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Classification of Monomial Curves (**)

Classificazione di curve monomiali

SUNTO. — Le equazioni cartesiane delle curve $X_1 = t^{n_1}$, $X_2 = t^{n_2}$, $X_3 = t^{n_3}$ e dei loro coni tangenti vengono classificate in termini dei residui di n_2 e $n_3 \bmod n_1$: Inoltre viene data una formula chiusa per le equazioni del cono tangente di una singola curva e viene descritto un facile algoritmo per calcolarlo.

INTRODUCTION

Monomial curves $X_1 = t^{n_1}$, $X_2 = t^{n_2}$, $X_3 = t^{n_3}$ are often used both to give examples and to test conjectures in Commutative Algebra. So an easy way to determine the equational pattern of them and of their tangent cone, which is the aim of this note, can be of a practical, interest; in this direction the reader may also consult [6] in which an algorithm is described to compute the equations of the projective closure of the same curves, which is very similar to the one here described for tangent cone equations.

These curves have been recently studied by Herzog and Kunz, using results in numerical semigroup theory; after the explicit description of their ideal given by Herzog [2], it was clear how to represent them as determinantal ideals [5]. This allowed Robbiano and Valla to use their results on the tangent cone of determinantal varieties and prove for instance that the tangent cone of these curves is generated by less than 4 elements iff it is Cohen-Macaulay; moreover, starting from their cartesian equations, they classified these curves with respect to the minimal number of generators of their tangent cone and gave a procedure to compute it [3].

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Herzog's results suggest also another approach to the study of these curves. It is possible to prove that if $X_2^a = X_1^b X_3^c$ is in the ideal of the curve $X_1 = t^{n_1}$, $X_2 = t^{n_2}$, $X_3 = t^{n_3}$ then b and c are uniquely determined by a, and c depends only on n_1 and on the residuals of n_2 and $n_3 \bmod n_1$. So with purely numerical observations, it is possible, for a family of curves $X_1 = t^{n_1}$, $X_2 = t^{\lambda n_1 + n_2}$, $X_3 = t^{\mu n_1 + n_3}$, n_1, n_2, n_3 assigned natural numbers, λ, μ natural parameters, to determine the values of the parameters for which the curve is a complete intersection (2.7) and to give a classification like the one of [3] but which doesn't require the knowledge of the cartesian equations (5.5). Moreover for a given curve it is possible to characterize its tangent cone (4.2) and to give a very easy algorithm to compute it (6.3, 6.4).

For an assigned λ, when μ grows, one has curves whose tangent cone is CM (Cohen-Macaulay), then curves whose tangent cone is not CM, then again curves with a CM tangent cone and so on (and among them also sporadical CI (complete intersections) whose tangent cone is not CI), and when μ is greater than a fixed value CI's whose tangent cone is CI.

The numerical nature of these results strongly suggests that a computer be used both for the classification of these curves and for the determination of the cartesian equations of the curves and of their tangent cones. All the algorithms here described have been implemented in BASIC on a PDP11-V03; for details on the algorithms see [4].

The use of a computer is not new in such situations and more generally in classification problems in Algebraic Geometry: see [1] which describes some FORTRAN programs which study monomial curves through the properties of the numerical semigroups associated to them.

1. - Definitions and preliminary results

1.1 Let $\mathfrak{C}$ be a curve in $A_3(k)$ with parametric equations

$$X_1 = t^{n_1}, \quad X_2 = t^{n_2}, \quad X_3 = t^{n_3}$$

$n_1 < n_2 < n_3$ natural numbers, g.c.d. $(n_1, n_2, n_3) = 1$.
$\sigma : k[X_1, X_2, X_3]_{(X_1, X_2, X_3)} \to k[t]_{(t)}$ be the homomorphism $\sigma(X_i) = t^{n_i}$.

$$I(\mathfrak{C}) = \operatorname{Ker} \sigma \quad \text{the ideal of the curve}.$$

1.2 As in [2] $\mathfrak{C} = \mathfrak{C}(S)$ is the curve associated to the numerical semigroup

$$S = (n_1, n_2, n_3).$$

If for any $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{N}^3$ $F(\nu)$ denotes the monomial

$$F(\nu) = X_1^{\nu_1} X_2^{\nu_2} X_3^{\nu_3}$$

and $\varrho : \mathbb{N}^3 \to S$ denotes the homorphism: $\varrho(\nu) = \sum \nu_i n_i$ then $\sigma F(\nu) = t^{\varrho(\nu)}$.

1.3 Let $M(S)$ denote the set

$$M(S) = \{v - w\colon v, w \in \mathbf{N}^3, \varrho(v) = \varrho(w)\} = \{(z_1, z_2, z_3) \in \mathbf{Z}^3\colon \sum z_i n_i = 0\}$$

If $v \in M(S)$, $v \neq 0$, $v = (z_1, z_2, z_3)$ there is a z_i such that either

$$z_i > 0, \quad z_j < 0 \quad \text{if } j \neq i$$

or

$$z_i < 0, \quad z_j > 0 \quad \text{if } j \neq i.$$

In this case, we say that v is of kind i.

Moreover if v is of kind i and for any $v' = (z_i')$ of kind i

$$|z_i| < |z_i'|$$

v is said *minimal of kind* i.

v is said *minimal* if it is minimal of any kind.

1.4 If $v = (z_1, z_2, z_3) \in M(S)$ is of kind i, define $v^+ = (z_1', z_2', z_3')$, $z_i' = |z_i|$, $z_j' = 0$ if $j \neq i$; $v^- = (z_1'', z_2'', z_3'')$, $z_i'' = 0$, $z_j'' = |z_j|$ if $j \neq i$.

Then $v = \pm(v^+ - v^-)$ and the polynomial $G(v) = F(v^+) - F(v^-)$ is obviously associated to v.

1.5 *Let c_i be the least natural number such that*

$$c_1 n_1 = r_{12} n_2 + r_{13} n_3$$

and c_2, c_3 be defined in a similar way.

Then $v_1 = (c_1, -r_{12}, -r_{13})$, $v_2 = (-r_{21}, c_2, -r_{23})$, $v_3 = (-r_{31}, -r_{32}, c_3)$ are minimal.

If such v_1, v_2, v_3 are chosen, then

1) *if $r_{ij} \neq 0$ for any i, j*

$$I(\mathfrak{L}) = (G(v_1), G(v_2), G(v_3))$$

and

$$c_1 = r_{21} + r_{31}, \quad c_2 = r_{12} + r_{32}, \quad c_3 = r_{13} + r_{23}$$

2) *otherwise, either*

$$2a) \quad I(\mathfrak{L}) = (G(v_1), G(v_2)) \quad v_2 = -v_3$$

$$2b) \quad I(\mathfrak{L}) = (G(v_1), G(v_2)) \quad v_1 = -v_3$$

$$2c) \quad I(\mathfrak{L}) = (G(v_1), G(v_3)) \quad v_1 = -v_2.$$

2. - Cartesian equations for a family of curves

2.1 To determine the cartesian equations of the curve $\mathfrak{C} = \mathfrak{C}(S)$, $S = (n_1, n_2, n_3)$ one has to find the minimal solutions in $\mathbf{Z}$ of the equation $\sum z_i n_i = 0$, and the problem can be further reduced to find the solutions of the equation

$$(1) \qquad z_2 n_2 \equiv z_3 n_3 \bmod n_1$$

if one aims to determine the cartesian equations of the family of curves $\mathfrak{C}_{\lambda\mu} = \mathfrak{C}(S_{\lambda\mu})$,

$$S_{\lambda\mu} = (n_1, n_1\lambda + n_2, n_1\mu + n_3)$$

n_1, n_2, n_3 natural numbers, g.c.d. $(n_1, n_2, n_3) = 1$, $n_2 < n_1$, $n_3 < n_1$, λ, μ natural parameters such that $n_1 < n_1\lambda + n_2 < n_1\mu + n_3$.

2.2 Let $q_1 =$ g.c.d. (n_1, n_2), $q_2 =$ g.c.d. (n_1, n_3). *Let* $R(1)$ *be such that* $0 < R(1) < n_1/q_2$

$$n_3 \equiv R(1) n_2/q_2 \bmod n_1/q_2 .$$

A set of solutions of (1) *is then* $(q_2 i, R(i))$, $0 < i < n_1/q_1 q_2$ *where*

$$\begin{aligned} &R(0) = n_1/q_2 \\ &R(i) \equiv iR(1) \quad 0 < R(i) < n_1/q_2 \quad \text{if } 1 < i < n_1/q_1 q_2 \\ &R(n_1/q_1 q_2) = 0 . \end{aligned}$$

2.3 *Let*

$$\begin{aligned} n'_2 &= n_1\lambda + n_2 \\ n'_3 &= n_1\mu + n_3 \\ A(i) &= (iq_2 n'_2 - R(i) n'_3)/n_1 \\ B(i) &= \left(iq_2 n'_2 + (n_1/q_2 - R(i)) n'_3\right)/n_1 \\ v(i) &= (-A(i), iq_2, -R(i)) \\ w(i) &= (-B(i), iq_2, n_1 q_2 - R(i)) . \end{aligned}$$

Then $v = (z_1, z_2, z_3)$, $|z_2| < n_1/q_1$, $|z_3| < n_1/q_2$

is of kind 1 *iff* $v = \pm w(i)$ *for some* i

is of kind 2 *iff* $v = \pm v(i)$ *for some* i *and* $A(i) > 0$

is of kind 3 *iff* $v = \pm v(i)$ *for some* i *and* $A(i) < 0$.

2.4 *Let*

$$I = \{i : 0 < i < n_1/q_1 q_2,\ R(i) < R(k) \text{ for any } k < i\}$$

$$K = \{i : 0 < i < n_1/q_1 q_2,\ R(i) > R(k) \text{ for any } k < i\}$$

$$K(i, h) = \{i : i \in K,\ R(i) + R(j) > n_1/q_2,\ i < h\}.$$

Then: if $v(i)$ is minimal $i \in I$; if $w(i)$ is minimal then $i \in K$.

PROOF: If $R(k) < R(i)$ and $k < i$, then

$$v(i) - v(k) = (A(k) - A(i),\ (i-k)q_2,\ R(k) - R(i))$$

is of the same kind as $v(i)$ and $i - k < i$, $R(i) - R(k) < R(i)$ where both equalities cannot occur. A similar argument proves the second assertion.

2.5 *Let $j \in I$ be such that $A(j) > 0$, $A(i) < 0$ if $i < j$, $i \in I$. Let l be the element in I immediately preceding j. Then:*

i) *$A(h) > 0$ for any $h > j$, $h \in I$.*

ii) *$v(j)$ is minimal of kind* 2.

iii) *If $A(j) = 0$ $v(j)$ is also minimal of kind* 3.

iv) *If $A(j) > 0$ $v(l)$ is minimal of kind* 3.

v) *$v(j) - v(l)$ is minimal of kind* 1.

PROOF: v) If $A(j) > 0$ it follows from 1.5 case 1). Otherwise let $w_1 = (-c_1, r_{12}, r_{13})$ be minimal of kind 1 such that $0 < r_{13} < c_3 = R(j) < R(l)$. There exists i such that $w_1 = w(i)$. Since

$$R(j) > r_{13} = n_1/q_2 - R(i) \quad \text{and} \quad R(i) + R(j) = R(i+j)$$

then

$$R(i) + R(j) = R(i+j) + n_1/q_2 \quad \text{and} \quad r_{13} = R(j) - R(i+j)$$

which implies $R(j) > R(i+j)$. Also,

$$w = v(j) - v(l) = (A(l),\ (j-l)q_1,\ R(l) - R(j))$$

is of kind 1 and

$$w_1 - w = (A(l) - c_1,\quad (i+l-j)q_1,\quad R(l) - R(i+j)),$$

where $A(l) - c_1 > 0$, $R(l) - R(i+j) = R(j) - R(i+l) > 0$ so $i + l > j$ as $j \in I$. But then necessarily $i = j - l$ and $w = w_1$.

2.6 Let the elements of I be indexed in increasing order: $I = \{i_0, \ldots, i_r\}$ and let $D(j) = q_2 i_j / R(i_j)$, for $j = 0, \ldots, r-1$, so that for any j, $D(j) > D(j-1)$ and $D(j)$ does not depend on λ and μ.

Then, combining the results of 2.5 with those of 1.4 one obtains:

If $n_2' = \lambda n_1 + n_2$, $n_3' = \mu n_1 + n_3$, $\mathfrak{C} = \mathfrak{C}_{\lambda\mu}$ *then:*

a) *if* $n_3' > D(r-1)n_2'$ *then*

$$I(\mathfrak{C}) = (G(u_1), G(u_2)) \qquad u_1 = v(n_1/q_1q_2)\,, \qquad u_2 = v(i_{r-1})$$

b) *if* $n_3' = D(r-1)n_2'$ *then*

$$I(\mathfrak{C}) = (G(u_1), G(u_2)) \qquad u_1 = v(n_1/q_1q_2)\,, \qquad u_2 = v(i_{r-1})$$

c) *if* $n_3' = D(j)n_2'$ *then*

$$I(\mathfrak{C}) = (G(u_1), G(u_2)) \qquad u_1 = v(i_j) - v(i_{j-1})\,, \qquad u_2 = v(i_j)$$

d) *if* $D(j-1)n_2' < n_3' < D(j)n_2'$, $1 < j < r-1$, *then*

$$I(\mathfrak{C}) = (G(u_1), G(u_2), G(u_3)) \qquad u_1 = v(i_j) - v(i_{j-1})\,, \qquad u_2 = v(i_j)\,, \quad u_3 = v(i_{j-1})$$

e) *if* $n_3' < D(1)n_2'$ *then*

$$I(\mathfrak{C}) = (G(u_1), G(u_2)) \qquad u_1 = v(1)\,, \qquad u_2 = v(0)\,.$$

2.7 *The curve* $\mathfrak{C} = \mathfrak{C}_{\lambda\mu}$ *is a complete intersection iff either:*

$$\mu > D(r-1)\lambda + (D(r-1)n_2 - n_3)/n_1$$

or:

$$\mu = D(j)\lambda + (D(j)n_2 - n_3)/n_1$$

or:

$$\mu < D(1)\lambda + (D(1)n_2 - n_3)/n_1\,.$$

3. - Some lemmata

3.1 The techniques of paragraph 2 can be applied also to the computation of the tangent cone of $\mathfrak{C}$. In this paragraph, some definitions are given and some lemmata are proved. In the following one the computation of the tangent cone of $\mathfrak{C}$ will be achieved.

3.2 If P is polynomial let $T(P)$ denote the initial form of P. The ideal

$$T(\mathfrak{C}) = ((T(P) : P \in I(\mathfrak{C})))$$

is the *tangent cone* of $\mathfrak{C}$.

3.3 Let $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in \mathbb{N}^3$.
Let $\|v\| = \sum v_i$.
Let $v \ll w$ iff $v_i < w_i$ for any i iff $F(w) \in (F(v))$.

3.4 *If* $v \in M(S)$ *let* $T(v) = v^+$ (*resp.* v, v^-) *iff* $\|v^+\| < \|v^-\|$ (*resp.* $=, >$).
Then:

$$T(G(v)) = K(T(v)) \quad \text{iff } T(v) \neq v$$
$$T(G(v)) = G(T(v)) \quad \text{iff } T(v) = v\,.$$

3.5 *If v is of kind 1, then $T(v) = v^-$. If v is of kind 3, then $T(v) = v^+$. If v is of kind 2, any of the three cases can occur.*

3.6 $v(i)$ is said *T-minimal* iff

$v(i)$ is of kind 2

$T(v(i)) \neq v(i)^-$

for any $k < i$, if $v(k)$ is of kind 2, then $T(v(k)) = v(k)^-$.

3.7 *Let $i \notin I$, $0 < i < n_1/q_1 q_2$ such that $v(i)$ is of kind 2. Let b be the greatest element in I such that $b < i$. Then:*

i) $v(i) - v(b) = v(i-b)$;

ii) $v(b)$ *is of kind* 2;

iii) $v(i-b)$ *is of kind* 2 *if* $A(i) > A(b)$, *of kind* 3 *if* $A(i) < A(b)$.

Proof. i) Since $0 < R(i) - R(b) < n_1/q_1$ and $R(i) - R(b) = R(i-b)$, then $R(i) - R(b) = R(i-b)$ and $A(i-b) = A(i) - A(b)$, so $v(i) - v(b) = v(i-b)$,

ii) if $v(b)$ were of kind 3, $v(i-b)$ would necessarily be of kind 2; let then b_1 be the greatest element in I such that $b_1 < i-b$. $b_1 < b$ and $v(b_1)$ is of kind 3; but this leads to an infinite descent.

3.8 *If $A(i) > A(b)$, then:*

i) *if* $T(v(i)) = v(i)^+$, $T(v(b)) \neq v(b)^+$ *then* $T(v(i-b)) = v(i-b)^+$;

ii) *if* $T(v(i)) = v(i)$, $T(v(b)) = v(b)^-$ *then* $T(v(i-b)) = v(i-b)^+$;

iii) *if* $T(v(i)) = v(i)$, $T(v(b)) = v(b)$ *then* $T(v(i-b)) = v(i-b)$;

iv) *if* $T(v(i)) = v(i)^-$, $T(v(b)) \neq v(b)^-$ *then* $T(v(i-b)) = v(i-b)^-$.

Proof: All the assertions follow easily from:

$$T(v(i)) = v(i)^+ \text{ (resp. } v(i), v(i)^-) \text{ iff } iq_2(n_2' - n_1) > R(i)(n_2' - n_1) \text{ (resp. } =, <).$$

3.9 *If $T(v(i)) \neq v(i)^-$ then $T(v(b)) = v(b)^+$.*
Therefore if $v(i)$ is T-minimal, $i \in I$.

Proof: If $A(i) < A(b)$, $v(i-b)$ is of kind 3, so $R(i) - R(b) < A(i) - A(b) + q_2(i-b)$. Therefore $0 < R(i) + A(i) - q_2 i < R(b) + A(b) - q_2 b$ so that $q_2 b < R(b) + A(b)$. If $A(i) > A(b)$ and $T(v(i)) = v(i)^+$, $T(v(b)) \neq v(b)^+$, then $T(v(i-b)) = v(i-b)^+$; if b_1 is as in 3.7, $T(v(b_1)) \neq v(b_1)^+$, since $b_1 < b$, and this leads to an infinite descent. If $A(i) > A(b)$ and $T(v(i)) = v(i)$, $T(v(b)) \neq v(b)^+$, then $T(v(i-b)) \neq v(i-b)^-$, $T(v(b_1)) \neq v(b_1)^+$, and infinite descent follows again.

4. - Equations of the tangent cone

4.1. We are now able to compute the tangent cone of the curve $\mathfrak{L}_{\lambda\mu} = \mathfrak{L}(S_{\lambda\mu})$, where $S_{\lambda\mu} = (n_1, n_1\lambda + n_2, n_1\mu + n_3)$.

Let:

$j \in I$ be such that $A(j) > 0$, $A(i) < 0$ for any $i \in I$, $i < j$

$l \in I$ be the element immediately preceding j

$k \in I$ be such that $v(k)$ is T-minimal

$I_1 = \{i \in I : j < i < k\}$

$I_2 = K(l, k)$

$\mathfrak{K} = \big(F(v(l)^+),\ T(G(v_k)),\ \{F(v(i)^-) : i \in I_1\},\ \{F(w(i)^-) : i \in I_2\}\big)$.

4.2 $T(\mathfrak{L}) = \mathfrak{K}$.

Proof:

1) $T(\mathfrak{L}) = \Big(\big\{T\big(G(v(i))\big)\big\}, \big\{T\big(G(w(i))\big)\big\}\Big)$

2) $\big\{T\big(G(w(i))\big)\big\} \subset \mathfrak{K}$

$w(i)$ is of kind 1, so $T(w(i)) = w(i)^-$. There are 4 cases:

2.i) If $i < k$ and $R(i) + R(l) < n_1/q_2$ then $w(i)^- \gg v(l)^+$.

2.ii) If $i < k$ and $R(i) + R(l) > n_1/q_2$ but there exists $b < i$ such that $R(b) > R(i)$ then $w(i)^- \gg w(b)^-$.

2.iii) If $i > k$ and $T(v(k)) = v(k)^+$, then $w(i)^- \gg v(k)^+$.

2.iv) If $i > k$ and $T(v(k)) = v(k)$, let u be such that $0 < i - uk < k$. Then:

$$F(w(i)^-) = X_1^{q_2(i-k)} X_2^{n_1/q_2 - R(i)} G(v(k)) + F(z)$$

where $z = (A(k),\ (i-k)q_2,\ n_1 q_2 - R(i) + R(k))$.
So $F(w(i)^-) \in \big(G(v(k)), F(z_1)\big)$, where

$$z_1 = \big(uA(k),\ (i-uk)q_2,\ n_1 q_2 - R(i) + uR(k)\big).$$

If $n_1/q_2 - R(i) + uR(k) > R(l)$ then $z_1 \gg v(l)^+$; otherwise

$$R(i) - uR(k) < n_1/q_2, \quad R(i) - uR(k) = R(i-uk) \quad \text{and} \quad z_1 \gg w(i-uk)^-,$$

where $i - uk < k$.

3) *If $v(i)$ is of kind 2, $i \in I$, then $T\big(G(v(i))\big) \in \mathcal{K}$.*
As $v(i)$ is of kind 2, $i > j$. If $j < i < k$ then $T\big(G(v(i))\big) \in \mathcal{K}$ by assumption. Otherwise:

3.i) If $T(v(k)) = v(k)^+$ then $v(i)^+ > v(k)^+$.

3.ii) If $T(v(k)) = v(k)$ then $F(v(i)^+) = X_2^{q_2(i-k)} G(v(k)) + F(z)$

where $z = (A(k),\ q_2(i-k),\ R(k))$.

$v(i) - v(k) = (-A(i) + A(k),\ q_2(i-k),\ R(k) - R(i))$ is of kind 1 so $T(v(i) - v(k)) = (v(i) - v(k))^- < z$ and $F(v(i)^+) \in \big(G(v(k)),\ F\big((v(i) - v(k))^-\big)\big)$.

4) *If $v(i)$ is of kind 2, $i \notin I$, then $T\big(G(v(i))\big) \in \mathcal{K}$.*
Let b be as in 3.7. There are 3 cases:

4.i) If $T(v(i)) = v(i)^+$, then $T(v(b)) = v(b)^+$, $v(i)^+ > v(b)^+$.

4.ii) If $T(v(i)) = v(i)^-$, then, if $R(i) > R(l)$, $v(i)^- > v(l)^+$.

Otherwise: if $A(i) < A(b)$, then $v(i-b)$ is of kind 3, $R(i-b) < R(i)$ so $v(i)^- > v(i-j)^+$; if $A(i) > A(b)$ either $T(v(b)) = v(b)^-$, $v(i)^- > v(b)^-$, or $T(v(i-j)) = v(i-j)^-$, $v(i)^- > v(i-j)^-$ and $F(v(i-j)^-) \in \mathcal{K}$ otherwise infinite descent leads to a contradiction.

4.iii) If $T(v(i)) = v(i)$, then $T(v(b)) = v(b)^+$, $v(i)^+ > v(b)^+$, and the same arguments as above prove that $F(v(i)^-) \in \mathcal{K}$.

5) *If $v(i)$ is of kind 3 then $T\big(G(v(i))\big) \in \mathcal{K}$.*

5. - Classification on monomial curves

5.1 The results of paragraphs 2 and 4 allow to classify monomial curves $\mathfrak{C} = \mathfrak{C}(S)$, $S = (n_1, n_2', n_3')$ with respect to the minimal number of generators of their ideals and of their tangent cones, given only the parametric equations. The invariants for this classification, i.e. the $D(j)$'s, depend only on n_1 and on the residuals of n_2' and $n_3' \bmod n_1$. This classification corresponds to the one of [3], which is given in terms of the cartesian equations.

5.2 $I_1 = \emptyset$ *iff* $j = k$.

5.3 *In cases c), d) of 2.6* $j - l \in I_2$.

Proof: $j - l < j < k$; $R(j-l) + R(l) = R(j) + n_1/q_2 > n_1/q_2$. If there exist $b < j - l$ such that $R(b) > R(j-l)$, then $R(b) + R(l) > n_1/q_2$,

$$R(l+b) = R(l) + R(b) - n_1/q_2 < R(j-l) + R(l) - n_1/q_2 = R(j)$$

and j would not be in I.

5.4 *If $j=k$, then in cases a), b), e) of 2.6, $I_2=\emptyset$; in cases c), d) $I_2=\{j-l\}$.*

Proof: In cases a), b) if $w(i)=(-B(i), iq_2, n_1/q_2-R(i))$ is in $M(S)$, then $B(i)>c_1=n_2'/q_1$. So $(-B(i)+c_1, iq_2-n_1/q_1, n_1/q_2-R(i))$ is of kind 3, therefore $R(i)+R(b)<n_1/q_2$ and $i\notin I_2$.

In cases c), d), if i is in I_2, $R(i)+R(l)>n_1/q_2$. Then

$$w(i)+v(l)=(-B(i)+A(l),\ (i+l)q_2,\ n_1/q_2-R(i)-R(l))$$

is in $M(S)$ and, since

$$(i+l)q_2>0\,,\quad -R(l)<n_1/q_2-R(i)-R(l)<0\,,$$

it cannot be of kind 3, therefore it is of kind 2.

So, $i+l>j$. But, if $i+l>j$, since $R(i)>R(j-l)$ as $i\in I_2$, and since $w(j-l)$ is minimal of kind 1,

$$w(i)-w(j-l)=(-B(i)+c_1,\ (i-j+l)q_2,\ R(j-l)-R(l))$$

is of kind 2, so $i-j+l>j$, $i>j=k$ and i cannot belong to I_2.

In case e), $k=j=1$ and $I_2=\emptyset$ obviously.

5.5 Let $S=(n_1, n_2', n_3')$, $\mathfrak{C}=\mathfrak{C}(S)$, $D(j)$ as in 2.6; α the minimal number of generators of $I(\mathfrak{C})$, β the minimal number of generators of $T(\mathfrak{C})$. Then the different possibilities are summarized in the following table.

6. - An algorithm to compute tangent cones

6.1 The computation of the equations of the tangent cone with the procedure of paragraph 4 is efficient whenever one's interest is in the classification of a family of curves, e.g. all monomial curves of given multiplicity, but it is less efficient to compute the equations of the tangent cone of *one* curve.

For this reason, here we show an algorithm to compute the tangent cone of a monomial curve (such that $\beta>3$) whose cartesian equations are known.

6.2 Let $j, l, v(j), v(l)$ as in paragraph 4. As $\beta>3$, $T(v(j))=v(j)^-$.

If $u\in M(S)$ we denote its i-th component with $u(j)$ $F_1(u)$ denotes $F(u)$ if $u\in \mathbb{N}^3$, $G(u)$ if $u\in M(S)$.

6.3 Initially: $I:=2$, $T_0:=v(l)^+$, $T_1:=(v(j)-v(l))^-$, $T_2:=v(j)^-$; $\mathcal{K}_2:=$ $:=(F_1(T_0), F_1(T_1), F_1(T_2))$; $w_{21}:=v(j)-v(l)$; $w_{22}:=-v(j)$, where the signs are chosen in such a way that $w_{2i}(3)>0$. The algorithm then proceeds by iteration of the following procedure.

Case			Case of 1.3	α	β	Min. of kind 3	Min. of kind 2	T-min.
A_{11}		$n_3' < D(1)n_2'$ $n_3' - n_1 < D(1)(n_2' - n_1)$	$2b$	2	2	$v(0)$	$v(1)$	$v(1)$
A_{1k}	$1 < k < r$	$n_3' < D(1)n_2'$ $D(k-1)(n_2' - n_1) < n_3' - n_1 < D(k)(n_2' - n_1)$	$2b$	2	$1 + k + \lvert K(0, i_k)\rvert$	$v(0)$	$v(1)$	$v(i_k)$
A_{1r}		$n_3' < D(1)n_2'$ $D(r-1)(n_2' - n_1) < n_3' - n_1$	$2b$	2	$1 + r + \lvert K(0, i_r)\rvert$	$v(0)$	$v(1)$	$v(i_r)$
A_{jj}	$1 < j < r$	$D(j-1)n_2' < n_3' < D(j)n_2'$ $n_3' - n_1 < D(j)(n_2' - n_1)$	1	3	3	$v(i_{j-1})$	$v(i_j)$	$v(i_j)$
A_{jk}	$1 < j < k < r$	$D(j-1)n_2' < n_3' < D(j)n_2'$ $D(k-1)(n_2' - n_1) < n_3' - n_1 < D(k)(n_2' - n_1)$	1	3	$2 + k - j + \lvert K(i_{j-1}, i_k)\rvert$	$v(i_{j-1})$	$v(i_j)$	$v(i_k)$
A_{jr}	$1 < j < r$	$D(j-1)n_2' < n_3' < D(j)n_2'$ $D(r-1)(n_2' - n_1) < n_3' - n_1$	1	3	$2 + r - j + \lvert K(i_{j-1}, i_r)\rvert$	$v(i_{j-1})$	$v(i_j)$	$v(i_r)$
B_{jj}	$1 < j < r$	$n_3' = D(j-1)n_2'$ $n_3' - n_1 < D(j)(n_2' - n_1)$	$2a$	2	3	$v(i_{j-1})$	$v(i_{j-1})$	$v(i_j)$
B_{jk}	$1 < j < k < r$	$n_3' = D(j-1)n_2'$ $D(k-1)(n_2' - n_1) < n_3' - n_1 < D(k)(n_2' - n_1)$	$2a$	2	$2 + k - j + \lvert K(i_{j-1}, i_k)\rvert$	$v(i_{j-1})$	$v(i_{j-1})$	$v(i_k)$
B_{jr}	$1 < j < r$	$n_3' = D(j-1)n_2'$ $D(r-1)(n_2' - n_1) < n_3' - n_1$	$2a$	2	$2 + r - j + \lvert K(i_{j-1}, i_r)\rvert$	$v(i_{j-1})$	$v(i_{j-1})$	$v(i_r)$
C		$n_3' > D(r-1)n_2'$	$2c$	2	2	$v(i_{r-1})$	$v(i_{r-1})$ or $V(i_r)$	$v(i_r)$

6.4 Let $I=j$.

Case 1: If w_{j2} is of kind 2 (in which case w_{j1} must be of kind 1), $T(w_{j2})=w_{j2}^-$, $w_{j2}(3)<w_{j1}(3)$, then:

$$I:=j+1,\quad w_{(j+1)1}:=w_{j2},\quad w_{(j+1)2}:=w_{j1}-w_{j2}\quad \text{(which is of kind 1);}$$

$$T_{j+1}:=w_{(j+1)2}^-,\quad \mathcal{K}_{j+1}:=\mathcal{K}_j+(F_1(T_{j+1})).$$

The procedure is repeated.

Case 2: If w_{j2} is of kind 2, $T(w_{j2})=w_{j2}^-$, $w_{j2}(3)>w_{j1}(3)$, then:

$$I:=j+1,\quad w_{(j+1)1}:=w_{j2},\quad w_{(j+1)2}:=w_{j2}-w_{j1}\quad \text{(which is of kind 2);}$$

$$T_{j+1}:=T(w_{(j+1)2}),\quad \mathcal{K}_{j+1}:=\mathcal{K}_j+(F_1(T_{j+1})).$$

The procedure is repeated.

Case 3: If w_{j2} is of kind 2, $T(w_{j2})\neq w_{j2}^-$, then the algorithm terminates. $T(\mathfrak{C})=\mathcal{K}_j$.

Case 4: If w_{j2} is of kind 1 (in which case w_{j1} is of kind 2), and $w_{j2}(3)>w_{j1}(3)$ then:

$$I:=j+1,\quad w_{(j+1)1}:=w_{j1},\quad w_{(j+1)2}:=w_{j2}-w_{j1}\quad \text{(which is of kind 1);}$$

$$T_{j+1}:=w_{(j+1)2}^-,\quad \mathcal{K}_{j+1}:=\mathcal{K}_j+(F_1(T_{j+1})).$$

The procedure is repeated.

Case 5: If w_{j2} is of kind 1 and $w_{j2}(3)<w_{j1}(3)$ then:

$$I:=j+1,\quad w_{(j+1)1}:=w_{j2},\quad w_{(j+1)2}:=w_{j1}-w_{j2}\quad \text{(which is of kind 2);}$$

$$T_{j+1}:=T(w_{(j+1)2}),\quad \mathcal{K}_{j+1}:=\mathcal{K}_j+(F_1(T_{j+1})).$$

The procedure is repeated.

6.5 Termination of the algorithm is ensured by the fact that for any j, $w_{j2}\in M(S)$ and $F_1(T_j)\notin\mathcal{K}_{j-1}$.

We omit here the proof of its correctness which relies on showing that if either a $v(i)$ or a $w(i)$ is not generated by the algorithm, then $i\notin I_1\cup I_2$ [4].

As for its complexity, it requires less than $6(\beta-2)$ assignations and less than $4(\beta-2)$ conditional statements, to check which of the third components of w_{j1} is less, and to check which is $T(w_{j2})$ if w_{j2} is of kind 2 (the kind of w_{j2} is not to be checked as it can be predicted at any step).

7. - Examples

7.1 Let $S_n = (3n+1, 5n+2, 12n+5)$, $\mathfrak{C}_n = \mathfrak{C}(S_n)$.

Then $q_1 = q_2 = 1$; $R(1) = 2n+1$, $R(2) = n+1$, $R(3) = 1$; $A(1) = -8n-3$, $A(2) = -4n-1$, $A(3) = 1$.

Then $I(\mathfrak{C}_n) = \big(G(v(2)), G(v(3)), G(v(3)-v(2))\big)$ where

$$v(2) = (4n+1, 2, -n-1)$$

$$v(3) = (-1, 3, -1)$$

$$v(3) - v(2) = (-4n-2, 1, n)$$

$$T(\mathfrak{C}_n) = (X_3^{n+1}, X_1X_2, \{X_2^{2i+1}X_3^{n-i}\ i = 1 \dots n\}, X_2^{2n+1}).$$

So $\beta = n+3$.

7.2 Let $S_n = (3n, 3n+1, 6n-1)$, $\mathfrak{C}_n = \mathfrak{C}(S_n)$.

Then $q_1 = q_2 = 1$; $R(i) = 3n - i$; $A(i) = 3i - 6n + 1$:

$$I(\mathfrak{C}_n) = \big(G(v(2n-1)), G(v(2n)), G(v(2n) - v(2n-1))\big)$$

where

$$v(2n-1) = (2, 2n-1, -n-1)$$

$$v(2n) = (-1, 2n, -n)$$

$$v(2n) - v(2n-1) = (-3, 1, 1)$$

$$T(\mathfrak{C}_n) = (X_3^{n+1}, X_2X_3, \{X_1^{2i+1}X_3^{n-i} : i = 0 \dots n-2\}, X_2^{2n-1} - X_1^{2n-2}X_3).$$

So $\beta = n+2$.

7.3 Example 7.1 is probably « minimal » and example 7.2 « maximal », in the sense that machine computation suggests that if a curve has multiplicity not greater than $3n$, its tangent cone can be generated by at most $n+2$ elements.

7.4 The results of 5.5 show that the classification of curves $\mathfrak{C}_{\lambda\mu} = \mathfrak{C}(S_{\lambda\mu})$ as regards the equations of the curves and of their tangent cone, can be reduced to the solutions of linear systems of inequalities in λ and μ. Such a classification can be easily performed by a computer. For details about the algorithm see [4]. The classification of all monomial curves of multiplicity 5, performed by a BASIC program (about 200 instructions) is given in the following table. The first two numbers give the values of α and β; the last letters indicate the appropriate equations of the curve and of the tangent cone, which can be found at the bottom of the table.

		$(5,\ 5\lambda,\ 5\mu+1)$				
2	2				a	A
		$(5,\ 5\lambda,\ 5\mu+2)$				
2	2				a	A
		$(5,\ 5\lambda,\ 5\mu+3)$				
2	2				a	A
		$(5,\ 5\lambda,\ 5\mu+4)$				
2	2				a	A
		$(5,\ 5\lambda+1,\ 5\mu)$				
2	2				b	B
		$(5,\ 5\lambda+1,\ 5\mu+1)$				
2	2				c	B
		$(5,\ 5\lambda+1,\ 5\mu+2)$				
3	3			$\lambda\leq\mu\leq 2\lambda-1$	d	C
2	2			$\mu>2\lambda$	e	B
		$(5,\ 5\lambda+1,\ 5\mu+3)$				
3	3			$\lambda\leq\mu\leq 3\lambda-2$	f	D
3	4			$\mu=3\lambda-1$	f	E
2	2			$\mu>3\lambda$	g	B
		$(5,\ 5\lambda+1,\ 5\mu+4)$				
3	3		$\lambda=2\nu$	$2\nu\leq\mu\leq 3\nu-1$	h	F
3	3	$1\leq\nu$	$\lambda=2\nu+1$	$2\nu+1\leq\mu\leq 3\nu$	h	F
2	3		$\lambda=2\nu+1$	$\mu=3\nu+1$	i	G
3	3		$\lambda=2\nu$	$3\nu\leq\mu\leq 8\nu-3$	l	G
3	3	$1\leq\nu$	$\lambda=2\nu+1$	$3\nu+2\leq\mu\leq 8\nu+1$	l	G
3	4			$4\lambda-2\leq\mu\leq 4\lambda-1$	l	H
2	2			$\mu>4\lambda$	m	B
		$(5,\ 5\lambda+2,\ 5\mu)$				
2	2				b	B
		$(5,\ 5\lambda+2,\ 5\mu+1)$				
3	3			$\lambda+1\leq\mu\leq 3\lambda-1$	f	D
3	4			$\mu=3\lambda$	f	E
2	2			$\mu>3\lambda+1$	g	B
		$(5,\ 5\lambda+2,\ 5\mu+2)$				
2	2				c	B
		$(5,\ 5\lambda+2,\ 5\mu+3)$				
3	3		$\lambda=2\nu$	$2\nu\leq\mu\leq 3\nu-1$	h	F
3	3		$\lambda=2\nu+1$	$2\nu+1\leq\mu\leq 3\nu+1$	h	F
2	3		$\lambda=2\nu$	$\mu=3\nu$	i	G
3	3		$\lambda=2\nu$	$3\nu+1\leq\mu\leq 8\nu-2$	l	G
3	3		$\lambda=2\nu+1$	$3\nu+2\leq\mu\leq 8\nu+2$	l	G
3	4			$4\lambda-1\leq\mu\leq 4\lambda$	l	H
2	2			$\mu>4\lambda+1$	m	B
		$(5,\ 5\lambda+2,\ 5\mu+4)$				
3	3			$\lambda\leq\mu\leq 2\lambda-1$	d	C
2	2			$\mu>2\lambda$	e	B
		$(5,\ 5\lambda+3,\ 5\mu)$				
2	2			$\mu>\lambda+1$	b	B
		$(5,\ 5\lambda+3,\ 5\mu+1)$				
3	3			$\lambda+1\leq\mu\leq 2\lambda$	d	C
2	2			$\mu>2\lambda+1$	e	B
		$(5,\ 5\lambda+3,\ 5\mu+2)$				
3	3		$\lambda=2\nu$	$2\nu+1\leq\mu\leq 3\nu$	h	F

3	3	$1<\nu$	$\lambda=2\nu+1$	$2\nu+2<\mu<3\nu+1$	h	F
2	3		$\lambda=2\nu+1$	$\mu=3\nu+2$	i	G
3	3		$\lambda=2\nu$	$3\nu+1<\mu<8\nu-1$	l	G
3	3		$\lambda=2\nu+1$	$3\nu+3<\mu<8\nu+3$	l	G
3	4			$4\lambda<\mu<4\lambda+1$	l	H
2	2			$\mu>4\lambda+2$	m	B
		$(5,\ 5\lambda+3,\ 5\mu+3)$				
2	2			$\mu>\lambda+1$	e	B
		$(5,\ 5\lambda+3,\ 5\mu+4)$				
3	3			$\lambda<\mu<3\lambda-1$	f	D
3	4			$\mu=3\lambda$	f	E
2	2			$\mu>3\lambda+1$	g	B
		$(5,\ 5\lambda+4,\ 5\mu)$				
2	2			$\mu>\lambda+1$	b	B
		$(5,\ 5\lambda+4,\ 5\mu+1)$				
3	3		$\lambda=2\nu$	$2\nu+1<\mu<3\nu$	h	F
3	3		$\lambda=2\nu+1$	$2\nu+2<\mu<3\nu+2$	h	F
2	3		$\lambda=2\nu$	$\mu=3\nu+1$	i	G
3	3		$\lambda=2\nu$	$3\nu+2<\mu<8\nu$	l	G
3	3		$\lambda=2\nu+1$	$3\nu+3<\mu<8\nu+4$	l	G
3	4			$4\lambda+1<\mu<4\lambda+2$	l	H
2	2			$\mu>4\lambda+3$	m	B
		$(5,\ 5\lambda+4,\ 5\mu+2)$				
3	3			$\lambda+1<\mu<3\lambda$	f	D
3	4			$\mu=3\lambda+1$	f	E
2	2			$\mu>3\lambda+2$	g	B
		$(5,\ 5\lambda+4,\ 5\mu+3)$				
3	3			$\lambda+1<\mu<2\lambda$	d	C
2	2			$\mu>2\lambda+1$	e	B
		$(5,\ 5\lambda+4,\ 5\mu+4)$				
2	2			$\mu>\lambda+1$	e	B

Patterns of Cartesian equations éelements of $M(S)$ are given; values of exponents of X_1 xhich are not given are easy to compute for a given curve)

a)	$(-r_{31},\ 0,\ 5)$	$(-r_{21},\ 1,\ 0)$	
b)	$(-r_{31},\ 0,\ 1)$	$(-r_{21},\ 5,\ 0)$	
c)	$(-r_{31},\ -1,\ 1)$	$(-r_{21},\ 5,\ 0)$	
d)	$(-r_{31},\ -1,\ 3)$	$(-r_{21},\ 2,\ -1)$	$(e_1,\ -1,\ -2)$
e)	$(-r_{31},\ -2,\ 1)$	$(-r_{21},\ 5,\ 0)$	
f)	$(-r_{31},\ -1,\ 2)$	$(-r_{21},\ 3,\ -1)$	$(e_1,\ -2,\ -1)$
g)	$(-r_{31},\ -3,\ 1)$	$(-r_{21},\ 5,\ 0)$	
h)	$(-r_{31},\ -2,\ 3)$	$(-r_{21},\ 3,\ -2)$	$(e_1,\ -1,\ -1)$
i)	$(0,\ -3,\ 2)$	$(e_1,\ -1,\ -1)$	
l)	$(-r_{31},\ -3,\ 2)$	$(-r_{21},\ 4,\ -1)$	$(e_1,\ -1,\ -1)$
m)	$(-r_{31},\ -4,\ 1)$	$(-r_{21},\ 5,\ 0)$	

Patterns of tangent cone equations (not given values of exponents of X_1 are easy to compute for a given curve)

A) X_2^2 X_3
B) X_2 X_3^2
C) X_2^2 $X_2X_3^2$ X_3^3 or $X_3^3-X_1X_3$
D) X_2^2 $X_2^2X_3$ X_3^3 or $X_3^3-X_1^2X_3$
E) X_2^2 $X_2^2X_3$ X_1X_3 X_3^3
F) X_2^2 X_2X_3 X_3^3 or $X_3^3-X_1X_3^2$
G) X_2^2 X_2X_3 X_3^4 or $X_3^4-X_1^2X_3$
H) X_2^2 X_2X_3 $X_1^4X_3$ X_3^3

REFERENCES

[1] C. Graetz, *Ein Program zur Berechnung von Invarianten eines lokalen Rings der Dimension* 1, Diplomarbeit, Regensburg, 1979.

[2] J. Herzog, *Generators and relations of Abelian semigroups and semigroup rings*, Manuscripta Math., **3** (1970), 175-193.

[3] L. Robbiano - G. Valla, *On the equations defining tangent cones*, Math. Proc. Camb. Phil. Soc., **88** (1980), 281-297.

[4] F. Mora, *Algorithms on monomial curves*, Rapporti Ist. Mat. Univ. Genova, **88** (1980).

[5] G. Valla, *On determinantal ideals which are set-theoretic complete intersections*, Comp. Math., **42** (1980), 3-12.

[6] H. Bresinsky - B. Renschuch, *Basisbestimmung Veronesescher Projektionsideale mit allgemeiner Nullstelle* $(t_0^m, t_0^{m-r}t_1^r, t_0^{m-s}t_1^s, t_1^m)$, Math. Nachr., **96** (1980), 257-289.

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