qquad (\mu \in \overline{R}).$$

Denote by ${}_{\overline{R}}E$ the $\overline{R}$-module obtained in this way. By a routine check, ${}_{\overline{R}}E$ is injective in $\overline{R}$-Mod.

Using the following isomorphisms of R-modules

$$(2) \qquad \mathrm{Hom}_R(\overline{R}, H) \cong \mathrm{Hom}_R(H \oplus R, H) \cong$$
$$\cong \mathrm{Hom}_R(H, H) \oplus \mathrm{Hom}_R(R, H) \cong R \oplus H$$

the elements of ${}_{\overline{R}}E$ are the couples $[r, h]$ with $r \in R$ and $h \in H$. As a morphism of $\overline{R}$ in H, $[r, h]$ acts as follows

$$[r, h](x, a) = rx + ah \qquad ((x, a) \in \overline{R}).$$

The scalar multiplication on ${}_{\overline{R}}E$ is given by

$$(y, b)[r, h] = [b_1 r, ry + bh]$$

where $b = b_1 + b_2$ according to (1).

The map $b \mapsto [0, b]$ identifies H with a submodule of ${}_{\bar R}E$, which we denote again with H, and it is easy to show that H is essential in ${}_{\bar R}E$. Since ${}_{\bar R}V$ is an essential submodule of ${}_{\bar R}H$, ${}_{\bar R}E$ coincides with the minimal injective cogenerator of $\bar R$-Mod.

It is also easy to show that ${}_{\bar R}E$ is τ-torsion so that ${}_{\bar R}E = {}_{\bar R}W$.

6.7. Remarks:

a) It is not difficult to show that the $\bar R$-submodule $K \oplus H$ of ${}_{\bar R}W$ is isomorphic to $\bar R/\mathfrak{K}$. The center of $\bar R$ coincides with the trivial extension $V \oplus 1K$, which is isomorphic to the endomorphism ring of $\bar R/\mathfrak{K}$.

b) The ring $\bar R = H \oplus 1R$ is l.c.d. In fact, using the natural inclusion $R \hookrightarrow \bar R$ $(r \mapsto (0, r))$ the R-module $H \oplus R$ is clearly l.c.d. On the other hand, every left ideal of $\bar R$ is an R-submodule of $H \oplus R$.

c) Let $A = \mathrm{End}\,({}_{\bar R}W)$ and let σ be the finite topology of A. Applying Corollary 5.2 to (A, σ) we see that W_A is l.c.d.

d) ${}_{\bar R}W$ is not l.c.d. since ${}_{\bar R}W/H$ is not l.c.d. Namely the action of $\bar R$ on W/H is the same as that of K, thus ${}_{\bar R}W/H$—as a left $\bar R$-module—is an infinite direct sum of simple modules.

e) Since ${}_{\bar R}W = {}_{\bar R}E$, the ring $\bar R$ gives a negative answer to Problem 2 of [13].

6.8. The ring $A = \mathrm{End}\,({}_{\bar R}W)$: It is easy to show that ${}_{\bar R}H$ is q.i.; hence ${}_{\bar R}H$ is a fully invariant submodule of its injective hull ${}_{\bar R}W$. Let $f \in A$. Since $Hf \subseteq H$, $f|_H$ acts on H as an element belonging to $\mathrm{End}_{\bar R}(H) = \mathrm{End}_R(H)$ and thus $f|_H$ is the left multiplication by a unique element $s_f \in R$. Then for every $b \in H$ we have:

$$[0, b]f = [0, s_f b]\,.$$

Let $r \in \bar R$. Taking account of the structure (2) of ${}_{\bar R}W$, it is straightforward to see that

$$[r, 0]f = [s_f r, \hat f(r)]$$

where $\hat f$ is a K-linear morphism $R \to V$, uniquely determined by f. Therefore for every $[r, b] \in {}_{\bar R}W$, we have

$$[r, b]f = [s_f r, \hat f(r) + s_f b]\,. \tag{3}$$

Then f may be represented by the couple $(s_f, \hat f)$ and conversely any such couple gives an element of A by means of (3).

The addition of these couples, as endomorphisms of ${}_{\bar R}W$, is the pointwise one, while the multiplication is given by

$$(s_f, \hat f)(s_g, \hat g) = (s_f s_g, \hat f s_g + s_f \hat g) \tag{4}$$

where, for every $r \in R$,

$$(\hat{f}s_\alpha)(r) = \hat{f}(r)s_\alpha = s_\alpha \hat{f}(r)\,, \tag{5}$$

$$(s_\beta \hat{g})(r) = \hat{g}(rs_\beta) = \hat{g}(s_\beta r)\,. \tag{6}$$

Put $\hat{R} = \mathrm{Hom}_R(R, V)$. The mapping $(\hat{f}, s_\alpha) \mapsto \hat{f}s_\alpha$ provides $\hat{R}$ of a structure of right R-module, while the mapping $(s_\beta, \hat{g}) \mapsto s_\beta \hat{g}$ gives a structure of left R-module to $\hat{R}$. Then by (4), (5), (6) A coincides with the trivial extension $A = R1 \oplus \hat{R}$.

Finally the finite topology σ of A coincides with the product topology of the discrete topology on R by the usual finite topology on $\hat{R}$. Clearly (A, σ) is l.c.d. and topologically noetherian. Obviously $J(A) = \mathcal{M} \oplus \hat{R}$, so A is local.

7. - The cobasic ring, the grade and the basic ring of a linearly compact ring

In the whole of this section (R, τ) denotes a fixed, but arbitrary, left l.c. ring, $(V_\gamma)_{\gamma \in \Gamma}$ a system of representatives of non isomorphic simple modules belonging to $\mathcal{C}_\tau$, $D_\gamma = \mathrm{End}_R(V_\gamma)$, ${}_R W$ the minimal injective cogenerator of $\mathcal{C}_\tau$, $A = \mathrm{End}({}_R W)$ and σ is the finite topology of A. Then (A, σ) is right l.c. and $A/J(A) \cong \prod_{\gamma \in \Gamma} D_\gamma$. By Lemma 2.1, $\sigma = \sigma_*$. Moreover $W_A = E_A\left(\bigoplus_{\gamma \in \Gamma} D_\gamma^{(\nu_\gamma)}\right)$. Set $\nu_\Gamma = (\nu_\gamma)_{\gamma \in \Gamma}$.

7.1. Definition: The right l.c. ring (A, σ) will be called the *cobasic ring* of (R, τ) and ν_Γ the *grade* of (R, τ).

7.2. Lemma: *Let (R, τ) be a left l.c. ring, τ_* its Leptin topology, ${}_R K$ the minimal injective cogenerator of $\mathcal{C}_{\tau_*}$. Then ${}_R K$ and ${}_R W$ are isomorphic in* R-Mod.

Proof: Since τ and τ_* are equivalent topologies, $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau_*}$ have the same left closed maximal ideals. Since such ideals are open, $(V_\gamma)_{\gamma \in \Gamma}$ is a system of representatives of non isomorphic simple modules in $\mathcal{C}_{\tau_*}$. Put $E = E\left(\bigoplus_{\gamma \in \Gamma} V_\gamma\right)$ in R-Mod. Then:

$${}_R K = t_{\tau_*}(E) < t_\tau(E) = {}_R W\,.$$

On the other hand, by Lemma 2.1, the W-topology of R coincides with τ_*, so that ${}_R W$ is a τ_*-torsion module. Thus $t_\tau(E) < t_{\tau_*}(E)$ and ${}_R K = {}_R W$.

An important consequence of Lemma 7.2 is that (R, τ) and (R, τ_*) have the same cobasic ring and the same grade.

Our purpose is to show that (R, τ_*) is uniquely determined, up to topological isomorphisms, by its cobasic ring and its grade.

7.3. LEMMA: *Let* (A, σ) *be a right l.c. ring such that* $A/J(A) \cong \prod_{\gamma \in \Gamma} D_\gamma$, *where the* D_γ*'s are division rings. Then the* D_γ*'s—as simple right* A*-modules—are a system of representatives of non isomorphic simple modules in* $\mathcal{C}_\sigma$.

PROOF: It is well known that $J(A)$ is the intersection of all right maximal open ideals of (A, σ), so that $J(A)$ is closed in (A, σ). Let $\bar{\sigma}$ be the quotient topology of $A/J(A)$. $(A/J(A), \bar{\sigma})$ is l.c., the simple modules belonging to $\mathcal{C}_\sigma$ are in a natural way simple $A/J(A)$-modules belonging to $\mathcal{C}_{\bar{\sigma}}$, and vice-versa. Thus we can suppose $A = \prod_{\gamma \in \Gamma} D_\gamma$. Since A is then semiprimitive and by 3.1 *a*), σ coincides with the product topology of the finite topologies of D_γ's. For a fixed $\gamma \in \Gamma$, it is clear that $\mathrm{Ann}_A(D_\gamma) \cong \prod_{\gamma' \neq \gamma} D_{\gamma'}$, so that the simple modules D_γ are pair wise not isomorphic, since they have different annihilators.

By the knowledge of the right maximal ideals of a cartesian product of division rings, we see that every maximal open (or closed) right ideal of (A, σ) is of the form $\prod_{\gamma' \neq \gamma} D_{\gamma'}$, since the other maximal right ideals of A contain $\bigoplus_{\gamma \in \Gamma} D_\gamma$ and therefore are dense.

7.4. LEMMA: *Let* $M \in R$-Mod *be a semisimple module and suppose that* $\mathrm{End}_R(M)$ *is the cartesian product of the division rings* D_γ, $\gamma \in \Gamma$. *Then* M *is a direct sum of non isomorphic simple modules.*

PROOF: Fix a decomposition of M as a direct sum of simple modules and let $M_1 \neq M_2$ be two direct simple summands of such a decomposition. Then $M_1 \not\cong M_2$: otherwise, putting $D = \mathrm{End}_R(M_1)$, $\mathrm{End}_R(M_1 \oplus M_2)$ is the ring of 2×2 matrices over D. This implies that $\mathrm{End}_R(M)$ contains non zero nilpotent elements, which is absurd.

7.5. THEOREM: *Let* (A, σ) *be a right l.c. ring, such that* $\sigma = \sigma_*$ *and* $A/J(A) = \prod D_\gamma$, *where the* D_γ*'s are division rings. Let* $\nu_\Gamma = (\nu_\Gamma)_{\gamma \in \Gamma}$ *be a* Γ*-tuple of cardinal numbers* ≥ 1. *Then there exists a left l.c. ring* (R, τ), *such that* $\tau = \tau_*$, *having grade* ν_Γ *and cobasic ring* (A, σ). *Such an* (R, τ) *is unique, up to topological isomorphisms.*

PROOF: Consider the right A-module

$$W_A = E_\sigma(\bigoplus D_\gamma^{(\nu_\gamma)}) .$$

W_A is an injective object in $\mathcal{C}_\sigma$ and, by Lemma 7.3, it contains a copy of every simple σ-torsion module. Thus W_A is an injective cogenerator, with essential socle, of $\mathcal{C}_\sigma$. By Lemma 2.1, $\sigma = \sigma_*$ coincides with the W-topology of A. Put

$$R = \mathrm{End}\,(W_A)$$

and let τ be the W-topology of R.

By Corollary 2.12 and Lemma 1.2 we have

1) (R, τ) is left l.c.

2) ${}_R W_A$ is faithfully balanced and ${}_R W$ is an injective cogenerator of $\mathfrak{C}_\tau$.

3) $\mathrm{Soc}({}_R W) = \mathrm{Soc}(W_A)$ and $\mathrm{Soc}({}_R W)$ is essential in ${}_R W$.

4) $\tau = \tau_*$.

By 2), ${}_R W$ has a submodule ${}_R K$ which is isomorphic to the minimal injective cogenerator of $\mathfrak{C}_\tau$. Let us prove that ${}_R K = {}_R W$. By Proposition 2.9

$$\mathrm{End}_R(\mathrm{Soc}({}_R W)) \simeq A/J(A) = \prod_{\gamma \in \Gamma} D_\gamma .$$

Thus, by Lemma 7.4, $\mathrm{Soc}({}_R W)$ is a direct sum of not isomorphic simple modules. Since $\mathrm{Soc}({}_R K) = {}_R K \cap \mathrm{Soc}({}_R W)$, it follows $\mathrm{Soc}({}_R K) = \mathrm{Soc}({}_R W)$.

By 3) we have now the essential inclusions

$$\mathrm{Soc}({}_R W) < {}_R K < {}_R W .$$

Thus, since ${}_R K$ is a direct summand of ${}_R W$, ${}_R K = {}_R W$.

Now ${}_R W = E_\tau\left(\bigoplus_{\lambda \in \Lambda} V_\lambda\right)$, where $(V_\lambda)_{\lambda \in \Lambda}$ is a system of representatives of not isomorphic simple modules belonging to $\mathfrak{C}_\tau$. Putting $S_\lambda = \mathrm{End}(V_\lambda)$, we have $V_\lambda = S_\lambda^{(\mu_\lambda)}$ where the μ_λ's are suitable cardinal numbers, and the S_λ's are not isomorphic simple right A-modules. Therefore

$$\mathrm{Soc}({}_R W) = \mathrm{Soc}(W_A) = \bigoplus_{\lambda \in \Lambda} S_\lambda^{(\mu_\lambda)} = \bigoplus_{\gamma \in \Gamma} D_\gamma^{(\nu_\gamma)}$$

where $S_\gamma^{(\mu_\gamma)}$ and $D_\lambda^{(\nu_\lambda)}$ are isotypical components.

Then, up to a bijection, and for every $\gamma \in \Gamma$,

$$S_\gamma^{(\mu_\gamma)} = D_\gamma^{(\nu_\gamma)} = V_\gamma .$$

It follows $S_\gamma = D_\gamma$, $\mu_\gamma = \nu_\gamma$ for every $\gamma \in \Gamma$. It is now proved that (R, τ) is a left l.c. ring, such that $\tau = \tau_*$, having grade $\nu_\Gamma = (\nu_\gamma)_{\gamma \in \Gamma}$ and cobasic ring (A, σ).

Finally, let (R, τ), (R', τ') be left l.c. rings such that $\tau = \tau_*$ and $\tau' = \tau'_*$. Suppose that they have the same grade and that the respective cobasic rings (A, σ), (A', σ') are topologically isomorphic. Let ${}_R W$, ${}_{R'} W$ be the minimal injective cogenerators of $\mathfrak{C}_\tau$ and $\mathfrak{C}_{\tau'}$ respectively. Then

$$W_A = E_\tau\Big(\bigoplus_{\gamma \in \Gamma} D_\gamma^{(\nu_\gamma)}\Big) \quad \text{and} \quad A/J(A) = \prod_{\gamma \in \Gamma} D_\gamma ,$$

$$W_{A'} = E_{\tau'}\Big(\bigoplus_{\gamma \in \Gamma} D_\gamma'^{(\nu_\gamma)}\Big) \quad \text{and} \quad A'/J(A') = \prod_{\gamma \in \Gamma} D'_\gamma .$$

Since (A, σ) is topologically isomorphic to (A', σ'), $D_\gamma \simeq D'_\gamma$ for every $\gamma \in \Gamma$. Let $\varphi: A \to A'$ be a topological isomorphism. Then φ induces a semilinear isomorphism of W_A onto $W_{A'}$. By the topological isomorphisms

$$(R, \tau) \simeq \operatorname{End}(W_A), \qquad (R', \tau') \simeq \operatorname{End}(W_{A'}),$$

where the endomorphism rings have their finite topologies, we obtain that (R, τ) and (R', τ') are topologically isomorphic.

7.6. Let (R, τ) be a left l.c. ring. Denote by $K_A = E_\sigma\left(\bigoplus_{\gamma \in \Gamma} D_\gamma\right)$ the minimal injective cogenerator of $\mathfrak{C}_\sigma$ and by $H_A = \bigoplus_{\gamma \in \Gamma} E_\sigma(D_\gamma)$ the minimal cogenerator of $\mathfrak{C}_\sigma$. Since (A, σ) is l.c. and $\sigma = \sigma_*$, σ coincides with the K_A-topology and with the H_A-topology (cf. Lemmata 2.1 and 2.2).

Since K_A is a direct summand of W_A, there exists in R an idempotent $\varepsilon: W_A \to K_A$ such that $\varepsilon(W_A) = K_A$, $\varepsilon|_{K_A} = 1_{K_A}$. Putting $T = \varepsilon R \varepsilon$, we have $T = \operatorname{End}(K_A)$. Let β be the K-topology of T. Then the bimodule ${}_T K_A$ is faithfully balanced, K_A and ${}_T K$ are both s.q.i., (T, β) is left l.c. and $\beta = \beta_*$.

(T, β) will be called the *basic ring* of (R, τ).

(T, β) is uniquely determined, up to topological isomorphisms, by (R, τ) since (T, β) is the cobasic ring of (A, σ). We have:

$$T/J(T) \simeq \prod_{\gamma \in \Gamma} D_\gamma \simeq A/J(A).$$

7.7. Proposition: *The topology β of $T = \varepsilon R \varepsilon$ coincides with the relative topology of τ_*.*

Proof: Let us prove that, for every $x \in {}_R W$,

$$\operatorname{Ann}_R(x) \cap T = \operatorname{Ann}_T(\varepsilon x),$$

from which the conclusion will follow.

$$r \in \operatorname{Ann}_R(x) \cap T \Rightarrow r = \varepsilon s \varepsilon \quad \text{with } s \in R \text{ and } rx = 0 \Rightarrow \varepsilon s \varepsilon x = 0 \Rightarrow$$
$$\Rightarrow \varepsilon s \varepsilon(\varepsilon x) = 0 \Rightarrow r \in \operatorname{Ann}_T(\varepsilon x).$$

Conversely let $t \in T$, $t = \varepsilon r \varepsilon$ with $r \in R$. Then:

$$t \in \operatorname{Ann}_T(\varepsilon x) \Rightarrow t\varepsilon x = 0 \Rightarrow (\varepsilon r \varepsilon)(\varepsilon x) = 0 \Rightarrow tx = 0.$$

7.8. Let $\gamma \in \Gamma$ and $e_\gamma: H_A \to H_A$ be the projection of H_A onto $E_\sigma(D_\gamma)$, whose kernel is $\bigoplus_{\gamma' \neq \gamma} E_\sigma(D_{\gamma'})$. Clearly $e_\gamma | E_\sigma(D_\gamma)$ is the identity on $E_\sigma(D_\gamma)$.

Looking at e_γ as a morphism of H_A onto $E_\sigma(D_\gamma)$, e_γ extends to an endomorphism $\bar{e}_\gamma$ of K_A, since K_A is q.i.

Consider the diagram:

$$W_A \xrightarrow{\varepsilon} K_A \xrightarrow{\bar{\varepsilon}_\gamma} K_A .$$

Then $\bar{\varepsilon}_\gamma \circ \varepsilon \in T$. Put $e_\gamma = \bar{\varepsilon}_\gamma \circ \varepsilon$.

7.9. Proposition: *The following conditions are equivalent:*

(*a*) *The family* $(e_\gamma)_{\gamma \in \Gamma}$ *is summable in* T (see Definition 8.3) *and* $\sum_{\gamma \in \Gamma} e_\gamma = \varepsilon$.

(*b*) $K_A = H_A$.

If (R, τ) *is s.l.c. the above conditions are fulfilled.*

Proof: (*a*) $\Rightarrow$ (*b*). Let $x \in K_A$, $x \neq 0$, $I = \mathrm{Ann}_T(x)$. I is open in (T, β). Then there exists a finite subset F of Γ such that $\sum_{\gamma \in F} e_\gamma - \varepsilon \in I$. This means:

$$x = \varepsilon x = \sum_{\gamma \in F} e_\gamma(x) .$$

Since $e_\gamma(x) \in H_A$, $x \in H_A$.

(*b*) $\Rightarrow$ (*a*). Let I be an open left ideal of (T, β). We can suppose that there exists a finite subset F of Γ such that

$$I = \bigcap_{\gamma \in F} \mathrm{Ann}_T(x_\gamma), \qquad x_\gamma \in E_A(D_\gamma) .$$

Let us prove that for every finite subset F' of Γ, $F' \supseteq F$, it is $\sum_{\gamma \in F'} e_\gamma - \varepsilon \in I$, from which the conclusion will follow.

Given $\bar{\gamma} \in F' \setminus F$, $e_{\bar{\gamma}}(x_\gamma) = 0$ by definition of e_γ. Then for every $\mu \in F$:

$$\Big(\sum_{\gamma \in F'} e_\gamma - \varepsilon\Big)(x_\mu) = e_\mu(x_\mu) - \varepsilon(x_\mu) = x_\mu - x_\mu = 0 .$$

It follows $\sum_{\gamma \in F'} e_\gamma - \varepsilon \in I$.

Finally, if (R, τ) is s.l.c., then (A, σ) is topologically noetherian, so that $K_A = H_A$.

7.10. Proposition: *Let* (R, τ) *be a left l.c. ring such that* $\tau = \tau_*$. *Then* (R, τ) *is s.l.c. iff its basic ring* (T, β) *is s.l.c.*

Proof: By Theorem 5.15, (R, τ) is s.l.c. iff (A, σ) is topologically noetherian. Since the bimodule ${}_T K_A$ is faithfully balanced and the topologies β and σ are the K-topologies, it follows by Lemma 2,13 that (A, σ) is topologically noetherian iff (T, β) is s.l.c.

7.11. Remark: The ring $\bar{R}$ considered in Section 6 coincides with its basic ring. The cobasic ring of $\bar{R}$ is A (6.8).

8. - Linearly compact primary rings

8.1. In this section we apply the preceding results, in particular the representation theorem and the notion of basic ring, to the study of l.c. primary rings.

A left l.c. ring (R, τ) is said to be primary if $R/J(R)$ is topologically isomorphic to an endomorphism ring $\mathrm{End}_D(V)$ where D is a division ring and V is a right vector space over D. Denote by ν the dimension of V over D.

In this case, comparing with our usual notations, Γ has a single element γ and $V_\gamma = V$, $D_\gamma = D$, $\nu_\gamma = \nu$.

The main results on l.c. primary rings are due to Leptin [8]:

a) Every left s.l.c. primary ring (R, τ) is topologically isomorphic to the ring of column-summable $\nu \times \nu$ matrices over a local s.l.c. ring.

b) The same result is valid when (R, τ) is a l.c. primary ring of finite grade.

Leptin proved *b*) requiring additionally $J(R)$ nilpotent, but recently Ánh [1] established *b*) without any restriction on $J(R)$. Both used lifting of idempotents.

Concerning *a*) Leptin asked in [8] whether every l.c. primary ring of infinite grade is necessarily s.l.c. (a l.c. matrix ring of infinite size is necessarily s.l.c. according to [8]). Ánh answered negatively displaying a counterexample in [1].

The results from Section 2 and Section 7 enable us to give a short proof of *a*) and *b*) and to give a necessary and sufficient condition for a l.c. primary ring to be s.l.c.

8.2. Let (R, τ) be a left l.c. primary ring. Then ${}_R W = E_\tau(V)$, hence the basic ring (A, σ) is a right local l.c. ring, since ${}_R W$ is an indecomposable injective object of $\mathfrak{C}_\tau$. Moreover A is primary, since $A/J(A) = D$. By Proposition 7.10 (R, τ) is s.l.c. iff (A, σ) is topologically noetherian.

Remark that the ring considered in 6.8 is l.c., primary and topologically noetherian, but not s.l.c. The example of a l.c. primary ring, which is not s.l.c., given in [1] is not topologically noetherian.

The basic ring (T, β) of (R, τ) is a left local (hence primary) l.c. ring. It is s.l.c. iff (R, τ) is s.l.c.

In general $W_A = E_\beta(D^{(\nu)})$, $W_A = E_\beta(D)^{(\nu)}$ if (R, τ) is s.l.c. or ν is finite.

Now we recall some facts about the endomorphism ring of a direct sum of copies of a module.

8.3. Let T be a left l.t. ring which is complete and Hausdorff and let Λ be a non empty set.

A family $(x_\lambda)_{\lambda \in \Lambda}$ of elements of T is said to be *summable* if for every left open ideal I of T there exists a finite subset F of Λ such that for every

$\lambda \in \Lambda \setminus F$, $x_\lambda \in I$. Setting $x_F = \sum_{\lambda \in F} x_\lambda$, the above family is summable iff the net (x_F) is convergent in T, when F runs over the set of finite subsets of Λ. The limit of (x_F) will be called the sum of the family $(x_\lambda)_{\lambda \in \Lambda}$.

Let $(x_\lambda)_{\lambda \in \Lambda}$ be a summable family of elements of T. If $(y_\lambda)_{\lambda \in \Lambda}$ is an arbitrary family of elements of T and if $t \in T$, then the families $(y_\lambda x_\lambda)_{\lambda \in \Lambda}$ and $(x_\lambda t)_{\lambda \in \Lambda}$ are summable.

Denotes by T_Λ the ring of column-summable $\Lambda \times \Lambda$ matrices over T. In T_Λ we consider the element-wise addition and the row-by-column multiplication. This product is well defined.

In what follows every ring of the form T_Λ will be endowed with the topology defined as follows.

Let F be a finite subset of Λ and let I be a left open ideal of T. Denote by $W(F; I)$ the left ideal of T_Λ consisting of all matrices whose μ-th column belongs to I^Λ for every $\mu \in F$. Denote by β the ring topology on T_Λ which has a basis of neighbourhoods of 0 the family $W(F; I)$.

8.4. Let B be a ring, $K_B \in$ Mod-B and let $(K_\lambda)_{\lambda \in \Lambda}$ be a family of isomorphic to K_B objects of Mod-B. Set $M = \bigoplus_{\lambda \in \Lambda} K_\lambda$ and identify each K_λ with a submodule of M using the canonical injections.

Fix an index $\lambda \in \Lambda$ and set for convenience $\lambda = 1$ and $K = K_1$. For every $\lambda \in \Lambda$ fix an isomorphism $i_\lambda \colon K \to K_\lambda$. For every $\lambda, \mu \in \Lambda$ consider the isomorphism $i_{\lambda\mu} \colon K_\mu \to K_\lambda$ given by $i_{\lambda\mu} = i_\lambda \circ i_\mu^{-1}$. Then $i_{\lambda\lambda} = 1_{K_\lambda}$ and $i_{\lambda\mu} \circ i_{\mu\nu} = i_{\lambda\nu}$ for $\lambda, \mu, \nu \in \Lambda$. For every $\lambda, \mu \in \Lambda$ let $\varepsilon_{\lambda\mu}$ be the endomorphism of M defined by:

$$\varepsilon_{\lambda\mu}|_{K_\mu} = i_{\lambda\mu}\,; \qquad \varepsilon_{\lambda\mu}|_{K_\nu} = 0 \qquad \text{if } \nu \neq \mu\,.$$

Set $\varepsilon_\lambda = \varepsilon_{\lambda\lambda}$. Let $R = \mathrm{End}_B(M_B)$ and endow R with the M-topology τ. Then (R, τ) is complete and Hausdorff. Moreover: *the family* $(\varepsilon_\lambda)_{\lambda \in \Lambda}$ *is summable in* (R, τ) *and* $\sum_\lambda \varepsilon_\lambda = 1$.

Set $T = \mathrm{End}_B(K_B)$ and $T_\lambda = \mathrm{End}_B(K_\lambda)$. Consider T and T_λ as subrings of T in the obvious way. Then $T_\lambda = \varepsilon_\lambda R \varepsilon_\lambda = \varepsilon_{\lambda\mu} R \varepsilon_{\mu\lambda} = \varepsilon_{\lambda 1} R \varepsilon_{1\lambda}$ for every $\lambda, \mu \in \Lambda$. For every $\lambda \in \Lambda$ there is an isomorphism $f \mapsto \varepsilon_{\lambda 1} \circ f \circ \varepsilon_{1\lambda}$ $(f \in T)$.

8.5. The relative topology of $T_\lambda = \varepsilon_\lambda R \varepsilon_\lambda$ coincides with the K_λ-topology (see [3], 4.3). Consider the application $\chi \colon R \to T_\Lambda$ given by

$$\chi(f) = (\varepsilon_{1\lambda} \circ f \circ \varepsilon_{\mu 1})_{\lambda,\mu \in \Lambda} \qquad (f \in R)\,.$$

8.6. Theorem ([3], Theorem 1.4): *The map* χ *is a topological isomorphism of* (R, τ) *onto* (T_Λ, β).

8.7. Theorem (Leptin [8]): *Let* (R, τ) *be a primary left l.c. ring with finite grade* n. *Then* (R, τ) *is topologically isomorphic to* T_n, *where* T *is the basic ring of* R *endowed with the relative topology.*

Proof: $W_A = E_\sigma(D^n) = E_\sigma(D)^n$ since n is finite. By Theorem 2.14 (R, τ_*) is topologically isomorphic to the ring End (W_A) endowed with the finite topology. By Theorem 8.6 and Proposition 7.7, End (W_A) is topologically isomorphic to T_n, where T has the relative topology of τ_*. On the other hand it was proved in [4] that if T is a subring of a topological ring (R, τ) such that $R \cong T_n$ algebraically, then $(R, \tau) \cong T_n$ topologically where T is endowed with the relative topology.

8.8. Remark: Let (T, β) be a left l.t. ring and A a finite set. It is easily checked that T is l.c. iff T_A is l.c. It was proved in [8] that, if A is infinite, then T_A is l.c. iff T_A is s.l.c. and in this case T is s.l.c.

8.9. Theorem (Leptin [8]): *Let (R, τ) be a primary left l.c. ring of infinite grade ν. Then the following conditions are equivalent:*

(a) (R, τ) is s.l.c.

(b) (R, τ) is topologically isomorphic to the ring of column-summable $\nu \times \nu$ matrices, over its basic ring T.

Proof: $(a) \Rightarrow (b)$. By Theorem 2.14, $(R, \tau) \cong$ End (W_A) and $W_A = E_\sigma(D)^{(\nu)}$. By Theorem 8.6, End $(W_A) \cong T_\nu$.

$(b) \Rightarrow (a)$. Follows by Remark 8.8.

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