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Dualities Over Compact Commutative Rings (**)

Dualità su anelli compatti e commutativi

1. - Introduction

The Pontryagin duality functor χ was characterized by Roeder [7] among the contravariant functors φ from the category $\mathfrak{L}$ of the locally compact Abelian groups into itself. In fact Roeder gave three such characterizations and one of them shows that if there exists a contravariant functor $\psi: \mathfrak{L} \to \mathfrak{L}$ so that $\varphi \circ \psi$ and $\psi \circ \varphi$ are naturally equivalent to $id_{\mathfrak{L}}$, then φ is naturally equivalent to χ. Dualities between categories of locally compact modules over a given discrete ring were studied by Prodanov [6].

In this paper we study dualities of a certain type on the category $\mathfrak{L}_S$ of the locally compact modules over a given compact commutative ring S with identity. More generally, following [6], we consider dualities between full subcategories of $\mathfrak{L}_S$ which satisfy some additional conditions, and prove that each duality of this type is naturally equivalent to the Pontryagin duality. The main tools used below are these developed in [7] and [6] as well as a theorem of Kaplansky [3] concerning the structure of the compact commutative rings. We use also some well known properties of the Pontryagin duality.

Let S be a compact commutative ring with identity. By $\mathfrak{L}_S$ we always denote the category of the locally compact S-modules and continuous S-homomorphisms. The following definitions are modifications of definitions from [6].

1.1. Definition: A subcategory $\mathcal{M}$ of $\mathfrak{L}_S$ is called c-admissible (resp. d-admissible) S-category, if it has the following properties:

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(i) $\mathcal{M}$ is a full subcategory of $\mathfrak{L}_S$, i.e. for each two objects X and Y of $\mathcal{M}$ the $\mathcal{M}$-morphisms $X \to Y$ are exactly the continuous homomorphisms of X in Y,

(ii) if $X \in \mathcal{M}$ and Y is topologically isomorphic to X, then $Y \in \mathcal{M}$,

(iii) if $X \in \mathcal{M}$ and Y is a closed submodule of X, then $Y \in \mathcal{M}$ and $X/Y \in \mathcal{M}$,

(iv) $\mathcal{M}$ contains all compact (resp. discrete) S-modules.

For example the category of the compact S-modules is c-admissible and the category of the discrete S-modules is d-admissible. The category $\mathfrak{L}_S$ as well as the category of all compact and all discrete S-modules are simultaneously c-admissible and d-admissible S-categories.

1.2. Definition: A couple (φ, ψ) of contravariant functors $\varphi: \mathcal{M} \to \mathcal{N}$ and $\psi: \mathcal{N} \to \mathcal{M}$, where $\mathcal{M}$ is a c-admissible S-category and $\mathcal{N}$ is a d-admissible S-category, is called a duality over S between $\mathcal{M}$ and $\mathcal{N}$, if it has the following peoperties:

(i) the compositions $\varphi\circ\psi$ and $\psi\circ\varphi$ are naturally equivalent to $id_{\mathcal{N}}$ and $id_{\mathcal{M}}$, respectively,

(ii) $\varphi(sf) = s\varphi(f)$ for each morphism f in $\mathcal{M}$ and each $s \in S$.

Clearly (i) and (ii) imply

(ii') $\psi(sg) = s\psi(g)$ for each morphism g in $\mathcal{N}$ and each $s \in S$.

We write briefly $\mathcal{M} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}$ to denote that (φ, ψ) is a duality between the S-categories $\mathcal{M}$ and $\mathcal{N}$.

1.3. Example (Pontryagin duality): If X is a locally compact S-module, then each continuous group homomorphism $u: X \to \mathbf{T}$, where $\mathbf{T}$ is the one-dimensional torus, is called a character of X. Let u be a character of X and $s \in S$. Define a character su by $su(x) = u(sx)$ for each $x \in X$. The group $\chi(X)$ of the characters of X, endowed with the compact-open topology, becomes a locally compact S-module under the multiplication $(s, u) \to su$. For each morphism $f: X \to Y$ in $\mathfrak{L}_S$ $\chi(f): \chi(Y) \to \chi(X)$ is defined by $\chi(f)(u) = u\circ f$. It is well known that $\chi: \mathfrak{L}_S \to \mathfrak{L}_S$ is a contravariant functor and the couple (χ, χ) is a duality in the sense of Definition 1.2. More generally, if $\mathcal{M}$ is a c-admissible S-category and $\mathcal{N}$ is the smallest d-admissible S-category which contains $\chi(X)$ for each $X \in \mathcal{M}$, then $\mathcal{M} \underset{\chi}{\overset{\chi}{\rightleftarrows}} \mathcal{N}$ is a duality over S. We call it Pontryagin duality.

The main theorem in the paper is the following.

1.4. Theorem: Let S be a compact commutative ring with identity and $\mathcal{M} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}$ be a duality over S. Then each of the functors φ and ψ is naturally equivalent to the Pontryagin duality functor χ.

The first seven sections of the paper are devoted to the proof of this theorem. Section 2 contains a note on the structure of the locally compact S-modules. Some elementary fact about dualities, taken from [6] and [7], are given in section 3. It is shown in section 4 that each duality over S generates in a natural way dualities over the quotient rings of S and some consequences from this are obtained. Section 5 deals with dualities over finite rings. It is shown in section 6 that for each duality (φ, ψ) over S the module $\varphi(S)$ is topologically isomorphic to $\chi(S)$. The main theorem is proved in section 7.

The last section contains a characterization of χ among the contravariant functors $\varphi: \mathcal{M} \to \mathfrak{L}_S$, where $\mathcal{M}$ is a c-admissible S-category. Our conditions for φ are similar to these in Axiom system I from [7].

All rings considered below are compact commutative with identity $(1 \neq 0)$, all modules are unitary and all topological rings and modules are Hausdorff. By homomorphism we mean an S-homomorphism. Recall that for each topological S-module X the multiplication $(s, x) \to sx$ is jointly continuous. If X and Y are topological S-modules, by $\mathrm{Hom}_S(X, Y)$ we denote the S-module (without topology) of the continuous homomorphisms $X \to Y$. The Pontryagin duality functor is always denoted by χ.

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2. - Remarks on the structure of the locally compact S-modules.

Let S be a compact commutative ring with 1. It is known that each compact S-module X has a local base of neighbourhoods of 0 consisting of open submodules of X. This follows, for example, from the fact that $\chi(X)$ is discrete and then for each $u \in \chi(X)$ the subgroup Su is compact and hence finite. In particular, $\chi(X)$ is a periodic Abelian group which implies (see [4], Theorem 24.21) that X is totally disconnected. It is easily seen now that the open submodules of X form a local base of neighbourhoods of 0.

2.1. Proposition: If S is a compact commutative ring with 1, then every locally compact S-module contains a compact open submodule.

Proof: Let X be a locally compact S-module and F be a compact neighbourhood of 0 in X. Set

$$Y = \bigcup_{n=1}^{\infty} S(\underbrace{F + F + \ldots + F}_{n}).$$

Clearly Y is an open submodule of X. Since Y is a countable union of compact sets, the structure theorem for locally compact Abelian groups (see [4], Theorem 9.14) shows that, as a topological group, Y is topologically isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m \times K$, where $\mathbf{R}$ is the additive group of the reals endowed with the standard topology, $\mathbf{Z}$ is the discrete group of the integers, K is a compact group and m and n are non-negative integers. Each element y of Y is contained in the compact subgroup Sy of Y. Then $m = n = 0$ and Y is compact.

By a theorem of Kaplansky ([3], Theorem 17) S has a representation of the form

$$S = \prod_{\alpha \in A} S_\alpha , \tag{1}$$

where A is a set, S_α is a compact local ring for each $\alpha \in A$ and $\prod_{\alpha \in A} S_\alpha$ is endowed with the product topology.

The following lemma will be used in section 6.

2.2. Lemma: The set K of the invertable elements of S is compact with respect to the relative topology.

Proof: Consider the representation (1) of S and denote by $\mathfrak{M}_\alpha$ the unique maximal ideal of S_α. Since there exist proper open ideals of S_α and each of them is contained in $\mathfrak{M}_\alpha$, then $\mathfrak{M}_\alpha$ is open. Therefore $S_\alpha \setminus \mathfrak{M}_\alpha$ is compact being an union of a finite number compact sets. Clearly $K = \prod_{\alpha \in A} (S_\alpha \setminus \mathfrak{M}_\alpha)$ and then K is also compact.

3. - Elementary properties of the dualities.

In this section we mention some properties of the dualities which are taken from [7] and [6].

Let $\mathcal{M} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}$ be a duality over S. There are canonical mappings

$$\varphi\colon \mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_S(\varphi(Y), \varphi(X))$$

for X and Y from $\mathcal{M}$ and

$$\psi\colon \mathrm{Hom}_S(Z, T) \to \mathrm{Hom}_S(\psi(T), \psi(Z))$$

for Z and T from $\mathcal{N}$, defined by $f \to \varphi(f)$ and $g \to \psi(g)$, respectively. It follows easily from Definition 1.2(i) that these mappings are bijections.

A morphism $f\colon X \to Y$ in $\mathfrak{L}_S$ is called a monomorphism if $\mathrm{Ker}(f) = 0$ and an epimorphism if $\mathrm{Im}(f)$ is dense in Y. The following statements can be obtained from the categorical definitions of a monomorphism and an epimorphism (see [1], ch. 1, § 2).

3.1. Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$ (resp. $\mathcal{N}$). Then

(a) f is a monomorphism iff $\varphi(f)$ (resp. $\psi(f)$) is an epimorphism;

(b) f is an epimorphism iff $\varphi(f)$ (resp. $\psi(f)$) is a monomorphism.

By an embedding we mean a homomorphism $f\colon X \to Y$ so that f is a homeomorphism between X and $f(X)$. If $f(X) = Y$, and f is an open map, we call f a quotient map. A sequence

$$(2) \qquad 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of morphism in $\mathfrak{L}_\mathcal{F}$ is called proper exact if $f(X) = \mathrm{Kern}\,(g)$, f is an embedding and g is a quotient map. It is easy to see ([6]) that (2) is proper exact iff $g \circ f = 0$, f is a monomorphism, g is an epimorphism and for any morphisms $h\colon A \to Y$ and $l\colon Y \to B$ with $l \circ f = 0$ and $g \circ h = 0$ there exist morphisms m and n so that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "m"'] & Y \arrow[r, "g"] \arrow[dr, "l"'] & Z \\ & A \arrow[u, "h"'] & B \arrow[u, "n"'] \end{tikzcd}$$

is commutative. A straightforward consequence from this is the following.

3.2. Let (2) be a sequence of morphisms in $\mathcal{M}$ (resp. $\mathcal{N}$). The (2) is proper exact iff the sequence

$$0 \leftarrow \varphi(X) \xleftarrow{\varphi(f)} \varphi(Y) \xleftarrow{\varphi(g)} \varphi(Z) \leftarrow 0$$

$$(\text{resp. } 0 \leftarrow \psi(X) \xleftarrow{\psi(f)} \psi(Y) \xleftarrow{\psi(g)} \psi(Z) \leftarrow 0)$$

is proper exact.

3.3. Let f be a morphism in $\mathcal{M}$ (resp. $\mathcal{N}$). Then

(a) f is an embedding iff $\varphi(f)$ (resp. $\psi(f)$) is a quotient map;

(b) f is a quotient map iff $\varphi(f)$ (resp. $\psi(f)$) is an embedding.

Clearly 3.3 follows immediately from 3.2. The proof of the following property is the same as the proof of Lemma 3.1 in [7].

3.4. If $X \times Y \in \mathcal{M}$ (resp. $\mathcal{N}$), then $\varphi(X \times Y)$ (resp. $\psi(X \times Y)$) is topologically isomorphic to $\varphi(X) \times \varphi(Y)$ (resp. $\psi(X) \times \psi(Y)$).

Finally we need the following statement whose proof is a part of the proof of Lemma 3.2 in [7].

3.5. If X and Y are compact $\mathcal{S}$-modules, then $\varphi(f+g) = \varphi(f) + \varphi(g)$, for any morphisms $f, g\colon X \to Y$ in $\mathcal{M}$.

3.6. Corollary: φ and ψ are additive functors.

Proof: It suffices to verify that ψ is additive. Suppose $f, g\colon X \to Y$ are morphisms in $\mathcal{N}$. Then $\psi(f)$, $\psi(g)$ and $\psi(f+g)$ are homomorphisms of $\psi(Y)$

in $\psi(X)$. Take an element $u \in \psi(Y)$ and set $B = Su$ and

$$A = S\psi(f)(u) + S\psi(g)(u) + S\psi(f+g)(u)\,.$$

Denote by f_1, g_1 and h the restrictions on B of $\psi(f)$, $\psi(g)$ and $\psi(f+g)$, respectively and let $i\colon A \to \psi(X)$ and $j\colon B \to \psi(Y)$ be the identical embeddings. Then the diagrams

$$\begin{tikzcd} \psi(X) & \psi(Y) \arrow[l, "\psi(f)"'] \\ A \arrow[u, "i"] & B \arrow[l, "f_1"'] \arrow[u, "j"'] \end{tikzcd} \qquad \begin{tikzcd} \psi(X) & \psi(Y) \arrow[l, "\psi(g)"'] \\ A \arrow[u, "i"] & B \arrow[l, "g_1"'] \arrow[u, "j"'] \end{tikzcd} \qquad \begin{tikzcd} \psi(X) & \psi(Y) \arrow[l, "\psi(f+g)"'] \\ A \arrow[u, "i"] & B \arrow[l, "h"'] \arrow[u, "j"'] \end{tikzcd}$$

are commutative and so the diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_X"'] & Y \arrow[d, "e_Y"] \\ \varphi\psi(X) \arrow[r, "\varphi\psi(f)"] \arrow[d, "\varphi(i)"'] & \varphi\psi(Y) \arrow[d, "\varphi(j)"] \\ \varphi(A) \arrow[r, "\varphi(f_1)"] & \varphi(B) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "e_X"'] & Y \arrow[d, "e_Y"] \\ \varphi\psi(X) \arrow[r, "\varphi\psi(g)"] \arrow[d, "\varphi(i)"'] & \varphi\psi(Y) \arrow[d, "\varphi(j)"] \\ \varphi(A) \arrow[r, "\varphi(g_1)"] & \varphi(B) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f+g"] \arrow[d, "e_X"'] & Y \arrow[d, "e_Y"] \\ \varphi\psi(X) \arrow[r, "\varphi\psi(f+g)"] \arrow[d, "\varphi(i)"'] & \varphi\psi(Y) \arrow[d, "\varphi(j)"] \\ \varphi(A) \arrow[r, "\varphi(h)"] & \varphi(B) \end{tikzcd}$$

where e_X and e_Y are the natural isomorphisms, are also commutative. Then $\varphi(f_1) + \varphi(g_1) = \varphi(h)$.

On the other hand, A and B are compact and 3.5 shows that $\varphi(f_1) + \varphi(g_1) = \varphi(f_1 + g_1)$. Hence $\varphi(f_1 + g_1) = \varphi(h)$ and then $f_1 + g_1 = h$. In particular,

$$\psi(f+g)(u) = h(u) = f_1(u) + g_1(u) = \psi(f)(u) + \psi(g)(u) = (\psi(f) + \psi(g))(u)\,.$$

Since u is an arbitrary element of $\psi(Y)$, we get $\psi(f+g) = \psi(f) + \psi(g)$.

4. - Quotients.

The observations of this section as well as those in the previous one, under some additional assumption for the categories $\mathcal{M}$ and $\mathcal{N}$, preserve their validity for arbitrary locally compact commutative rings S with identity. We restrict our considerations for compact rings S.

Let I be a closed ideal of S. Consider the quotient ring $R = S/I$ endowed with the quotient topology.

If X is a locally compact S-module with $I \cdot X = 0$, then X possesses a natural structure of an R-module—the multiplication is defined by $(s+I)x = sx$

Conversely, for each locally compact R-module Y the canonical mappings

$$S \times Y \to R \times Y \to Y$$

define on Y a structure of a locally compact S-module. Roughly speaking, the category $\mathfrak{L}_R$ of the locally compact R-modules can be identified with the full subcategory of $\mathfrak{L}_S$ consisting of those $X \in \mathfrak{L}_S$ for which $I \cdot X = 0$. Clearly for each X and Y from $\mathfrak{L}_R$ a map $f\colon X \to Y$ is a morphism in $\mathfrak{L}_R$ iff it is a morphism in $\mathfrak{L}_S$.

Suppose $\mathcal{M} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}$ is a duality over S. Setting $\mathcal{M}_R = \mathcal{M} \cap \mathfrak{L}_R$ and $\mathcal{N}_R = \mathcal{N} \cap \mathfrak{L}_R$, we obtain a c-admissible R-category and a d-admissible R-category, respectively.

4.1. Lemma: If $X \in \mathcal{M}_R$ (resp. $Y \in \mathcal{N}_R$), then $\varphi(X) \in \mathcal{N}_R$ (resp. $\psi(Y) \in \mathcal{M}_R$). The restrictions of φ on $\mathcal{M}_R$ and of ψ on $\mathcal{N}_R$ define a duality over R.

Proof: Assume $X \in \mathcal{M}_R$, i.e. $I \cdot X = 0$. Taking an element i from I we have $i \cdot id_X = 0 \cdot id_X$ and then $i \cdot id_{\varphi(X)} = 0 \cdot id_{\varphi(X)} = 0$. Hence $I \cdot \varphi(X) = 0$ which means that $\varphi(X) \in \mathcal{N}_R$.

For $Y \in \mathcal{N}_R$ the same reasoning show that $\psi(Y) \in \mathcal{M}_R$. The proof of the second part is straightforward.

We use the notation $\mathcal{M}_R \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}_R$ for the duality between $\mathcal{M}_R$ and $\mathcal{N}_R$ generated by $\mathcal{M} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}$. Below we consider some applications of Lemma 4.1 which will be useful later.

4.2. Lemma: If X is a finite S-module, then $\varphi(X)$ and $\psi(X)$ are also finite.

Proof: Let I be an open ideal of S and $R = S/I$. Then R is finite and $\varphi(R) \in \mathcal{N}_R$. For $x \in \varphi(R)$ there exists an ideal J_x of R so that $R/J_x = Rx$. Letting

$$J = \bigcap \{J_x / x \in \varphi(R)\}$$

we have $J \cdot \varphi(R) = 0$, and by Lemma 4.1, $J \cdot \psi \circ \varphi(R) = 0$, i.e. $J \cdot R = 0$ and $J = 0$. Since R is finite, there exists a finite number of elements $x_1, x_2, \ldots, x_n$ of R so that $\bigcap\limits_{i=1}^{n} J_{x_i} = 0$. Then R can be embedded in $\prod\limits_{i=1}^{n} (R/J_{x_i})$ which is isomorphic to the submosule $\prod\limits_{i=1}^{n} Rx_i$ of $(\varphi(R))^*$. So there is a monomorphism $R \to (\varphi(R))^*$, and therefore an epimorphism $R^* \to \psi(R)$ in accordance with 3.1 and 3.4. Hence $\psi(R)$ is finite. Similar reasonings prove that $\varphi(R)$ is finite too.

If X is a finite S-module, then X is a homomorphic image of a finite product of the type $\prod\limits_{K} (S/I_K)$ for some open ideals I_K of S. According to 3.1, 3.4 and the assertion proved above we find that $\varphi(X)$ and $\psi(X)$ are finite.

4.3. Let I be a closed ideal of S and $p_I\colon S \to S/I$ be the quotient map. Then $\varphi(p_I)\colon \varphi(S/I) \to \varphi(S)$ and $\chi(p_I)\colon \chi(S/I) \to \chi(S)$ are embeddings. Set

$$X_I = \operatorname{Im}(\varphi(p_I)) \qquad \text{and} \qquad Y_I = \operatorname{Im}(\chi(p_I))\,.$$

Lemma 4.2 shows that these modules are finite iff I is open in S.

The following two lemmata study some properties of X_I. Similar properties are possessed by the modules Y_I.

4.4. Lemma: For each closed submodule X of $\varphi(S)$ there is a unique closed ideal I of S with $X_I = X$.

Proof: Consider the identical embedding $i\colon X \to \varphi(S)$. By 3.3(a) $\psi(i)\colon \psi\varphi(S) \to \psi(X)$ is a quotient map. Denote by e the natural topological isomorphism $\psi\varphi(S) \to S$ and set $I = \operatorname{Kern}(\psi(i)\circ e^{-1})$. Clearly I is a closed ideal of S and there exists a topological isomorphism f so that the diagram

$$\begin{tikzcd}
\psi\varphi(S) \arrow[r, "\psi(i)"] \arrow[d, "e"'] & \psi(X) \\
S \arrow[r, "p_I"] & S/I \arrow[u, "f"']
\end{tikzcd}$$

is commutative. Then the diagram

$$\begin{tikzcd}
\varphi(S) & X \arrow[l, "i"'] \\
\varphi\psi\varphi(S) \arrow[u, "e_1"] & \varphi\psi(X) \arrow[l, "\varphi\psi(i)"'] \arrow[u, "e_2"'] \arrow[d, "\varphi(f)"] \\
\varphi(S) \arrow[u, "\varphi(e)"] & \varphi(S/I) \arrow[l, "\varphi(p_I)"']
\end{tikzcd}$$

where e_1 and e_2 are the natural isomorphisms, is also commutative. Hence $X = \operatorname{Im}(i) = e_1\circ\varphi(e)\bigl(\operatorname{Im}(\varphi(p_I))\bigr) = e_1\circ\varphi(e)(X_I)$.

On the other hand, each topological isomorphism $\varphi(S) \to \varphi(S)$ has the form $s\cdot id_{\varphi(S)}$ for some invertible $s \in S$ and therefore $e_1\cdot\varphi(e) = s\cdot id_{\varphi(S)}$ for some invertable $s \in S$. Then $X = s\cdot X_I = X_I$.

Suppose I and J are closed ideals of S with $X_I = X_J$. Since $X_I \cong \varphi(S/I)$ (see 4.3), Lemma 4.1 shows that $I\cdot X_I = 0$. Then $I\cdot X_J = 0$ and, according to Lemma 4.1 again, we get $I\cdot S/J = 0$ and $I \subset J$. Similarly, $J \subset I$ and therefore $I = J$.

4.5. Lemma: Let I and J be closed ideals of S. Then:

(a) $\operatorname{Im}(f) \subset X_I$ for each quotient map $f\colon \varphi(S/I) \to \operatorname{Im}(f) \subset \varphi(S)$;

(*b*) X_I is the unique submodulr of $\varphi(S)$ which is topologically isomorphic to $\varphi(S/I)$;

(*c*) $I\subset J$ iff $X_I\supset X_J$;

(*d*) $X_I\cap X_J=X_{I+J}$ and $X_I+X_J=X_{I\cap J}$.

Proof: (*c*) Suppose $I\subset J$ and consider the canonical epimorphism $p:S/I\to\to S/J$. Since $p_J=p\circ p_I$, we have $\varphi(p_J)=\varphi(p_I)\circ\varphi(p)$ and therefore $\mathrm{Im}\,(\varphi(p_I))\supset\supset\mathrm{Im}\,(\varphi(p_J))$. So $X_J\subset X_I$.

If $X_J\subset X_I$, Lemma 4.1 implies $I\cdot X_J=0$ and then $I\cdot S/J=0$. Thus $I\subset J$.

(*a*) Since $\mathrm{Im}\,(f)$ is a closed submodule of $\varphi(S)$, Lemma 4.4 shows that $\mathrm{Im}\,(f)=X_J$ for some closed ideal J of S. Clearly $I\cdot\mathrm{Im}\,(f)=0$ and then $I\cdot X_J=0$. According to Lemma 4.1 we get $I\subset J$ and (*c*) implies $X_I\supset X_J$. Hence $\mathrm{Im}\,(f)\subset X_I$.

(*b*) Follows immediately from (*a*).

(*d*) By Lemma 4.4, $X_I\cap X_J=X_L$ for some closed ideal L of S. Applying (*c*) we have $I\subset L$ and $J\subset L$. Then $I+J\subset L$. Again by (*c*), $X_I\cap X_J=X_L\subset X_{I+J}$.

On the other hand, $I\subset I+J$ and $J\subset I+J$ imply $X_{I+J}\subset X_I$ and $X_{I+J}\subset X_J$. Hence $X_{I+J}\subset X_I\cap X_J=X_L$.

The proof of the second part of (*d*) is similar.

5. - Dualities over finite rings

Throughout this section R will be a finite commutative ring with identity. Our aim here is to prove that for each duality (φ,ψ) over R, $\varphi(R)\simeq\chi(R)$.

Let us note first that if R is a local ring and $F=R/\mathfrak{M}$, where $\mathfrak{M}$ is the unique maximal ideal of R, then $\varphi(F)$ and $\psi(F)$ are isomorphic to F for each duality (φ,ψ) over R. Indeed, $\varphi(F)$ and $\psi(F)$ are simple modules, because F has no non-trivial quotient modules.

5.1. Lemma: For any local ring R the following are equivalent:

(i) $\varphi(R)\simeq R$ for each duality (φ,ψ) over R;

(ii) there exists a duality (φ,ψ) over R with $\varphi(R)\simeq R$;

(iii) there exists an unique minimal non-zero ideal of R.

Proof: Since Pontryagin duality is a duality, (i) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (iii) set $F=R/\mathfrak{M}$, where $\mathfrak{M}$ is the unique maximal ideal of R. Clearly there are minimal non-zero ideals of R and each of them is isomorphic to F. Assume I and J are such ideals with $I\neq J$. Then $I+J\simeq\simeq F\times F$. Let (φ,ψ) be a duality over R with $\varphi(R)\simeq R$. Since $\varphi(F)\simeq F$, 3.4 and 3.1 show that there is an epimorphism $R \to F \times F$ which is impossible.

It remains to probe (iii) $\Rightarrow$ (i). Suppose I is the unique non-zero minimal ideal of R and (φ, ψ) is a duality over R. To show that there is a monomorphism $R \to \varphi(R)$ assume the contrary. Then for each $x \in \varphi(R)$ there is a non-zero ideal J of R with $J \cdot x = 0$. Since $I \subset J$, we have $I \cdot x = 0$. So $I \cdot \varphi(R) = 0$ and by Lemma 4.1, $I \cdot R = 0$ and $I = 0$, contradiction. That is why there is a monomorphism $R \to \varphi(R)$ and hence an epimorphism $f \colon R \to \psi(R)$. Lemma 4.1 shows that $\operatorname{Kern}(f) = 0$.

5.2. Proposition: Let R be a finite commutative ring with identity. Then $\varphi(R) \cong \chi(R)$ for each duality (φ, ψ) over R.

Proof: Remember that χ is the Pontryagin duality functor.

Suppose first that R is a local ring. In this case we use an induction on $n = \operatorname{card}(R)$. Assume that the statement is true for all local rings with cardinality less than n, $n > 2$. Let $\operatorname{card}(R) = n$ and (φ, ψ) be a duality over R. There are two cases.

Case 1. $\varphi(R) \cong R$. By Lemma 5.1 we have also $\chi(R) \cong R$ and then $\varphi(R) \cong \chi(R)$.

Case 2. $\varphi(R)$ is not isomorphic to R. Lemma 5.1 shows now that there are two different non-zero minimal ideals I and J of R. Cleraly $I \cap J = 0$ and, according to Lemma 4.5(d), we find

$$X_I + X_J = X_{I \cap J} = X_0 = \varphi(R) \tag{3}$$

and $X_I \cap X_J = X_{I+J}$. Similarly,

$$Y_I + Y_J = \chi(R) \tag{4}$$

and $Y_I \cap Y_J = Y_{I+J}$.

By Lemma 5.1, (φ, ψ) defines a duality over R/I and the inductive hypothesis implies that $\varphi(R/I) \cong \chi(R/I)$. The definitions of X_I and Y_I (see 4.3) show that $X_I \cong Y_I$. Similarly, $X_J \cong Y_J$. Taking arbitrary isomorphisms $f \colon X_I \to Y_I$ and $g \colon X_J \to Y_J$ we get, according to Lemma 4.5(b),

$$f(X_{I+J}) = g(X_{I+J}) = Y_{I+J}.$$

Since every automorphism of $R/(I+J)$ is a multiplication by an invertable element of R, there is an invertable $r \in R$ with $f(u) = rg(u)$ for $u \in X_{I+J}$. Set $h = r \cdot g$, then $h \colon X_J \to Y_J$ is an isomorphism and $f(u) = h(u)$ for any $u \in X_{I+J}$. Now (3) and (4) show that it is possible to define an isomorphism $\omega \colon \varphi(R) \to \chi(R)$ which coincides with f on X_I and with h on X_J. Hence $\varphi(R) \cong \chi(R)$.

To prove the general case note first that $R = \prod_i R_i$, where R_i is a finite set of local rings. By Lemma 4.1, (φ, ψ) induces a duality over R_i for each j and the above statement implies $\varphi(R_i) \cong \chi(R_i)$. Using 3.4 we find $\varphi(R) \cong \chi(R)$.

6. - The structure of $\varphi(S)$

Let S be a compact commutative ring with identity and suppose $\mathcal{M} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}$ is a duality over S. We are going to prove that $\varphi(S)$ is topologically isomorphic to $\chi(S)$.

Remember that a direct system in $\mathfrak{L}_S$ is a set $\{X_\alpha, f_{\alpha\beta}\}_A$, where A is a directed set, $X_\alpha \in \mathfrak{L}_S$ for $\alpha \in A$ and if $\alpha < \beta$, then $f_{\alpha\beta}\colon X_\alpha \to X_\beta$ is a continuous homomorphism and for each $\alpha < \beta < \gamma$ $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$. The inductive limit $\varinjlim \{X_\alpha, f_{\alpha\beta}\}_A$ is also an element of $\mathfrak{L}_S$. Note that it is discrete if X_α is discrete for each α. If $f_{\alpha\beta}\colon X_\beta \to X_\alpha$ are continuous homomorphisms for $\alpha < \beta$ and $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ whenever $\alpha < \beta < \gamma$, then $\{X_\alpha, f_{\alpha\beta}\}_A$ is called an inverse system. Its projective limit is denoted by $\varprojlim \{X_\alpha, f_{\alpha\beta}\}_A$. If X_α is compact for each α, then $\varprojlim \{X_\alpha, f_{\alpha\beta}\}_A$ is also compact. All facts concerning inductive and projective limits which are used below can be found in [2].

6.1. Lemma: (*a*) Let $\{F_\alpha, f_{\alpha\beta}\}$ be an inverse system in $\mathcal{M}$ so that F_α is finite for each α and $F = \varprojlim \{F_\alpha, f_{\alpha\beta}\}_A$. Then $\varphi(F)$ is topologically isomorphic to $\varinjlim \{\varphi(F_\alpha), \varphi(f_{\alpha\beta})\}_A$ and hence it is discrete.

(*b*) Let $\{E_\alpha, g_{\alpha\beta}\}_A$ be a direct system in $\mathcal{N}$ so that E_α is finite for each α and $E = \varinjlim \{E_\alpha, g_{\alpha\beta}\}_A$. Then $\psi(E)$ is topologically isomorphic to $\varprojlim \{\psi(E_\alpha)$ $\psi(g_{\alpha\beta})\}_A$ and hence it is compact.

The proof is the same as the proof of Lemma 4.3 from [7]. We use also Lemma 4.2 above and the fact that $F \in \mathcal{M}$ as a compact S-module and $E \in \mathcal{N}$ as a discrete S-module.

Below we use again the modules X_I and Y_I which are defined in 4.3.

6.2. Corollary: The module $\varphi(S)$ is discrete and

$$\varphi(S) = \bigcup \{X_I \mid I \in \mathfrak{F}\}, \tag{5}$$

where $\mathfrak{F}$ is the set of all open ideals of S.

Proof: For any two elements I and J from $\mathfrak{F}$ with $I \subset J$ let $p_{I,J}\colon S/I \to S/J$ be the canonical epimorphism. Since $\mathfrak{F}$ is a local base of neighbourhoods of 0 for S (see section 2), S is topologically isomorphic to $\varprojlim \{S/I, p_{I,J}\}_{\mathfrak{F}}$. By Lemma 6.1(*a*)

$$\varphi(S) \cong \varinjlim \{\varphi(S/I), \varphi(p_{I,J})\}_{\mathfrak{F}}$$

and hence it is discrete. Furthermore, Lemma 4.5(b) the above show that (5) holds.

Clearly we have also

$$\chi(S) = \bigcup \{Y_I / I \in \mathcal{F}\}. \tag{6}$$

This is known and follows, for example, from Corollary 6.2 with replacing φ by χ.

The following lemma is the central moment in this section.

6.3. Lemma: There exist isomorphisms $g_I \colon X_I \to Y_I$ for each $I \in \mathcal{F}$ so that the diagram

$$\begin{array}{ccc} X_I & \xrightarrow{g_I} & Y_I \\ \cup & & \cup \\ X_J & \xrightarrow{g_J} & Y_J \end{array} \tag{7}$$

is commutative whenever $I \subset J$.

Proof: The ring S/I is finite for each $I \in \mathcal{F}$ and Lemma 4.1 and Proposition 5.2 show that $\varphi(S/I) \simeq \chi(S/I)$. Since X_I is isomorphic to $\varphi(S/I)$ and Y_I —to $\chi(S/I)$, we have $X_I \simeq Y_I$. Choose an isomorphism $f_I \colon X_I \to Y_I$ for each $I \in \mathcal{F}$.

Let I and L be two elements of $\mathcal{F}$ with $L \subset I$, then X_I is contained in X_L by Lemma 4.5(c). Since

$$f_L(X_I) \simeq X_I \simeq Y_I,$$

and Y_I is the unique submodule of $\chi(S)$ which is isomorphic to $\chi(S/I)$ (see for example Lemma 4.5(b)), we obtain $f_L(X_I) = Y_I$. Then f_I and the restriction of f_L on X_I are two isomorphisms between X_I and Y_I. Since each automorphism of S/I has the form $s \cdot id_{S/I}$ for some invertable element s of S, there is an invertable element $s(L, I)$ of S so that

$$f_L(u) = s(L, I) \cdot f_I(u) \qquad \text{for any } u \in X_I. \tag{8}$$

Set $s(L, I) = 1$ for $L \not\subset I$. Thus we have a map $(L, I) \to s(L, I)$ of $\mathcal{F} \times \mathcal{F}$ into the set K of the invertable elements of S.

Consider $K^{\mathcal{F}}$ endowed with the product topology. By Lemma 2.2 K is compact and then the Tichonov's theorem shows that $K^{\mathcal{F}}$ is also compact. Letting $x_L = (s(L, I))_{I \in \mathcal{F}}$ we obtain a net $\{x_L\}_{L \in \mathcal{F}}$. Here $\mathcal{F}$ is being considered as a directed set under $\subset$. The compactness of $K^{\mathcal{F}}$ shows that there is a density point $x = (s(I))_{I \in \mathcal{F}}$ of $\{x_L\}_{L \in \mathcal{F}}$ in $K^{\mathcal{F}}$. Set $g_I = s(I) \cdot f_I$. Since $s(I)$ is an invertable element of S, then $g_I \colon X_I \to Y_I$ is an isomorphism.

Let I and J be two elements of $\mathcal{F}$ with $I \subset J$. In order to prove the commutativity of (7) consider the neighbourhood

$$U = [(s(I) + I) \cap K] \times [(s(J) + J) \cap K] \times K^{\mathcal{F} / \{I, J\}}$$

of x in $K^{\mathcal{F}}$. Since x is a density point of $\{x_L\}_{L\in\mathcal{F}}$, there is an element L of $\mathcal{F}$ with $L\subset I$ and $x_L\in U$. The last means that

(9) $$s(L, I)\in s(I)+I$$

and

(10) $$s(L, J)\in s(J)+J.$$

Since $I\cdot Y_I=0$, according to (8) and (9) we have

$$f_L(u)=s(L, I)\cdot f_I(u)=s(I)\cdot f_I(u)=g_I(u).$$

Likewise, (8), (10) and $J\cdot Y_J=0$ imply

$$f_L(u)=s(L, J)\cdot f_J(u)=s(J)\cdot f_J(u)=g_J(u).$$

Hence $g_I(u)=g_J(u)$ for $u\in X_I$ which means that (7) is commutative.

6.4. Corollary: $\varphi(S)$ and $\chi(S)$ are topologically isomorphic.

Proof: Taking isomorphism $g_I\colon X_I\to Y_I$ for $I\in\mathcal{F}$ so that (7) is commutative whenever $I\subset J$, using (5) and (6) we are able to define an isomorphism $\omega\colon \varphi(S)\to\chi(S)$ which coincides with g_I on X_I for each $I\in\mathcal{F}$. Since $\varphi(S)$ and $\chi(S)$ are discrete (see Corollary 6.2), ω is a topological isomorphism.

7. - Construction of natural equivalences

Let $\mathcal{M} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathcal{N}$ be a duality over S. In this section we will construct natural equivalences λ between φ and χ and μ between ψ and χ which will complete the proof of Theorem 1.4.

By Corollary 6.4 there exists a topological isomorphism $\omega\ :\ \varphi(S)\to\chi(S)$. For $X\in\mathcal{N}$ consider the following chain of algebraical isomorphisms ([6])

(11) $$\psi(X)\xrightarrow{p}\mathrm{Hom}_s(S, \psi(X))\to\mathrm{Hom}_s(\varphi\psi(X), \varphi(S))\xrightarrow{q}\mathrm{Hom}_s(X, \chi(S)),$$

where p is defined by $p_x(s)=sx$ and q is determined by ω and the natural isomorphism $\varphi\psi(X)\to X$. In a similar way are defined the isomorphisms

(12) $$\chi(X)\to\mathrm{Hom}_s(S, \chi(X))\xrightarrow{s}\mathrm{Hom}_s(\chi\chi(X), \chi(S))\to\mathrm{Hom}_s(X, \chi(S)).$$

By (11) and (12) we get an isomorphism $\mu_X\colon \psi(X)\to\chi(X)$. It is easy to see that μ is natural in the sense that if $f\colon X\to Y$ is a morphism in $\mathcal{N}$, then the diagram

$$\begin{tikzcd} \psi(X) \arrow[r, "\mu_X"] & \chi(X) \\ \psi(Y) \arrow[u, "\psi(f)"] \arrow[r, "\mu_Y"] & \chi(Y) \arrow[u, "\chi(f)"'] \end{tikzcd}$$

is commutative. We omit the detailed verification of this fact.

To prove that μ is a natural equivalence we have to show that μ_X is a homeomorphism for any $X \in \mathcal{N}$. To do this we need the following lemma.

7.1. Lemma: The functors φ and ψ take compact modules to discrete and discrete modules to compact.

Proof: Each compact S-module F is a projective limit of an inverse system $\{F_\alpha, f_{\alpha\beta}\}$ with finite modules F_α, since F has a local base of neighbourhoods of 0 consisting of open submodules (see section 2). Then, by Lemma 6.1(a), $\varphi(F)$ is discrete.

Similarly, according to Lemma 6.1(b), we find that $\psi(E)$ is compact for any discrete S-module E.

Suppose now that $X \in \mathcal{M}$ and X is discrete. By Proposition 2.1 there is a proper exact sequence

$$0 \to Y \xrightarrow{i} \varphi(X) \xrightarrow{j} E \to 0 \tag{13}$$

of morphism in $\mathcal{N}$ so that Y is compact and E is dicrete. Then by 3.2 the sequence

$$0 \leftarrow \psi(Y) \xleftarrow{\psi(i)} \psi\varphi(X) \xleftarrow{\psi(j)} \psi(E) \to 0$$

is also exact. Since $\psi\varphi(X)$ is discrete, as it is topologically isomorphic to X, $\psi(E)$ is dicrete too. On the other hand, it follows from above that $\psi(E)$ is compact. Hence $\psi(E)$ is finite and Lemma 4.2 shows that E is also finite. Now the exactness of (13) and the compactness of Y show that $\varphi(X)$ is compact.

In a similar way one can prove that $\psi(Y)$ is discrete for each compact $Y \in \mathcal{N}$.

The next statement is possibly known.

7.2. Proposition: For $X \in \mathfrak{L}_X$ let $f_X \colon X \to \chi\chi(X)$ be the natural isomorphism defined by $f_X(x)(u) = u(x)$, where $x \in X$ and $u \in \chi(X)$. Then $(f_{\chi(X)})^{-1} = \chi(f_X)$.

Proof: It is sufficient to show that $f_{\chi(X)} \circ \chi(f_X) = id_{\chi\chi\chi(X)}$.

If $u \in \chi\chi\chi(X)$, then $u \colon \chi\chi(X) \to \mathbf{T}$ is a continuous group homomorphism and $\chi(f_X)(u) = u \circ f_X$. So we have to prove that

$$f_{\chi(X)}(u \circ f_X) = u\,. \tag{14}$$

For $v \in \chi\chi(X)$ there exists an $x \in X$ with $f_X(x) = v$. Then

$$u(v) = u(f_X(x)) \tag{15}$$

and

$$f_{\chi(X)}(u \circ f_X)(v) = v(u \circ f_X) = f_X(x)(u \circ f_X) = u \circ f_X(x)\,. \tag{16}$$

Clearly (15) and (16) imply (14).

We are able to prove our main theorem.

Proof of Theorem 1.4: We are going to show that μ_X is a topological isomorphism for any $X \in \mathcal{N}$.

Suppose first that X is a discrete S-module. Then $\psi(X)$ and $\chi(X)$ are compact. Since $\chi(X)$ has a local base of neighbourhoods of 0 consisting of open submodules (see section 2), to prove that μ_X is continuous it is sufficient to show that $\mu_X^{-1}(V)$ is open in $\psi(X)$ for each open submodule V of $\chi(X)$.

Let V be an open submodule of $\chi(X)$ and $U = \mu_X^{-1}(V)$. Clearly U is a submodule of $\psi(X)$ with a finite index in it (which is equal to the index of V in $\chi(X)$). So to prove that U is open in $\psi(X)$ it sufficies to see that U is closed in $\psi(X)$.

Consider the identical embedding $i\colon V \to \chi(X)$. Then $\chi(i)\colon \chi\chi(X) \to \chi(V)$ is a quotient map. Setting $p = \chi(i) \circ f_X$, where $f_X\colon X \to \chi\chi(X)$ is the natural isomorphism, we obtain the commutative diagram

$$\begin{tikzcd} 0 & \chi(V) \arrow[l] & \chi\chi(X) \arrow[l, "\chi(i)"'] \\ & & X \arrow[ul, "p"] \arrow[u, "f_X"'] \end{tikzcd} \tag{17}$$

The naturality of μ shows that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \psi\chi(V) \arrow[r, "\psi(p)"] \arrow[d, "\mu_{\chi(V)}"'] & \psi(X) \arrow[d, "\mu_X"] \\ 0 \arrow[r] & \chi\chi(V) \arrow[r, "\chi(p)"] & \chi(X) \end{tikzcd} \tag{18}$$

is commutative too.

To verify that $\mathrm{Im}\,(\chi(p)) = V$ consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "i"] \arrow[d, "f_V"'] & \chi(X) \arrow[d, "f_{\chi(X)}"] \\ 0 \arrow[r] & \chi\chi(V) \arrow[r, "\chi\chi(i)"] \arrow[dr, "\chi(p)"'] & \chi\chi\chi(X) \arrow[d, "\chi(f_X)"] \\ & & \chi(X) \end{tikzcd}$$

where f_Y and $f_{\chi(X)}$ are the natural isomorphisms. The commutativity of the square follows from the properties of χ and this of the triangle—from the commutativity of (17). By Proposition 7.2, $\chi(f_X)\circ f_{\chi(X)} = id_{\chi(X)}$. Hence $\operatorname{Im}(\chi(p)) = \operatorname{Im}(i) = V$. Now (18) shows that $U = \operatorname{Im}(\psi(p))$ and then U is compact. Thus U is closed in $\psi(X)$. We have proved that μ_X is continuous. Since $\psi(X)$ is compact, it is a homeomorphism.

Let now Y be an arbitrary element of $\mathcal{N}$. Since Y is locally compact, according to Proposition 2.1, we find a proper exact sequence

$$0 \to Z \xrightarrow{f} Y \xrightarrow{g} X \to 0$$

of morphism in $\mathcal{N}$ so that Z is compact and X is discrete. The naturality of μ and the properties of ψ and χ imply that the diagram

$$\begin{tikzcd}
0 & \psi(Z) \arrow[l] \arrow[d, "\mu_Z"'] & \psi(Y) \arrow[l, "\psi(f)"'] \arrow[d, "\mu_Y"'] & \psi(X) \arrow[l, "\psi(g)"'] \arrow[d, "\mu_X"'] & 0 \arrow[l] \\
0 & \chi(Z) \arrow[l] & \chi(Y) \arrow[l, "\chi(f)"'] & \chi(X) \arrow[l, "\chi(g)"'] & 0 \arrow[l]
\end{tikzcd}$$

is commutative. It follows from above that μ_Z is a topological isomorphism. Since $\psi(Z)$ and $\chi(Z)$ are discrete, we obtain that μ_Y is also a topological isomorphism.

Thus we have shown that μ is a natural equivalence between ψ and χ. It remains to construct a natural equivalence between φ and χ.

Let e and f be natural equivalence between $\psi\circ\varphi$ and $id_{\mathcal{M}}$ and between $id_{\mathfrak{C}_S}$ and $\chi\circ\chi$, respectively. For each $X\in\mathcal{M}$ we have $\varphi(X)\in\mathcal{N}$ and then $\mu_{\varphi(X)}$: $\psi\varphi(X)\to\chi\varphi(X)$ is a topological isomorphism. Hence the composition $\mu_{\varphi(X)}\circ e_X$ is a topological isomorphism which implies that

$$\chi(\mu_{\varphi(X)}\circ e_X):\ \chi\chi\varphi(X)\to\chi(X)$$

is also a topological isomorphism. Finally, set

$$\lambda_X = \chi(\mu_{\varphi(X)}\circ e_X)\circ f_{\varphi(X)}.$$

Then λ_X: $\varphi(X)\to\chi(X)$ is a topological isomorphism. A straightforward verification shows that λ is a natural equivalence between φ and χ.

Remark: It follows from Theorem 1.4 that for compact rings S each duality between the category $\mathfrak{C}_S$ of the compact S-modules and the category $\mathfrak{D}_S$ of the discrete S-modules is naturally equivalent to the duality $\mathfrak{C}_S \underset{\chi}{\overset{\chi}{\rightleftarrows}} \mathfrak{D}_S$. Prodanov [6] showed that the analog of this statement for topological groups (i.e. for $S = \mathbf{Z}$ with the discrete topology) is not true.

8. - Another characterization of the Pontryagin duality

Let S be a compact commutative ring with identity and $\mathcal{M}$ be a c-admissible S-category. In this section we characterize the Pontryagin duality functor χ on $\mathcal{M}$ as the only contravariant functor $\varphi: \mathcal{M} \to \mathfrak{L}_s$ which satisfies the following axiom system, similar to the Axiom system I in [7].

(A) $\varphi(S) \cong \chi(S)$,

(B) $\varphi(sf) = s\varphi(f)$ for each $s \in S$ and each morphism f in $\mathcal{M}$,

(C) φ takes proper exact sequences in $\mathcal{M}$ into proper exact sequences,

(D) if X is a compact S-module and $j_U: X \to X/U$ are the canonical quotient maps, then $\varphi(X)$ is discrete and

$$\varphi(X) = \bigcup_U \operatorname{Im}(\varphi(j_U))$$

where U runs over the open submodules of X. Likewise, if $Y \in \mathfrak{M}$ and Y is discrete and for each submodule E of Y $i_E: E \to Y$ is the canonical embedding, then $\varphi(Y)$ is compact and

$$\bigcap_E \operatorname{Kern}(\varphi(i_E)) = 0$$

whenever E runs over the finite submodules of Y.

Roughly speaking, (D) says that φ takes $\varprojlim\{F_\alpha, f_{\alpha\beta}\}_A$ with finite F_α, to $\varinjlim\{\varphi(F_\alpha), \varphi(f_{\alpha\beta})\}_A$ and $\varinjlim\{E_\alpha, g_{\alpha\beta}\}_A$, if it belongs to $\mathcal{M}$ and E_α are finite, to $\varprojlim\{\varphi(E_\alpha), \varphi(g_{\alpha\beta})\}_A$.

Throughout this section $\varphi: \mathcal{M} \to \mathfrak{L}_s$ will be a contravariant functor which satisfies the conditions (A), (B), (C) and (D).

8.1. If $X \times Y \in \mathcal{M}$, then $\varphi(X \times Y) \cong \varphi(X) \times \varphi(Y)$.
The proof is the same as the proof of Lemma 3.1 in [7].

8.2. Lemma: φ is an additive functor.

Proof: Let $f, g: X \to Y$ be morphisms in $\mathcal{M}$. For compact X and Y the equality $\varphi(f+g) = \varphi(f) + \varphi(g)$ can be established as in the proof of Lemma 3.2 in [7].

Suppose X and Y are discrete. Take a finite submodule E of X and let $F = f(E) + g(E)$. There exist morphism f_1 and g_1 so that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_E}\uparrow & & \uparrow{\scriptstyle i_F} \\ E & \xrightarrow{f_1} & F \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle i_E}\uparrow & & \uparrow{\scriptstyle i_F} \\ E & \xrightarrow{g_1} & F \end{array}$$

where i_E and i_F are the identical embeddings, are commutative. Setting $h = \varphi(f) + \varphi(g) - \varphi(f+g)$ and using that $\varphi(f_1) + \varphi(g_1) - \varphi(f_1 + g_1) = 0$ we have that the diagram

$$\begin{array}{ccc}
\varphi(X) & \xleftarrow{\;h\;} & \varphi(Y) \\
{\scriptstyle \varphi(i_E)}\downarrow & & \downarrow{\scriptstyle \varphi(i_F)} \\
\varphi(E) & \xleftarrow{\;0\;} & \varphi(F)
\end{array}$$

is commutative and then $\mathrm{Im}\,(h) \subset \mathrm{Kern}\,(\varphi(i_E))$. By (D), $\bigcap_E \mathrm{Kern}\,(\varphi(i_E)) = 0$ where E runs over the finite submodules of X. Hence $\mathrm{Im}\,(h) = 0$ and $\varphi(f+g) = \varphi(f) + \varphi(g)$.

In the general case let K be a compact open submodule of X and L be a compact open submodule of Y with $f(K) + g(K) \subset L$. The existence of K and L follows from Proposition 2.1. There are morphisms f_1, f_2, g_1 and g_2 so that the diagrams

$$\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{i} & X & \xrightarrow{j} & X/K & \to & 0 \\
 & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f_2}\downarrow & & \\
0 & \to & L & \xrightarrow{m} & Y & \xrightarrow{n} & Y/L & \to & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \to & K & \xrightarrow{i} & X & \xrightarrow{j} & X/K & \to & 0 \\
 & & {\scriptstyle g_1}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle g_2}\downarrow & & \\
0 & \to & L & \xrightarrow{m} & Y & \xrightarrow{n} & Y/L & \to & 0
\end{array}$$

where i, j, m and n are the canonical maps, are commutative. Now it follows from above that $\varphi(f_i + g_i) = \varphi(f_i) + \varphi(g_i)$ for $i = 1, 2$ and therefore the diagram

$$\begin{array}{ccccccccc}
0 & \leftarrow & \varphi(K) & \xleftarrow{\varphi(i)} & \varphi(X) & \xleftarrow{\varphi(j)} & \varphi(X/K) & \leftarrow & 0 \\
 & & {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle \varphi(f+g)-\varphi(f)-\varphi(g)} & & \uparrow{\scriptstyle 0} & & \\
0 & \leftarrow & \varphi(L) & \xleftarrow{\varphi(m)} & \varphi(Y) & \xleftarrow{\varphi(n)} & \varphi(Y/L) & \leftarrow & 0
\end{array}$$

is commutative. Hence $\varphi(f+g) = \varphi(f) + \varphi(g)$.

Let $\omega: \varphi(S) \to \chi(S)$ be a topological isomorphism, such exists by (A). For each $X \in \mathcal{M}$ consider the following sequence of algebrical homomorphisms

$$\text{(19)} \qquad X \xrightarrow{p} \mathrm{Hom}_R\,(S, X) \xrightarrow{s} \mathrm{Hom}_R\,(\varphi(X), \varphi(S)) \xrightarrow{q} \mathrm{Hom}_R\,(\varphi(X), \chi(S)) \to$$
$$\xrightarrow{\chi} \mathrm{Hom}_R\,(\chi\chi(S), \chi\varphi(X)) \xrightarrow{r} \mathrm{Hom}_R\,(S, \chi\varphi(X)) \xrightarrow{t} \chi\varphi(X)$$

where p and t are the canonical isomorphisms, q and r are induced by ω and the natural isomorphism $\chi\chi(S) \to S$, respectively. By (19) we find homomorphisms $\lambda_X: X \to \chi\varphi(X)$ so that for each morphism $f: X \to Y$ in $\mathcal{M}$ the diagram

$$\begin{tikzcd} X \arrow[r, "\lambda_X"] \arrow[d, "f"'] & \chi\varphi(X) \arrow[d, "\chi\varphi(f)"] \\ Y \arrow[r, "\lambda_Y"] & \chi\varphi(Y) \end{tikzcd}$$

is commutative.

We are going to prove that λ_X are topological isomorphisms. To do this we need some technical preparation.

8.3. Let I be a closed ideal of S. As in the proof of Lemma 4.1 we see that if $X \in \mathcal{M}$ and $I \cdot X = 0$, then $I \cdot \varphi(X) = 0$.

8.4. Lemma: For any finite S-module X

$$\operatorname{card}(\varphi(X)) = \operatorname{card}(X). \tag{20}$$

Proof: We will use the representation (1) of S (see section 2) with compact local rings S_α. Denote by $\mathfrak{M}_\alpha$ the unique maximal ideal of S_α.

Taking $\alpha \in \mathcal{A}$ we want to show first that $\varphi(S_\alpha/\mathfrak{M}_\alpha) \neq 0$. Assume the contrary. Then $\varphi(F) = 0$ for each finite S_α-module F. To prove this we use an induction on $n = \operatorname{card}(F)$. Suppose the statement is true for all finite S_α-modules with cardinality less than n and let $\operatorname{card}(F) = n$. If F is a simple S_α-module, then $F \cong S_\alpha/\mathfrak{M}_\alpha$ and $\varphi(F) = 0$. Supposing F is not simple, we find an exact sequence

$$0 \to E \xrightarrow{u} F \xrightarrow{\pi} F/E \to 0$$

with $\operatorname{card}(E) < n$ and $\operatorname{card}(F/E) < n$. By (C) and the inductive hypothesis we get $\varphi(F) = 0$. In particular, $\varphi(S_\alpha/I) = 0$ for each open ideal I of S_α. It follows now from (D) that $\varphi(S_\alpha) = 0$.

On the other hand, $S = S_\alpha \times \prod_{\beta \neq \alpha} S_\beta$ and (A) and 8.1 imply

$$\chi(S) \cong \varphi(S) \cong \varphi(S_\alpha) \times \varphi\Big(\prod_{\beta \neq \alpha} S_\beta\Big) = \varphi\Big(\prod_{\beta \neq \alpha} S_\beta\Big).$$

According to 8.3, we have $S_\alpha \cdot \varphi\Big(\prod_{\beta \neq \alpha} S_\beta\Big) = 0$ and then $S_\alpha \cdot \chi(S) = 0$. The last means $S_\alpha = 0$ which is a contradiction.

Hence $\varphi(S_\alpha/\mathfrak{M}_\alpha) \neq 0$. Set $T_\alpha = \mathfrak{M}_\alpha \times \prod_{\beta \neq \alpha} S_\beta$, then $S_\alpha/\mathfrak{M}_\alpha \cong S/T_\alpha$. It follows from (A) and (C) that there is a monomorphism

$$f\colon \varphi(S/T_\alpha) \to \chi(S).$$

By 8.3, $T_\alpha \cdot \varphi(S/T_\alpha) = 0$ and then $\operatorname{Im}(f) = Y_{T_\alpha}$ (see 4.3 and 4.5(b)). On the other hand, card $(Z) = \text{card}\,(\chi(Z))$ for each finite S-module Z. So we have

$$\text{card}\,(\varphi(S/T_\alpha)) = \text{card}\,(Y_{T_\alpha}) = \text{card}\,(\chi(S/T_\alpha)) = \text{card}\,(S/T_\alpha)\,.$$

Thus we have proved (20) for $X = S_\alpha/\mathfrak{M}_\alpha$ and arbitrary $\alpha \in A$.

Further we use an induction on $n = \text{card}\,(X)$. Suppose (20) is true for card $(X) < n$ and let Y be an S-module with card $(Y) = n$. If Y is simple, then $Y \cong S_\alpha/\mathfrak{M}_\alpha$ for some $\alpha \in A$ and the above reasonings imply card $\varphi(Y) =$ $=$ card (Y). Assume Y is not simple, then there is a non-zero proper submodule X of Y. By the inductive hypothesis card $(\varphi(X)) = \text{card}\,(X)$ and card $(\varphi(Y/X)) = \text{card}\,(Y/X)$. Since there is an exact sequence

$$0 \leftarrow \varphi(X) \leftarrow \varphi(Y) \leftarrow \varphi(Y/X) \leftarrow 0$$

we have

$$\text{card}\,(\varphi(Y)) = \text{card}\,(\varphi(X)) \cdot \text{card}\,(\varphi(Y/X)) = \text{card}\,(X) \cdot \text{card}\,(Y/X) = \text{card}\,(Y)\,.$$

8.5. Corollary: If $X \in \mathcal{M}$ and $\varphi(X) = 0$, then $X = 0$.

Proof: For finite X this follows immediately from Lemma 8.4.

Let X be an arbitrary element of $\mathcal{M}$ with $\varphi(X) = 0$. By Proposition 2.1 X contains a compact open submodule Y. Since there exists an epimorphism $\varphi(X) \to \varphi(Y)$, we have $\varphi(Y) = 0$. If $Y \neq 0$, then there will be an epimorphism of the type $Y \to F$ with finite $F \neq 0$ which is clearly impossible. Thus X is discrete. All finite submodules of X are 0 and therefore $X = 0$.

We are able to show that $\lambda_X\colon X \to \chi\varphi(X)$ is a monomorphism for each $X \in \mathcal{M}$.

8.6. Lemma: If f is a morphism in $\mathcal{M}$ with $\varphi(f) = 0$, then $f = 0$.

Proof: Let $f\colon X \to Y$ be a morphism in $\mathcal{M}$ with $\varphi(f) = 0$ and suppose $f \neq 0$. Then $f(x) \neq 0$ for some $x \in X$. Define a homomorphism $j\colon S \to X$ with $j(s) = sx$ and set $I = \text{Kern}\,(f \circ j)$. Obviously, I is a proper closed ideal of S and there exists a unique monomorphism h in $\mathcal{M}$ so that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ S \arrow[u, "j"] \arrow[r, "p_I"] & S/I \arrow[u, "h"'] \end{tikzcd}$$

where p_I is the canonical epimorphism, is commutative. Then the diagram

$$\begin{tikzcd} \varphi(X) \arrow[d, "\varphi(j)"'] & \varphi(Y) \arrow[l, "\varphi(f)"'] \arrow[d, "\varphi(h)"] \\ \varphi(S) & \varphi(S/I) \arrow[l, "\varphi(p_I)"] \end{tikzcd}$$

is also commutative which shows that $\varphi(p_I)\circ\varphi(b) = 0$. Since $\varphi(p_I)$ is a monomorphism, we get $\varphi(b) = 0$. On the other hand, $\varphi(b)$ is an epimorphism and therefore $\varphi(S/I) = 0$. Contradiction with Corollary 8.5. Hence $f = 0$.

Now looking at the definition of $\lambda_X\colon X \to \chi\varphi(X)$ we see that it is a monomorphism for each $X \in \mathcal{M}$. This enables us to prove the main theorem in this section.

8.7. Theorem: Let S be a compact commutative ring with identity, $\mathcal{M}$ be a c-admissible S-category and $\varphi\colon \mathcal{M} \to \mathfrak{C}_s$ be a contravariant functor which satisfies the conditions (A), (B), (C) and (D). Then φ is naturally equivalent to the Pontryagin duality functor χ.

Proof: We have mentioned above that the algebraical homomorphisms $\lambda_X\colon X \to \chi\varphi(X)$, constructed before, are monomorphisms for any $X \in \mathcal{M}$. It follows now from Lemma 8.4 that λ_X is an isomorphism for any finite X.

Suppose $X \in \mathcal{M}$ and X is compact. Let $p_U\colon X \to X/U$ be the canonical epimorphisms. By (D) and the properties of χ we have that $\{\mathrm{Kern}\,(\chi\varphi(p_U))\}_U$ form a local base of neighbourhoods of 0 for $\chi\varphi(X)$, when U runs over the open submodules of X. The commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda_X} & \chi\varphi(X) \\ {\scriptstyle p_U}\big\downarrow & & \big\downarrow{\scriptstyle \chi\varphi(p_U)} \\ X/U & \xrightarrow{\lambda_{X/U}} & \chi\varphi(X/U) \end{array}$$

and the fact that $\lambda_{X/U}$ are isomorphisms imply

$$\lambda_X(U) \subset \mathrm{Kern}\,(\chi\varphi(p_U))$$

and then λ_X is continuous. Hence $\mathrm{Im}\,(\lambda_X)$ is compact. Since

$$\chi\varphi(X) = \mathrm{Im}\,(\lambda_X) + \mathrm{Kern}\,(\chi\varphi(p_U))$$

for each open submodule U of X, we find $\mathrm{Im}\,(\lambda_X) = \chi\varphi(X)$ and therefore λ_X is a topological isomorphism.

Similar reasoning show that for any discrete module X, λ_X is again a topological isomorphism. Let X be an arbitrary element of $\mathcal{M}$. By Proposition 2.1 there is a proper exact sequence

$$0 \to Y \xrightarrow{f} X \xrightarrow{g} Z \to 0$$

in $\mathcal{M}$ with compact Y and discrete Z. Then the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \xrightarrow{f} & X & \xrightarrow{g} & Z & \to & 0 \\ & & {\scriptstyle \lambda_Y}\big\downarrow & & {\scriptstyle \lambda_X}\big\downarrow & & {\scriptstyle \lambda_Z}\big\downarrow & & \\ 0 & \to & \chi\varphi(Y) & \xrightarrow{\chi\varphi(f)} & \chi\varphi(X) & \xrightarrow{\chi\varphi(g)} & \chi\varphi(Z) & \to & 0 \end{array}$$

is commutative and the sequence on the second line is also proper exact. It follows from above that λ_Y and λ_Z are topological isomorphisms. This clearly implies that λ_X is also a topological isomorphism.

Thus we have shown that λ is a natural equivalence between $id_{\mathscr{M}}$ and $\chi\varphi$. Let f be a natural equivalence from $id_{\mathfrak{C}_s}$ to $\chi\chi$. Setting $\mu_X = \chi(\lambda_X) \circ f_{\chi(X)}$ we obtain a natural equivalence between φ and χ on $\mathscr{M}$.

Remark: Let $\mathscr{M}$ be a d-admissible S-category. Using reasonings similar to these in this section one can prove that any contravariant functor $\varphi : \mathscr{M} \to \mathfrak{C}_s$ which satisfies (B), (C), (D) and

(A') $\qquad \varphi(S/I) \simeq \chi(S/I)$ for every open ideal I of S,

is naturally equivalent to χ on $\mathscr{M}$.

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