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Stability under Localization of the Phragmén-Lindelöf Principle (**)

Stabilità del principio di Phragmén-Lindelöf rispetto alla localizzazione

SUNTO. Si confronta il principio di Phragmén-Lindelöf su un germe di varietà analitica con quello sul cono tangente. Se ne deduce che la risolubilità analitica su un aperto convesso di un operatore differenziale (omogeneo) a coefficienti costanti implica quella degli operatori ottenuti per localizzazione del polinomio caratteristico in punti reali e si individuano alcune condizioni geometriche imposte dalla risolubilità analitica deducendole da analoghe condizioni per la risolubilità di operatori degeneri.

INTRODUCTION

We consider a partial differential operator $P = P(D)$ in $\mathbb{R}^n$ with constant coefficients and the canonically associated polynomial $P = P(\zeta)$, $\zeta \in \mathbb{C}^n$, an open set Ω of $\mathbb{R}^n$, and the space $A(\Omega)$ of real analytic functions on Ω.

Our purpose is to compare the global analytic solvability on Ω of the operator P with the solvability of the operators P_ξ obtained by localizing the characteristic polynomial P at real points ξ. In the analogous problem of comparing the solvability of P with that of its principal part P_∞ (which is P's localization at ∞), a positive answer is given in [3] by Hörmander. In fact he characterizes the open convex sets Ω for which $PA(\Omega) = A(\Omega)$, as the sets which admit a « Phragmén-Lindelöf principle » on the asymptotic variety $V(P_\infty)$ ($\{P_\infty(\zeta) = 0,\ \zeta \in \mathbb{C}^n\}$). Under this criterion, to which we'll always refer, we assume without loss of generality P homogeneous (and Ω convex).

Since the cone $V(P_\xi)$ ($\{P_\xi(\zeta) = 0,\ \zeta \in \mathbb{C}^n\}$) is fairly close to $V(P)$'s germ at ξ we must prove a result of stability under perturbation of such a Ph. L. principle.

First we need a definition of Ph. L. principle in neighbourhoods of ξ on $V(P)$ which is, roughly speaking, linearly dependent on the diameter of the

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neighbourhoods. In fact our object is a translation of estimates for functionsde-fined on $V(P)$ near ξ into analogous estimates for functions defined on $V(P_\xi)$ (and conversely) with small residues depending on the local divarication of $V(P)$ from $V(P_\xi)$; so we need to approach ξ more and more in order to get a better approximation of $V(P)$ by means of $V(P_\xi)$.

Next we prove that the Ph. L. principle on $V(P)$ in neighbourhoods of ξ and that on $V(P_\xi)$ (in neighbourhoods of the origin) can be made as close as we like provided we correspondingly contract the neighbourhoods. However the homogeneity of $V(P_\xi)$ in the variable $\zeta-\xi$ and the non homogeneity of $V(P)$ give asymmetrical conclusions; i.e. the Ph. L. on $V(P)$ at ξ implies that on $V(P_\xi)$ but not the converse. The last was not unexpected because it is very easy to find operators which are not solvable on any convex open set even if all their localizations are solvable on everyone. We believed however that the conclusions were symmetrical when $V(P)$'s germ at ξ is locally hyperbolic i.e. when it can be normalized in order that every (finite) fiber is real if it has real projection.

In this case indeed, much more than in general, we relate the Ph. L. on $V(P)$ at ξ to that on $V(P_\xi)$. In this connection however, we have found the following counterexemple in $\mathbb{R}^4$ (Theorem 3.4):

$$P(\zeta)=\zeta_4^2\zeta_3^2-\zeta_4^2\zeta_1^2-\zeta_2^4, \qquad \xi=(0,0,0,1),$$

$$P_\xi(\zeta)=\zeta_3^2-\zeta_1^2, \qquad \Omega=\{x\in\mathbb{R}^4: x_1<0\}.$$

We show that *the half space Ω doesn't admit the Ph. L. on $V(P)$ at ξ even if it does on $V(P_\xi)$ and even if $V(P)$ is locally hyperbolic at ξ* (probably the reducibility of $V(P_\xi)$ plays an essential role here).

Summarizing up our results from the point of view of the existence of real analytic solutions for differential equations we obtain the following statement (Theorem 2.5)

$PA(\Omega)=A(\Omega)$ *implies* $P_\xi A(\Omega)=A(\Omega)$ *for every localization at real points* ξ.

Since P_ξ has a non empty lineality (i.e. it depends only on the variables of the space orthogonal to ξ), then strong geometrical conditions for Ω arise from the global solvability of P_ξ that are, because of our theorem, conditions for the solvability of P too. We infer when P has some second order irreducible localization at non null real points and Ω has C^1 boundary (Theorem 2.6)

$PA(\Omega)=A(\Omega)$ *implies* Ω *unbounded*.

On the other hand when P (real) has irreducible regular germs at every real non null point (and therefore localizations which are products of real linear terms), then easily (Theorem 3.3)

$PA(\Omega)=A(\Omega)$ *for every open convex set* Ω.

It seems therefore senseless to look for real analytic solutions on bounded regular open sets unless we are concerned with (real) simply characteristic operators or with regular irreducible germs at every non null real characteristic.

Next we deal with operators which have hyperbolic localizations of degree $\leqslant 2$. We prove that from the solvability of P on Ω, precise geometrical conditions for Ω arise, conditions which are related to the conical (convex) supports of the fundamental solutions of the localizations. This is a consequence, via the previous theorem, of the following statement relative to a second order irreducible degenerate hyperbolic (with respect to some direction $\pm v$) form (Lemma 2.7)

$PA(\Omega) = A(\Omega)$ (if and) only if either $x + \Gamma^*(P, v) \cap \Omega = \emptyset$ or $x - \Gamma^* \cdot (P, v) \cap \Omega = \emptyset$ for every x on Ω's boundary.

We wish to thank professor M. Nacinovich who initiated us in this subject.

1. - A local version of the Phragmén-Lindelöf principle.

The results of this paper rely on the following criterion of analytic solvability on open convex sets.

Let P_∞ be the principal part of P; K, K' be compact convex sets of $\mathbb{R}^n$. We say that the (global) Phragmén-Lindelöf principle on the variety $\{P_\infty(\zeta) = 0\}$ ($V(P_\infty)$ for brevity) relative to K, K' holds, if there is a constant δ s.t. for every weakly plurisubharmonic function φ on $V(P_\infty)$ the following implication is true:

$$(1) \qquad \begin{cases} \varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta|\zeta| & \text{when} \quad P_\infty(\zeta) = 0\,, \\ \varphi(\zeta) \leqslant 0 \quad \text{when} \quad P_\infty(\zeta) = 0\,,\ \zeta \in \mathbb{R}^n \end{cases}$$

implies

$$(2) \qquad \varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) \quad \text{when} \quad P_\infty(\zeta) = 0\,.$$

We say that an open convex set Ω admits the (global) Ph. L. principle on $V(P_\infty)$ if for every compact convex set $K \subset\subset \Omega$ we can find another compact convex set K', with $K \subset K' \subset \Omega$, and a positive constant δ s.t. the implication (1) $\Rightarrow$ (2) holds $\forall \varphi$ w.p.s.h. on $V(P_\infty)$.

In [3] Hörmander proves that $PA(\Omega) = A(\Omega)$ if and only if Ω admits the Ph. L. principle on $V(P_\infty)$ so proving among other things that the lower order terms of P don't play any role in analytic convexity; therefore P will be homogeneous in the following. Hörmander gives again in [3] a local version of the Ph. L. principle proving that the implication (1) $\Rightarrow$ (2) is equivalent to analogous implications in neighbourhoods of every unit real zero ξ of P. That which matters to us is to explain how δ depends on the ray r of such neighbourhoods in order to exclude that $\delta = o(r)$, $r \to 0$. In fact we'll deal with better and better approximations of P by means of P_ξ and this requires $r \to 0$; so we need a Ph. L. principle which filters homogeneously to ξ.

LEMMA 1.1. *There is a (universal) constant* $\alpha > 0$ *for which the following statement holds. Let be given* δ, $K \subset\subset \mathbf{R}^n$, $r \leqslant \frac{1}{2}$, $\theta \in \mathbf{C}^n$ *s.t.* $P(\theta) = 0$, $|\mathrm{Re}\,\theta| = 1$, $|\mathrm{Im}\,\theta| < r/2$; $\forall \varphi$ *w.p.s.h. on* $\{P(\zeta) = 0,\ |\zeta - \theta| < r\}$ *and there verifying*:

$$\varphi(\zeta) \leqslant H_K(\mathrm{Im}\,\zeta) + \alpha\delta r\,,$$

$$\varphi(\zeta) \leqslant 0\,, \qquad \zeta \in \mathbf{R}^n\,,$$

we can find ψ *w.p.s.h. on* $\{P(\zeta) = 0\}$ *there verifying*:

$$\psi(\zeta) \leqslant H_K(\mathrm{Im}\,\zeta) + \delta|\zeta|\,,$$

$$\psi(\zeta) \leqslant 0\,, \qquad \zeta \in \mathbf{R}^n\,,$$

$$\psi(\theta) \geqslant \varphi(\theta)\,.$$

PROOF. Take $\chi \in C_c^\infty(\mathbf{R}^n)$ s.t.: supp χ is contained in the unit ball with center at the origin; $\chi \geqslant 0$; χ is even and unit in L^1 norm. Set $\phi(\zeta) = \log|\tilde{\chi}(\zeta)|$ where $\tilde{\chi}$ is the Laplace transform of χ:

$$\tilde{\chi}(\zeta) = \int_{\mathbf{R}^n} \exp\left[-i\langle x, \zeta\rangle\right] \chi(x)\,dx\,.$$

It follows:

$$\phi(\zeta) \leqslant |\mathrm{Im}\,\zeta|\,, \quad \zeta \in \mathbf{C}^n\,; \qquad \phi(\zeta) \lneqq |\mathrm{Im}\,\zeta|\,, \quad \zeta \in \mathbf{C}^n - \{0\}\,.$$

So there is a $\sigma > 0$ s.t.:

$$\phi(\zeta) - |\mathrm{Im}\,\zeta| < -\sigma\,, \qquad \tfrac{1}{2} < |\zeta| < \tfrac{3}{2}\,.$$

Consider the p.s.h. function $\phi_r(\zeta) = r\phi(\zeta/r)$. Obviously:

$$\phi_r(\zeta) \leqslant |\mathrm{Im}\,\zeta|\,, \quad \zeta \in \mathbf{C}^n\,; \qquad \phi_r(\zeta) - |\mathrm{Im}\,\zeta| < -\sigma r\,, \qquad r/2 < |\zeta| < 3r/2$$

where we can suppose w.l.o.g. $\sigma \leqslant 1$. Set $\alpha = \frac{2}{5}\sigma$; if φ is as in the hypothesis define:

$$\psi_1(\zeta) = \varphi(\zeta) + \tfrac{2}{5}\delta\phi_r(\zeta - \mathrm{Re}\,\theta)\,, \qquad P(\zeta) = 0\,, \qquad \zeta \in B(\theta, r) = \{|\zeta - \theta| < r\}\,.$$

Clearly:

$$\psi_1(\zeta) \leqslant H_K(\mathrm{Im}\,\zeta) + \delta r \leqslant H_K(\mathrm{Im}\,\zeta) + \delta(1 - r)\,, \qquad P(\zeta) = 0\,,\ \zeta \in B(\theta, r)\,,\ r \leqslant \tfrac{1}{2}\,.$$

If ζ is a point on the boundary of $B(\theta, r)$ we have $r/2 \leqslant |\zeta - \mathrm{Re}\,\theta| \leqslant 3r/2$ and therefore $\phi_r(\zeta - \mathrm{Re}\,\theta) < |\mathrm{Im}\,\zeta| - \sigma r$. So for ζ near the boundary of $B(\theta, r)$:

$$\psi_1(\zeta) \leqslant H_K(\mathrm{Im}\,\zeta) + \tfrac{2}{5}\sigma\delta r + \tfrac{2}{5}\delta|\mathrm{Im}\,\zeta| - \tfrac{2}{5}\delta\sigma r = H_K(\mathrm{Im}\,\zeta) + \tfrac{2}{5}\delta|\mathrm{Im}\,\zeta|\,.$$

We define a function on the whole $V(P)$ variety by setting:

$$\psi(\zeta) = \max\left(\psi_1(\zeta), H_K(\operatorname{Im}\zeta) + \tfrac{2}{5}\delta|\operatorname{Im}\zeta|\right) \qquad \text{if } P(\zeta) = 0\,,\ \zeta \in B(\theta, r)\,,$$

$$\psi(\zeta) = H_K(\operatorname{Im}\zeta) + \tfrac{2}{5}\delta|\operatorname{Im}\zeta| \qquad \text{when } P(\zeta) = 0\,,\ \zeta \notin B(\theta, r)\,.$$

ψ is clearly w.p.s.h. and has all the required properties.

In fact:

$$\psi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta(1-r) \leqslant H_K(\operatorname{Im}\zeta) + \delta|\zeta|\,, \qquad P(\zeta) = 0\,,\ \zeta \in B(\theta, r)$$

(and the same estimate holds for $\zeta \notin B(\theta, r)$ also).

Besides $\psi(\theta) \geqslant \varphi(\theta)$ since $\phi_r(\theta - \operatorname{Re}\theta) = \phi_r(i\operatorname{Im}\theta) \geqslant 0$.

Let K, K' be compact convex sets of $\mathbf{R}^n$ and let K' cover the ε-neighbourhood K_ε of K for some ε; let α be as in Lemma 1.1.

LEMMA 1.2. *Assume that the global Ph. L. principle on $V(P)$ relative to K, K', (δ) holds. Then $\forall r \leqslant \frac{1}{2}$, $\forall \theta \in \mathbf{C}^n$, $P(\theta) = 0$, $|\operatorname{Re}\theta| = 1$, $|\operatorname{Im}\theta| < r/2$:*

$$\varphi(\theta) \leqslant H_{K'}(\operatorname{Im}\theta) \tag{1}$$

if φ is w.p.s.h. on $\{P(\zeta) = 0,\ \zeta \in B(\theta, r)\}$ there satisfying:

$$\begin{cases} \varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \alpha\delta r\,, \\ \varphi(\zeta) \leqslant 0\,, \qquad \zeta \in \mathbf{R}^n\,. \end{cases} \tag{2}$$

On the contrary suppose there are $r \leqslant \frac{1}{2}$ and δ' s.t. $\forall \theta \in \mathbf{C}^n$, $P(\theta) = 0$, $|\operatorname{Re}\theta| = 1$, $|\operatorname{Im}\theta| < r/2$:

$$\varphi(\theta) \leqslant H_{K'}(\operatorname{Im}\theta) \tag{1'}$$

if φ is w.p.s.h. on $\{P(\zeta) = 0,\ \zeta \in B(\theta, r)\}$ there satisfying:

$$\begin{cases} \varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta' r\,, \\ \varphi(\zeta) \leqslant 0\,, \qquad \zeta \in \mathbf{R}^n\,; \end{cases} \tag{2'}$$

then the global Ph. L. principle on $V(P)$ relative to K, K', (δ) holds with

$$\delta = \min\left(\tfrac{4}{7}\delta' r, \varepsilon(1 + 2/r)^{-1}\right).$$

PROOF. Let φ w.p.s.h. on $\{P(\zeta)=0,\ \zeta \in B(\theta, r)\}$ verify (2). For Lemma 1.1 there exists ψ w.p.s.h. on $\{P(\zeta) = 0\}$ with $\psi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta|\zeta|$; $\psi(\zeta) \leqslant 0$, $\zeta \in \mathbf{R}^n$; $\psi(\theta) \geqslant \varphi(\theta)$. It follows:

$$\varphi(\theta) \leqslant \psi(\theta) \leqslant H_{K'}(\operatorname{Im}\theta)\,.$$

Viceversa let φ be w.p.s.h. on $\{P(\zeta)=0\}$ and there verify: $\varphi(\zeta)\leqslant H_K(\operatorname{Im}\zeta)+$ $+\delta|\zeta|$; $\varphi(\zeta)\leqslant 0$, $\quad \zeta\in\mathbb{R}^n$, where

$$\delta=\min\left(\tfrac{4}{7}\delta' r,\ \varepsilon\left(1+\frac{2}{r}\right)^{-1}\right), \qquad r\leqslant\frac{1}{2}.$$

If $P(\zeta)=0$ and $|\operatorname{Im}\zeta|\geqslant(r/2)|\operatorname{Re}\zeta|$ then

$$\varphi(\zeta)\leqslant H_K(\operatorname{Im}\zeta)+\delta\left(1+\frac{2}{r}\right)|\operatorname{Im}\zeta|\leqslant H_{K'}(\operatorname{Im}\zeta)+\varepsilon|\operatorname{Im}\zeta|\leqslant H_{K'}(\operatorname{Im}\zeta)$$

(since $K_\varepsilon\subset K'$).

If on the contrary $|\operatorname{Im}\zeta|<(r/2)|\operatorname{Re}\zeta|$ set $\theta=\zeta|\operatorname{Re}\zeta|^{-1}$ and define $\psi(\eta)=$ $=|\operatorname{Re}\zeta|^{-1}\varphi(|\operatorname{Re}\zeta|\eta)$, $\quad \eta\in\mathbb{C}^n$, $P(\eta)=0$. ψ is w.p.s.h. on $\{P(\eta)=0\}$ and, when $\eta\in B(\theta,r)$, it verifies (2′) since $\delta\leqslant\frac{4}{7}\delta' r$. It follows $\psi(\theta)\leqslant H_{K'}(\operatorname{Im}\theta)$ from which $\varphi(\zeta)\leqslant H_{K'}(\operatorname{Im}\zeta)$.

Lemma 1.3. *Assume that the global Ph. L. principle on $V(P)$ relative to K, K', (δ) holds; set $\delta'=\min(\alpha\delta/2,\varepsilon/4)$. Then $\forall 0<r^1<r^2\leqslant\frac{1}{2}$ and $\forall\xi\in\mathbb{R}^n$, $|\xi|=1$, $P(\xi)=0$, we have*

$$\varphi(\zeta)\leqslant K_{K'}(\operatorname{Im}\zeta)\,,\qquad P(\zeta)=0\,,\qquad \zeta\in B(\xi,r^1)$$

if φ is w.p.s.h. on $\{P(\zeta)=0,\ \zeta\in B(\xi,r^2)\}$ there satisfying:

$$\varphi(\zeta)\leqslant H_K(\operatorname{Im}\zeta)+\delta'\frac{r^2-r^1}{1+r^1},$$

$$\varphi(\zeta)\leqslant 0\,,\qquad\qquad \zeta\in\mathbb{R}^n\,.$$

On the contrary assume that $\forall\xi$ as above there are $0<r^1_\xi<r^2_\xi\leqslant\frac{1}{2}$ and δ_ξ (1) s.t.:

$$\varphi(\zeta)\leqslant H_{K'}(\operatorname{Im}\zeta)\,,\qquad P(\zeta)=0\,,\qquad \zeta\in B(\xi,r^1_\xi)$$

for every φ w.p.s.h. on $\{P(\zeta)=0,\ \zeta\in B(\xi,r^2_\xi)\}$ there verifying:

$$\varphi(\zeta)\leqslant H_K(\operatorname{Im}\zeta)+\delta_\xi r^2_\xi\,,$$

$$\varphi(\zeta)\leqslant 0\,,\qquad \zeta\in\mathbb{R}^n\,.$$

If then

$$B=\bigcup_{i=1}^{s}B(\xi_i,r^1_{\xi_i})\supset\{\zeta:\ P(\zeta)=0,\ |\zeta|=1,\ \zeta\in\mathbb{R}^n\}$$

set

$$0<\sigma=\inf\{|\operatorname{Im}\zeta|:\ P(\zeta)=0,\ |\operatorname{Re}\zeta|=1,\ \zeta\notin B\}$$

(1) The index ξ denotes the eventual dependence on ξ.

and

$$\delta = \inf\left(\varepsilon\left(1+\frac{1}{\sigma}\right)^{-1}, \tfrac{2}{3}\inf_{i=1,\dots,s}\delta_{\xi_i} r^2_{\xi_i}\right).$$

It follows that the global Ph. L. on $V(P)$ relative to K, K', (δ) holds.

PROOF. (*a*) From Lemma 1.2 it follows that $\forall\zeta$ with $|\mathrm{Im}\,\zeta| < (r/2)|\mathrm{Re}\,\zeta|$, $r \leqslant \frac{1}{2}$:

$$\varphi(\zeta) \leqslant H_{K'}(\mathrm{Im}\,\zeta)$$

if φ is w.p.s.h. on $\{P(\eta) = 0,\ \eta \in B(\zeta, r|\mathrm{Re}\,\zeta|)\}$ there satisfying:

$$\varphi(\eta) \leqslant H_K(\mathrm{Im}\,\eta) + \alpha\delta r|\mathrm{Re}\,\zeta|\ ; \qquad \varphi(\eta) \leqslant 0\ , \quad \eta \in \mathbf{R}^n\ .$$

In fact if φ is as before it's enough to apply the lemma to the function:

$$\psi(\eta) = |\mathrm{Re}\,\zeta|^{-1}\varphi(|\mathrm{Re}\,\zeta|\eta)\ , \qquad P(\eta) = 0\ ,\ \eta \in B(\zeta|\mathrm{Re}\,\zeta|^{-1}, r)\ .$$

(*b*) Let $r = (r^2 - r^1)/(1 + r^1)$ (clearly $r \leqslant \frac{1}{2}$); ξ and φ as in hypothesis. Then φ is w.p.s.h. on $\{P(\eta) = 0,\ \eta \in B(\zeta, r|\mathrm{Re}\,\zeta|)\}$ if $\zeta \in B(\xi, r^1)$.

If $|\mathrm{Im}\,\zeta| < (r/2)|\mathrm{Re}\,\zeta|$ we have $\varphi(\zeta) \leqslant H_{K'}(\mathrm{Im}\,\zeta)$ by (*a*).

If on the contrary $|\mathrm{Im}\,\zeta| \geqslant (r/2)|\mathrm{Re}\,\zeta| \geqslant r/4$ then

$$\varphi(\zeta) \leqslant H_K(\mathrm{Im}\,\zeta) + \delta'\frac{r^2 - r^1}{1 + r^1} \leqslant$$
$$H_K(\mathrm{Im}\,\zeta) + \frac{\varepsilon}{4}\frac{r^2 - r^1}{1 + r^1}\frac{4}{r}|\mathrm{Im}\,\zeta| \leqslant H_K(\mathrm{Im}\,\zeta) + \varepsilon|\mathrm{Im}\,\zeta| \leqslant H_{K'}(\mathrm{Im}\,\zeta)\ .$$

Vice versa suppose $P(\zeta) = 0$ and $\zeta|\mathrm{Re}\,\zeta|^{-1} \notin \bigcup_{i=1}^{s} B(\xi_i, r^1_{\xi_i})$ then $|\mathrm{Im}\,\zeta| \geqslant \sigma\cdot$ $\cdot|\mathrm{Re}\,\zeta|$ and so

$$\varphi(\zeta) \leqslant H_K(\mathrm{Im}\,\zeta) + \varepsilon\left(1 + \frac{1}{\sigma}\right)^{-1}|\zeta| \leqslant H_{K'}(\mathrm{Im}\,\zeta)\ .$$

Otherwise $\zeta|\mathrm{Re}\,\zeta|^{-1} \in B(\xi_i, r^1_{\xi_i})$ for some i; set $\psi(\eta) = |\mathrm{Re}\,\zeta|^{-1}\varphi(|\mathrm{Re}\,\zeta|\eta)$, $\eta \in \mathbf{C}^n$, $P(\eta) = 0$. Then the function ψ verifies on $\{P(\eta) = 0,\ \eta \in B(\xi_i, r^2_{\xi_i})\}$ the following estimates:

$$\psi(\eta) \leqslant H_K(\mathrm{Im}\,\eta) + \tfrac{2}{3}\delta_{\xi_i} r^2_{\xi_i}|\eta| \leqslant H_K(\mathrm{Im}\,\eta) + \delta_{\xi_i} r^2_{\xi_i}\ ,$$
$$\psi(\eta) \leqslant 0\ , \qquad \eta \in \mathbf{R}^n\ .$$

Hence by hypothesis: $\psi(\zeta|\mathrm{Re}\,\zeta|^{-1}) \leqslant |\mathrm{Re}\,\zeta|^{-1}H_{K'}(\mathrm{Im}\,\zeta)$, from which:

$$\varphi(\zeta) \leqslant H_{K'}(\mathrm{Im}\,\zeta)\ .$$

Setting $r^2 = r$, $r^1 = r/2$, $r \leqslant \frac{1}{2}$, Lemma 1.3 can be paraphrased as follows. The (global) Ph. L. on $V(P)$ relative to K, K', (δ), implies, for every unit real zero ξ of P and for every $r \leqslant \frac{1}{2}$:

$$(1) \qquad \varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta)\,, \qquad P(\zeta) = 0\,, \qquad \zeta \in B\left(\xi, \frac{r}{2}\right)$$

if φ is w.p.s.h. on $\{P(\zeta) = 0,\ \zeta \in B(\xi, r)\}$ there verifying:

$$(2) \qquad \varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta' r\,; \qquad \varphi(\zeta) \leqslant 0\,, \qquad \zeta \in \mathbb{R}^n\,.$$

where $\delta' = \frac{2}{5}\min(\alpha\delta/2, \varepsilon/4)$ if α is as in Lemma 1.1 and if $K_\varepsilon \subset K'$.

Conversely if for every ξ we can find r_ξ and δ_ξ s.t. for $r \leqslant r_\xi$ and for $\delta' = \delta_\xi$ the implication (2) $\Rightarrow$ (1) is fulfilled, then the (global) Ph. L. on $V(P)$ relative to K, K', (δ) holds with

$$\delta = \inf\left(\varepsilon\left(1 + \frac{1}{\sigma}\right)^{-1}, \frac{2}{3} \inf_{i=1,\dots,s} \delta_{\xi_i} r_{\xi_i}\right)$$

if the union B of the balls with center at the points ξ_i, $i = 1, \dots, s$ and of radius $r_{\xi_i}/2$ covers the unit real part of $V(P)$ and if σ bounds «from below» the imaginary part of every ζ s.t. $P(\zeta) = 0$, $|\operatorname{Re}\zeta| = 1$, $\zeta \notin B$.

Let then ξ be a unit real zero of P, and K, K' be compact convex sets with $K \subset \mathring{K}'$. It's natural to give the following definitions.

Definition 1.1. *We say that the (local) Ph. L. principle on (the germ of) $V(P)$ at ξ relative to K, K' holds, if there are positive constants δ and r_0 s.t. $\forall r \leqslant r_0$:*

$$(1) \qquad \varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta), \qquad P(\zeta) = 0\,, \qquad \zeta \in B\left(\xi, \frac{r}{2}\right)$$

whenever φ is w.p.s.h. on $\{P(\zeta) = 0,\ \zeta \in B(\xi, r)\}$ there verifying:

$$(2) \qquad \varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta r\,; \qquad \varphi(\zeta) \leqslant 0\,, \qquad \zeta \in \mathbb{R}^n\,.$$

Definition 1.2. *We say that an open convex set Ω admits the (local) Ph. L. principle on (the germ of) $V(P)$ at ξ if $\forall K \subset\subset \Omega$ there are $K' \subset\subset \Omega$, δ and r_0 so that the implication (2) $\Rightarrow$ (1) is fulfilled $\forall r \leqslant r_0$.*

From Lemma 1.3 we infer that the global Ph. L. relative to K, K', (δ) implies the local one at every real unit zero ξ relative to K, K', $(\delta', \frac{1}{2})$ with $\delta' = \frac{2}{5}\min(\alpha\delta/2, \varepsilon/4)$. And conversely the local Ph. L. at every ξ relative to K, K', $(\delta_\xi, r_{0\xi})$ implies the global one relative to K, K', (δ) with δ determined as already shown.

Obviously we could also have given an equivalent definition by putting radius r/m, m real > 1 instead of $r/2$ in (1) of Definition 1.1.

2. - Localization of the Phragmén-Lindelöf principle.

Let F be the germ at a point $\xi \in \mathbf{C}^n$ of a function which is analytic in some neighbourhood of ξ. Denote by F_ξ the localization of F in ξ i.e. the first non vanishing term of the expansion of F at ξ. If m is the multiplicity of F, that is the degree of F_ξ, we have:

$$F_\xi(\zeta) = \lim_{t \to 0} t^{-m} F(\xi + t\zeta), \qquad \zeta \in \mathbf{C}^n.$$

We shall say that the germ F is normalized in the ζ_n direction if $F(\xi + (0, \ldots, \zeta_n))$ does not vanish identically. We shall say that the (homogeneous) polynomial F_ξ is normalized in the ζ_n direction if its germ at 0 is such, or equivalently if $F_\xi(0, \ldots, 1) \neq 0$ (in this hypothesis the germ F also is normalized in the ζ_n direction).

Our aim is to show the relation between the analytic set $\{F(\zeta) = 0, \ \zeta \in \mathbf{C}^n\}$ and its tangent cone $\{F_\xi(\zeta) = 0, \ \zeta \in \mathbf{C}^n\}$.

Lemma 2.1. *Assume F_ξ normalized in the ζ_n direction and with degree m; denote by ζ' the variables in the orthogonal to the ζ_n axis.*

There are k and r s.t. the equation, in ζ_n, $F(\xi + (\zeta', \zeta_n)) = 0$, $|\zeta'| < r$ has exactly m zeros $\mu_i(\zeta')$ with $|\mu_i(\zeta')| < kr$ and for such zeros we have the estimation:

$$|\mu_i(\zeta')| < k|\zeta'|, \qquad |\zeta'| < r, \ i = 1, \ldots, m.$$

Proof. Since $F_\xi(0, \ldots, 1) \neq 0$ there exists k so that the zeros $\mu_i^0(\zeta')$ of the equation, in ζ_n, $F_\xi(\zeta', \zeta_n) = 0$ verify:

$$|\mu_i^0(\zeta')| < k|\zeta'|, \qquad i = 1, \ldots, m.$$

Therefore, supposing F_ξ monic in ζ_n,

$$|F_\xi(\zeta', \zeta_n)| = \prod_1^m {}_i \, |(\zeta_n - \mu_i^0(\zeta'))| \geqslant (\varepsilon|\zeta'|)^m \qquad \text{if } |\zeta_n| \geqslant (k + \varepsilon)|\zeta'|.$$

On the other hand:

$$|F(\xi + \zeta) - F_\xi(\zeta)| < (\varepsilon'|\zeta|)^m, \qquad |\zeta| < r'$$

merely by the definition of localization.

It follows:

$$|F(\xi + \zeta) - F_\xi(\zeta)| < (\varepsilon|\zeta'|)^m, \qquad |\zeta'| < r, \quad |\zeta_n| \leqslant (k + \varepsilon)|\zeta'|$$

from which:

$$|F_\xi(\zeta', \zeta_n)| \geqslant (\varepsilon|\zeta'|)^m > |F(\xi + \zeta) - F_\xi(\zeta)|, \qquad |\zeta'| < r, \ |\zeta_n| = (k + \varepsilon)|\zeta'|.$$

So by Rouché's theorem, when $|\zeta'| < r$, $F(\xi + (\zeta', \zeta_n))$ has exactly m zeros $\mu_i(\zeta')$ verifying $|\mu_i(\zeta')| < (k + \varepsilon)|\zeta'|$.

We'll always suppose in the following that the germ $F(\xi + (\zeta', \zeta_n))$ is in the form $\prod_1^m{}_i\, (\zeta_n - \mu_i(\zeta'))$ because the last has the same small zeros and so the same localization (apart from a multiplicative constant); therefore we'll refer to $F(\xi + (\zeta', \zeta_n))$ as a Weierstrass polynomial with respect to ζ_n.

Lemma 2.2. *For every ε there exists r $(= r(\varepsilon))$ s.t. $\forall$ zero $\mu_i^0(\zeta')$ of $F_\xi(\zeta', \zeta_n)$ of multiplicity α_i there are at least α_i zeros $\mu_i(\zeta')$ of $F(\xi + (\zeta', \zeta_n))$ with*

$$|\mu_i(\zeta') - \mu_i^0(\zeta')| < \varepsilon|\zeta'|\,, \qquad |\zeta'| < r\,.$$

Proof. Choose r so that:

$$|F(\xi + \zeta) - F_\xi(\zeta)| < \left(\frac{\varepsilon}{2m}\right)^m |\zeta'|^m\,, \qquad |\zeta'| < r\,, \qquad |\zeta_n| \leqslant (k + \varepsilon)|\zeta'|$$

(which is possible because F_ξ is a polynomial of degree m). Take a zero $\mu_i^0(\zeta')$ of $F_\xi(\zeta', \zeta_n)$; if its distance from the other ones is $\geqslant (\varepsilon/m)|\zeta'|$ we have therefore:

$$|F_\xi(\zeta)| \geqslant \left(\frac{\varepsilon}{2m}\right)^m |\zeta'|^m > |F(\xi + \zeta) - F_\xi(\zeta)|\,, \qquad |\zeta'| < r\,, \ |\zeta_n - \mu_i^0(\zeta')| = \frac{\varepsilon}{2m}|\zeta'|\,.$$

Otherwise share the remainder into two sets $\{\mu_j^0(\zeta')\}_j$, $\{\mu_k^0(\zeta')\}_k$ so that:

$$(1) \qquad |\mu_i^0(\zeta') - \mu_j^0(\zeta')| < \frac{\varepsilon}{m}|\zeta'| \quad \text{for some } j\,;$$

$$\forall j \ |\mu_j^0(\zeta') - \mu_{j'}^0(\zeta')| < \frac{\varepsilon}{m}|\zeta'| \quad \text{for some } j'\,;$$

$$(2) \qquad |\mu_k^0(\zeta') - \mu_i^0(\zeta')| \geqslant \frac{\varepsilon}{m}|\zeta'| \quad \forall k\,; \qquad |\mu_k^0(\zeta') - \mu_j^0(\zeta')| \geqslant \frac{\varepsilon}{m}|\zeta'| \quad \forall j, k\,.$$

Consider the open connected set:

$$B = B\left(\mu_i^0(\zeta'), \frac{\varepsilon}{2m}|\zeta'|\right) \bigcup_j B\left(\mu_j^0(\zeta'), \frac{\varepsilon}{2m}|\zeta'|\right).$$

If $\zeta_n \in \partial B$, $|\zeta'| < r$ we have:

$$|F_\xi(\zeta)| \geqslant \left(\frac{\varepsilon}{2m}|\zeta'|\right)^m > |F(\xi + \zeta) - F_\xi(\zeta)|\,.$$

And so in B there are as many zeros $\mu_i(\zeta')$ of $F(\xi + (\zeta', \zeta_n))$ as the sum of the multiplicities of $\mu_i^0(\zeta')$ and of all the $\mu_j^0(\zeta')$. Since the diameter of B is $< \varepsilon|\zeta'|$, we conclude.

Consider the decomposition of F_ξ's germ at 0 in irreducible germs i.e. factorize $F_\xi = \prod_i Q_i^{\alpha_i}$ with Q_i irreducible in the local ring of analytic germs. Every Q_i being homogeneous, it is then entire and so it is a polynomial. Consider the discriminant Δ of the product of the Q_i thought as a (monic) polynomial in ζ_n with ζ' as parameters. We have

$$\Delta(\zeta') = \prod_{i'<i''} (\mu_{i'}^0(\zeta') - \mu_{i''}^0(\zeta'))^2$$

where $\mu_i^0(\zeta')$ are the different zeros of $F_\xi(\zeta', \zeta_n)$; set $\deg \Delta = d$.

Lemma 2.3. *For every ε there exist ε' and r $(= r(\varepsilon))$ s.t. if $|\Delta(\zeta')| > \varepsilon' |\zeta'|^d$, $|\zeta'| < r$, then $\forall$ zero $\mu_i^0(\zeta')$ of multiplicity α_i there are exactly α_i zeros $\mu_i(\zeta')$ with:*

$$|\mu_i(\zeta') - \mu_i^0(\zeta')| < \varepsilon|\zeta'|\,.$$

Proof. Clearly $\exists k$ s.t. $|\mu_i^0(\zeta')| < k|\zeta'|$ $\forall i$ by which if $|\Delta(\zeta')| > \varepsilon' |\zeta'|^d$ it follows

$$|\mu_i^0(\zeta') - \mu_j^0(\zeta')|^2 = \frac{|\Delta(\zeta')|}{\prod\limits_{h<k,\,(h,k)\neq(i,j)} |\mu_h^0(\zeta') - \mu_k^0(\zeta')|^2} \geqslant \frac{(\varepsilon'|\zeta'|)^d}{(2k|\zeta'|)^{d-2}} \geqslant (2\varepsilon|\zeta'|)^2\,.$$

Therefore if $|F(\xi+\zeta) - F_\xi(\zeta)| < (\varepsilon|\zeta'|)^m$ when $|\zeta'| < r$, $|\zeta_n| \leqslant (k+\varepsilon)|\zeta'|$ it follows

$$|F_\xi(\zeta)| \geqslant (\varepsilon|\zeta'|)^m > |F(\xi+\zeta) - F_\xi(\zeta)|\,, \qquad |\Delta(\zeta')| > \varepsilon' |\zeta'|^d\,,$$
$$|\zeta'| < r\,,\ |\zeta_n - \mu_i^0(\zeta')| = \varepsilon|\zeta'|$$

since for $|\Delta(\zeta')| > \varepsilon' |\zeta'|^d$ the zeros $\mu_i^0(\zeta')$ have distance from eachother $\geqslant 2\varepsilon|\zeta'|$. Again we conclude by Rouché's theorem.

Let us come back to a (homogeneous) polynomial P, and consider its germ at $\xi \in \mathbb{R}^n$, $|\xi| = 1$, supposing, as always, that P_ξ is normalized in the ζ_n direction. Observe that when r is small then $P(\zeta) = 0$, $|\zeta' - \xi'| < r$ implies $|\zeta_n - \xi_n| < k|\zeta' - \xi'|$ and so $|\zeta - \xi| < (1 + k^2)^{\frac{1}{2}} r$ unless ζ is bounded away from ξ; besides the number of such roots close to ξ is equal to $m = \deg P_\xi$. In this situation, we can state in a (seemingly) different way the local Ph. L. principle.

Definition 2.1. *The (local) Ph. L. principle on (the germ of) $V(P)$ at ξ relative to K, K', holds if there are δ and r_0 s.t. $\forall r \leqslant r_0$:*

(1) $$\varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta)\,, \qquad P(\zeta) = 0\,,\ |\zeta' - \xi'| < \frac{r}{2}\,,\ |\zeta_n - \xi_n| < k|\zeta' - \xi'|$$

whenever φ is w.p.s.h. on $\{P(\zeta) = 0,\ |\zeta' - \xi'| < r,\ |\zeta_n - \xi_n| < k|\zeta' - \xi'|\}$ there verifying

(2) $$\varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta r\,; \qquad \varphi(\zeta) \leqslant 0\,, \quad \zeta \in \mathbb{R}^n\,.$$

Lemma 2.4. *Definitions* 2.1 *and* 1.1 *are equivalent in the sense that the (global) Ph. L. on $V(P)$ relative to* K, K′ *holds if and only if for every real unit zero* ξ, P_ξ *is normalized in some direction (e.g. the* ζ_n *direction) so that the implication* (2) $\Rightarrow$ (1) *for the same* K, K′ *is fulfilled.*

Proof. Let φ be defined on the r-neighbourhood of ξ on $V(P)$ there verifying:

$$\varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta' r\,; \qquad \varphi(\zeta) \leqslant 0\,, \quad \zeta \in \mathbb{R}^n\,.$$

Then φ is defined on $\{P(\zeta)=0,\ |\zeta'-\xi'|<r/(1+k^2)^{\frac{1}{2}},\ |\zeta_n-\xi_n|<k|\zeta'-\xi'|\}$ also. Setting $r'=r/(1+k^2)^{\frac{1}{2}}$, then φ verifies the inequalities (2) if δ' is taken so that $\delta'(1+k^2)^{\frac{1}{2}} \leqslant \delta$. It follows:

$$\varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta)\,, \qquad P(\zeta)=0\,,\ |\zeta-\xi|<\frac{r}{2(1+k^2)^{\frac{1}{2}}}$$

where obviously r is so small that $P(\zeta)=0$, $|\zeta_n-\xi_n| \geqslant k|\zeta'-\xi'|$ implies $|\zeta_n-\xi_n|>r'/2$. Then we conclude by observing that in Definition 1.1 we could have put balls of radius r/m, m real >1, instead of balls of radius $r/2$.

The converse is just a repetition of Lemmas 1.1, 1.2, 1.3 (omitted for brevity).

Observe that we again obtain an equivalent definition by putting r/m, m real >1, instead of $r/2$ in (1).

Suppose now that the Ph. L. on $V(P)$ at ξ (in the form of Definition 2.1) relative to K, K', (δ, r_0) holds. Set $d=\deg \Delta$ and define:

$$C_\varepsilon(r)=\{\zeta\colon P(\zeta)=0,\ |\zeta'-\xi'|<r,\ \zeta_n \text{ near } \xi_n,\ |\Delta(\zeta'-\xi')|>\varepsilon\}\,, \qquad r<r_0\,.$$

Lemma 2.5. *There exists a constant* M *which doesn't depend on* ε *and* r *s.t.*:

$$\text{(1)} \qquad \varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) + M\frac{\log|\Delta(\zeta'-\xi')|/r^d}{\log\varepsilon}\,r\,, \qquad \zeta \in C_{\varepsilon r^d}\left(\frac{r}{2}\right)$$

if φ *is a w.p.s.h. function on* $C_{\varepsilon r^d}(r)$ *verifying*:

$$\text{(2)} \qquad \varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta r\,; \qquad \varphi(\zeta) \leqslant 0\,, \quad \zeta \in \mathbb{R}^n\,.$$

Proof. If $b>0$, set:

$$\psi(\zeta)=\max\left(\varphi(\zeta)+b\log\frac{|\Delta(\zeta'-\xi')|}{r^d},\,0\right), \qquad \zeta \in C_{\varepsilon r^d}(r)\,,$$

$$\psi(\zeta)=0\,, \qquad P(\zeta)=0\,,\ |\zeta'-\xi'|<r\,,\ |\Delta(\zeta'-\xi')| \leqslant \varepsilon r^d\,.$$

Let r be so small that $P(\xi+\zeta)=0$, $|\zeta'|<r$, ζ_n small, implies $|\zeta_n|<k|\zeta'|$ (Lemma 2.1). If then M is a constant s.t. $H_K(\operatorname{Im}\zeta)+\delta \leqslant M$, $|\zeta'| \leqslant 1$, $|\zeta_n| \leqslant k$ it follows: $H_K(\operatorname{Im}\zeta)+\delta r \leqslant Mr$, $P(\xi+\zeta)=0$, $|\zeta'|<r$. In order that ψ be

w.p.s.h. it is enough to take b such that $b \log \varepsilon < -Mr$. It is not restrictive to assume $|\Delta(\zeta')| \leqslant 1$, $\forall |\zeta'|=1$; it follows $b \log |\Delta(\zeta'-\xi')|/r^d \leqslant 0$ when $|\zeta'-\xi'|<r$ from which ψ also satisfies the estimations (2). By the local Ph. L. (as stated in Definition 2.1) we have:

$$\varphi(\zeta) \leqslant K_{K'}(\operatorname{Im}\zeta) - b \log \frac{|\Delta(\zeta'-\xi')|}{r^d}, \qquad \zeta \in C_{\varepsilon r^d}\left(\frac{r}{2}\right)$$

provided that $-b < Mr/\log \varepsilon$. Thus, with M' arbitrarily close to M:

$$\varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) + M' \frac{\log |\Delta(\zeta'-\xi')|/r^d}{\log \varepsilon} r, \qquad \zeta \in C_{\varepsilon r^d}\left(\frac{r}{2}\right).$$

Remark. Obviously we have analogous conclusions for functions which are w.p.s.h. on $\{\zeta : P_\xi(\zeta)=0,\ |\zeta'|<r,\ |\Delta(\zeta')| > \varepsilon r^d\}$ in the case when the Ph. L. on $V(P_\xi)$ at 0 holds.

Let, as always, $\xi \in \mathbb{R}^n$, $|\xi| = 1$, $P(\xi) = 0$ and assume that P_ξ be normalized in the ζ_n direction.

Theorem 2.1. *Suppose that the Ph. L. on $V(P)$ at ξ relative to K, K', (δ, r_1) holds. For every bounded C and for every ε there is r_0 $(= r_0(\varepsilon))$ s.t. $\forall r \leqslant r_0$ we have:*

$$\text{(1)} \qquad \varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) + \varepsilon r, \qquad P_\xi(\zeta) = 0, \quad |\zeta'| < \frac{r}{4}$$

if φ is w.p.s.h. on $\{P_\xi(\zeta) = 0,\ |\zeta'| < r\}$, $r \leqslant r_0$ there verifying:

$$\text{(2)} \qquad \varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \frac{\delta}{2} r; \qquad \varphi(\zeta) \leqslant C|\operatorname{Im}\zeta|.$$

Proof. Clearly r_1 has been chosen so that for $|\zeta'| < r_1$ there are exactly $\deg P_\xi$ zeros of $P(\xi + (\zeta', \zeta_n))$ which are $\leqslant k|\zeta'|$, whereas the other ones are $\geqslant kr_1$.

Fix ε and let ε' and r_0 $(\leqslant r_1)$ be the numbers which correspond to ε as in Lemma 2.3. If $|\Delta(\zeta')| > \varepsilon' |\zeta'|^d$, $|\zeta'| < r_0$ then for every small zero $\mu_i(\zeta')$ of $P(\xi + (\zeta', \zeta_n))$ there exists one and only one zero $\mu_i^0(\zeta')$ of $P_\xi(\zeta', \zeta_n)$ with:

$$|\mu_i^0(\zeta') - \mu_i(\zeta')| < \varepsilon|\zeta'|.$$

It is unique because $|\Delta(\zeta')| > \varepsilon' |\zeta'|^d$ implies $|\mu_i^0(\zeta') - \mu_j^0(\zeta')| \geqslant 2\varepsilon|\zeta'|$, $j \neq i$. It exists because when $|\zeta'| < r_0$, the small zeros μ_i with corresponding multiplicities, are as many as the zeros μ_i^0; use then Lemma 2.3.

Let φ w.p.s.h. on $\{P_\xi(\zeta)=0,\ |\zeta'|<r\}$, $r \leqslant r_0$ be as in hypothesis and define on $\{P(\xi+\zeta) = 0,\ |\Delta(\zeta')| > \varepsilon' |\zeta'|^d,\ |\zeta'| < r\}$:

$$\psi\big(\xi + (\zeta', \mu_i(\zeta'))\big) = \varphi(\zeta', \mu_i^0(\zeta'))$$

if $\mu_i^0(\zeta')$ is the only zero of $P_\xi(\zeta', \zeta_n)$ for which $|\mu_i^0(\zeta') - \mu_i(\zeta')| < \varepsilon|\zeta'|$.

We claim that ψ is w.p.s.h.; in fact when the discriminant of the product of the irreducible factors of P's germ at ξ doesn't vanish on $\xi' + \zeta'$ then ψ is obviously w.p.s.h. near $\xi + \zeta$; otherwise use the Riemann extension theorem (see Lemma 4.4 of [3]).

Let c be the radius of a ball which covers K (and K' also). We have:

$$\psi(\xi + (\zeta', \mu_i(\zeta'))) \leqslant H_K(\operatorname{Im}(\zeta', \mu_i^0(\zeta'))) + \frac{\delta}{2} r \leqslant H_K(\operatorname{Im}(\zeta', \mu_i(\zeta'))) + c\varepsilon r + \frac{\delta}{2} r \leqslant$$
$$H_K(\operatorname{Im}(\zeta', \mu_i(\zeta'))) + \delta r, \qquad |\varDelta(\zeta')| > \varepsilon' |\zeta'|^d \ (\varepsilon \text{ small}), \quad |\zeta'| < r.$$

Besides:

$$\psi\big(\xi + (\zeta', \mu_i(\zeta'))\big) \leqslant C|\operatorname{Im}(\zeta', \mu_i(\zeta'))| + C\varepsilon r, \qquad |\varDelta(\zeta')| > \varepsilon' |\zeta'|^d, \ |\zeta'| < r.$$

Application of Lemma 2.5 to the function $\psi - C\varepsilon r$ gives:

$$\psi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) + C\varepsilon r + M \frac{\log |\varDelta(\zeta' - \xi')|/r^d}{\log \varepsilon'} r, \qquad \zeta \in C_{\varepsilon' r^d}\left(\frac{r}{2}\right).$$

Thus merely by the definition of ψ:

$$\varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) + (c + C)\varepsilon r + M \frac{\log |\varDelta(\zeta')|/r^d}{\log \varepsilon'} r,$$

$$P_\xi(\zeta) = 0, \quad |\varDelta(\zeta')| > \varepsilon' r^d, \quad |\zeta'| < \frac{r}{2}.$$

Let $P_\xi(\zeta) = 0$, $|\zeta'| < r/4$; let $\gamma > 0$ and $|\varDelta| \leqslant T$ on the boundary of the polydisc $\{\eta \in \mathbf{C}^{n-1}: |\eta_i - \zeta_i| < \gamma r, \ i = 1, \dots, n-1\}$. Cauchy's inequalities give:

$$\sum_{|\alpha| = d} |\varDelta^\alpha(\zeta')| \leqslant C_d \frac{T}{(\gamma r)^d} \qquad \text{where } \varDelta^\alpha = D^\alpha \varDelta.$$

Thus there are a_1 (depending only on the coefficients of $\varDelta$) and $|\theta_1| = \gamma r$ s.t.

$$|\varDelta(\zeta' + \theta_1)| \geqslant a_1 (\gamma r)^d.$$

Furthermore by Lemma 3.1.6 of [2] we can find $0 \leqslant k \leqslant d$ s.t. setting $\theta = (k/d)\theta_1$ we have

$$\inf_{|\tau| = 1} |\varDelta(\zeta' + \tau\theta)| \geqslant a(\gamma r)^d$$

where a depends only on a_1 and on d.

By Lemma 4.5 of [3] all branches of the curve defined by the equation $P_\xi\big(\zeta/r + (\tau(\theta/r), t)\big) = 0$ (in the indeterminates τ and t) passing through the origin satisfy the inequality:

$$|t| \leqslant A\gamma^{1/m}, \qquad \gamma < 1$$

with A depending only on the coefficients of P_ξ. More, such estimate holds on the connected component of the origin in the curve

$$P_\xi\left(\frac{\zeta}{r}+\left(\tau\left(\frac{\theta}{r}\right), t\right)\right)=0\,, \qquad |\tau|\leqslant 1\,.$$

Thus there exists an analytic function $\tau \mapsto (\tau, t(\tau))$, $\forall|\tau|\leqslant 1$ in a Riemann surface which parametrizes the connected component of the origin in $P_\xi(\zeta+(\tau\theta, t))=0$, $|\tau|\leqslant 1$ and which satisfies the conditions:

$$0\in t(0)\,; \qquad |t(\tau)|\leqslant A\gamma^{1/m} r\,, \qquad |\tau|\leqslant 1\,.$$

Then define the subharmonic function:

$$\sup_{t\in t(\tau)} \varphi(\zeta+(\tau\theta, t))\,, \qquad |\tau|\leqslant 1\,.$$

If $a\gamma^d>\varepsilon'$, $\gamma<\frac{1}{4}$, this function can be estimated when $|\tau|=1$ by:

$$H_{K'}(\operatorname{Im}\zeta)+(\gamma+A\gamma^{1/m})cr+(c+C)\varepsilon r+M\frac{\log(a\gamma^d)}{\log\varepsilon'}r:$$

So by the maximum principle for subharmonic functions:

$$\varphi(\zeta)\leqslant H_{K'}(\operatorname{Im}\zeta)+\left[c(\gamma+A\gamma^{1/m})+(c+C)\varepsilon+M\frac{\log(a\gamma^d)}{\log\varepsilon'}\right]r$$

provided that

$$P_\xi(\zeta)=0\,, \qquad |\zeta'|<\frac{r}{4}\,, \qquad a\gamma^d>\varepsilon'\,, \qquad \gamma<\frac{1}{4}\,.$$

Choosing $a\gamma^d=-1/\log\varepsilon'$ we can conclude by observing that the expression between brackets is infinitesimal for $\varepsilon\to 0$ (and so $\varepsilon'\to 0$).

Note now that P_ξ is independent of the variable on the straight line parallel to ξ (i.e. $P_\xi(\zeta+t\xi)=P_\xi(\zeta)$ $\forall t$ and $\forall\zeta$); so the (global) Ph. L. on $V(P_\xi)$ implies that on $V(P_\xi)$'s germ at ξ, and therefore that on $V(P_\xi)$'s germ at 0. Then we can prove, just repeating the previous demonstration and recalling the remark at the end of Lemma 2.5

THEOREM 2.2. *Assume that the Ph. L. on $V(P_\xi)$ at 0 relative to K, K', (δ, r_1) (or the global one with exactly alike compact sets (and new constant δ')) holds. For every bounded C and for every ε there is r_0 $(=r_0(\varepsilon))$ s.t. $\forall r\leqslant r_0$ we have:*

$$\varphi(\zeta)\leqslant H_{K'}(\operatorname{Im}\zeta)+\varepsilon r\,, \qquad P(\zeta)=0\,, \quad |\zeta'-\xi'|<\frac{r}{4}\,, \quad \zeta_n-\xi_n \textit{ small}$$

whenever φ *is w.p.s.h. on* $\{P(\zeta)=0,\ |\zeta'-\xi'|<r,\ \zeta_n-\xi_n \text{ small}\}$ *there verifying*

$$\varphi(\zeta)\leqslant H_K(\operatorname{Im}\zeta)+\frac{\delta}{2}r\,;\qquad \varphi(\zeta)\leqslant C|\operatorname{Im}\zeta|\,.$$

Thus we infer from the two previous theorems that the Ph. L. on $V(P)$ at ξ implies that on $V(P_\xi)$ at 0, and conversely, apart from a residue εr where ε can be made as small as we like provided that $r\to 0$. We dedicate the following to showing that in the first case the homogeneity of P_ξ at 0 enables us to avoid the contraction of r (for ε tending to 0) and allows us to conclude that the Ph. L. on $V(P)$ at ξ implies the (global) Ph. L. on $V(P_\xi)$.

LEMMA 2.6. *Suppose that for every (bounded)* C *and for every* ε *there is* r_0 $(=r_0(\varepsilon))$ *s.t.* $\forall r<r_0$ *the implication* (2) $\Rightarrow$ (1) *of Theorem* 2.1 *is fulfilled when* φ *is w.p.s.h. on* $\{P_\xi(\zeta)=0,\ |\zeta'|<r\}$. *Let* E *be an arbitrary positive number. Then every* φ *w.p.s.h. on* $V(P_\xi)$ *which verifies there*:

$$(1')\qquad \varphi(\zeta)\leqslant H_K(\operatorname{Im}\zeta)+\frac{\delta}{2(1+k^2)^{\frac12}}|\zeta|\,;\qquad \varphi(\zeta)\leqslant C|\operatorname{Im}\zeta|+E|\zeta|$$

verifies also:

$$(2')\qquad \varphi(\zeta)\leqslant H_{K'}(\operatorname{Im}\zeta)+4(1+k^2)^{\frac12}E|\zeta|\,.$$

PROOF. C and E being fixed, let $\varepsilon\to 0$. Let $P_\xi(\zeta)=0$, $\zeta\in\mathbf{C}^n$; define:

$$\psi(\eta)=\frac{r}{4|\zeta|}\varphi\left(\frac{4|\zeta|}{r}\eta\right),\qquad P_\xi(\eta)=0\,,\qquad r\leqslant r_0(\varepsilon)\,.$$

Obviously ψ is w.p.s.h. on $V(P_\xi)$ and there verifies (1'). Remembering that $(1+k^2)^{\frac12}$ is a constant for which $P_\xi(\eta)=0$ implies $|\eta|\leqslant(1+k^2)^{\frac12}|\eta'|$ it follows:

$$\psi(\eta)\leqslant H_K(\operatorname{Im}\eta)+\frac{\delta}{2}r\,,\qquad P_\xi(\eta)=0\,,\qquad |\eta'|<r\,,$$

$$\psi(\eta)\leqslant C|\operatorname{Im}\eta|+(1+k^2)^{\frac12}Er\,,\qquad P_\xi(\eta)=0\,,\qquad |\eta'|<r\,.$$

Thus applying Theorem 2.1 to the function $\psi-(1+k^2)^{\frac12}Er$:

$$\psi(\eta)\leqslant H_{K'}(\operatorname{Im}\eta)+(1+k^2)^{\frac12}Er+\varepsilon r\,,\qquad P_\xi(\eta)=0\,,\qquad |\eta'|<\frac{r}{4}\,.$$

Using the last inequality for $\eta=(r/4|\zeta|)\,\zeta$ [2] we have:

$$\frac{r}{4|\zeta|}\varphi(\zeta)\leqslant\frac{r}{4|\zeta|}H_{K'}(\operatorname{Im}\zeta)+(1+k^2)^{\frac12}Er+\varepsilon r$$

[2] It is clear that (1) of Theorem 2.1 is true even if $|\eta|=r/4$.

from which:

$$\varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) + 4(1+k^2)^{\frac{1}{2}} E|\zeta| + 4\varepsilon|\zeta| .$$

Letting $\varepsilon \to 0$ the lemma follows.

Summarizing up we proved that the Ph. L. on $V(P)$ at ξ relative to K, K', (δ) implies for a new constant δ' the following statement: $\forall C, E$, if φ is a w.p.s.h. function on $V(P_\xi)$ which verifies there:

$$\varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta' |\zeta| ; \qquad \varphi(\zeta) \leqslant C|\operatorname{Im}\zeta| + E|\zeta|$$

in consequence it also verifies:

$$\varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta) + 4(1+k^2)^{\frac{1}{2}} E|\zeta| .$$

This statement can be rephrased, via the «fundamental principle» of Ehrenpreis, in the following way. Set

$$\begin{aligned} A &= \{x+\zeta : x \in K', \zeta \in \mathbf{C}^n, |\zeta| < 5(1+k^2)^{\frac{1}{2}} E\} , \\ A_1 &= \{x+\zeta : x \in K, \zeta \in \mathbf{C}^n, |\zeta| < \delta'\} , \\ A_2 &= \{x+\zeta : x \in \mathbf{R}^n, |x| < C, \zeta \in \mathbf{C}^n, |\zeta| < E\} . \end{aligned}$$

Then every bounded (complex) analytic solution of the equation $P_\xi u = 0$ on A can be written as a sum $u_1 + u_2$ with u_i analytic solution of the same homogeneous equation on A_i and with

$$\sup_{i=1,2} \sup_{A_i} |u_i| \leqslant c(C, E) \sup_{A} |u| .$$

If then we suppose to be concerned with an open convex set Ω and to refer to K, K' as subsets of Ω $(K \subset K' \subset \Omega)$ we have the following conclusions just repeating step by step the demonstrations of Theorems 1.2, 1.3 of [3].

THEOREM 2.3. *Suppose that the Ph. L. on $V(P)$ at the real point ξ relative to K, K' holds. Then for every $f \in A(\Omega)$ there exists $u \in C^\infty(\Omega)$, analytic on K, resolving the equation $P_\xi u = f$ (³) in Ω.*

THEOREM 2.4. *If Ω admits the Ph. L. on $V(P)$ at the real point ξ then $P_\xi A(\Omega) = A(\Omega)$.*

THEOREM 2.5. *If $PA(\Omega) = A(\Omega)$ then $P_\xi A(\Omega) = A(\Omega)$ for every real ξ (⁴).*

Naturally the converse of Theorem 2.5 (supposing ξ non null) doesn't hold as happens for the polynomial in three variables: $P(\zeta) = (\zeta_1^2 + \zeta_2^2)^2 + 4\zeta_1^2\zeta_3^2$. Here the localizations P_ξ $(\xi \neq 0)$ are either constants or multiples

(³) Here we mean differential operators instead of associated polynomials.

(⁴) Observe that if ξ is not a zero of P then P_ξ is a constant; if $\xi \neq 0$ is not unit then $P_\xi = cP_{\xi/|\xi|}$; and if $\xi = 0$ then $P_\xi = P$ (since P is homogeneous).

of ζ_1^2. Therefore $P_\xi A(\Omega)=A(\Omega)$ $\forall\xi$ and $\forall\Omega$ (see the first part of the demonstration of Lemma 2.7) although for no (convex) Ω, $PA(\Omega)=A(\Omega)$. In fact the real characteristic variety $\{\zeta\in\mathbb{R}^3: P(\zeta)=0\}$ is a straight line and so its codimension disagrees with Theorem 6.3 of [3].

The matter is that even if P has hyperbolic localizations, nevertheless it is not locally hyperbolic which is the necessary and sufficient condition to get $PA(\mathbb{R}^3)=A(\mathbb{R}^3)$ and so the necessary condition to get $PA(\Omega)=A(\Omega)$, $\Omega\subset\mathbb{R}^3$. So in Section 3 we'll only be concerned with locally hyperbolic operators and we'll see under what conditions, Theorem 2.5 admits inversion. Before doing so we want to deduce from Theorem 2.5 necessary conditions for analytic P-convexity of open convex regions.

THEOREM 2.6. *Let P have some second order irreducible localization at non null real points and Ω be an open convex set with C^1 boundary.*

If $PA(\Omega)=A(\Omega)$ then Ω is unbounded.

Factorize P_ξ in the product $\prod_i (P_{\xi,i})^{\alpha_i}$ of irreducible (homogeneous) polynomials. If $P_{\xi,i}$ is hyperbolic with respect to $v(\xi,i)$ it is well known how the cones $\Gamma_{\xi,i}=\Gamma(P_{\xi,i}, v(\xi,i))$ and $\Gamma^*_{\xi,i}=\Gamma^*(P_{\xi,i}, v(\xi,i))$ are defined; the first is the component of $v(\xi,i)$ in the set $\{\eta\in\mathbb{R}^n: P_{\xi,i}(\eta)\neq 0\}$ the second is its dual cone namely $\{x\in\mathbb{R}^n: \langle x,\eta\rangle\geqslant 0,\ \forall\eta\in\Gamma_{\xi,i}\}$. Suppose to be concerned with a polynomial P s.t. $\forall\xi\neq 0$ every irreducible factor $P_{\xi,i}$ is a hyperbolic form of degree $\leqslant 2$; in particular we can consider locally hyperbolic operators with multiplicities $\leqslant 2$ (everywhere in $\mathbb{R}^n-\{0\}$) since they have, as known, hyperbolic localizations.

THEOREM 2.7. *Let P be as above and Ω be convex. If $PA(\Omega)=A(\Omega)$ then $\forall\xi\neq 0$ and $\forall i$ the following condition holds:*

$$\text{for every } x\in\partial\Omega \begin{cases} \text{either } & x+\Gamma^*_{\xi,i}\cap\Omega=\emptyset\,, \\ \text{or} & x-\Gamma^*_{\xi,i}\cap\Omega=\emptyset\,. \end{cases}$$

PROOF OF THEOREMS 2.6, 2.7. We know that $PA(\Omega)=A(\Omega)$ implies $P_\xi A(\Omega)=A(\Omega)$. For the first observe then that P_ξ, $\xi\neq 0$, is independent of the variable on the ξ-line. So for some ξ, P_ξ is an irreducible degenerate quadratic form. By Theorem 6.6 of [3] P_ξ must be real indefinite; then apply Theorem 6.7 again of [3].

For the second theorem note that if $P_{\xi,i}$ is linear, being even real, then the corresponding $\Gamma^*_{\xi,i}$ is a bicharacteristic ray; so the condition of Theorem 2.7 is trivially fulfilled because of Ω's convexity. Otherwise the theorem ensues from the following lemma which again arises from Theorem 6.7 of [3].

LEMMA 2.7. *Let P be a second order irreducible degenerate hyperbolic (with respect to v) form; set $\Gamma^*=\Gamma^*(P,v)$. Then $PA(\Omega)=A(\Omega)$ if and only if*

$$\text{for every } x\in\partial\Omega \begin{cases} \text{either } & x+\Gamma^*\cap\Omega=\emptyset\,, \\ \text{or} & x-\Gamma^*\cap\Omega=\emptyset\,. \end{cases}$$

PROOF. The sufficiency holds in more general hypotheses. We could suppose that P has principal part IE with I hyperbolic and E elliptic and that Ω is a generic (even non convex) open set. If then $\forall x \in \partial\Omega$ either $x + \Gamma^* \cdot (I) \cap \Omega = \emptyset$ or $x - \Gamma^*(I) \cap \Omega = \emptyset$ it follows that $PA(\Omega) = A(\Omega)$. This is precisely our result in [5], even though there we make use of it only for polynomials in two variables.

Conversely suppose P in Sylvester's canonical form $P(\zeta) = \zeta_n^2 - \sum_{k}^{n-1}{}_i \zeta_i^2$ where $k \leqslant n-2$, otherwise P is reducible, and $k \geqslant 2$, otherwise it is non degenerate. If $x \in \partial\Omega$ the tangent half space to Ω at x has been defined in [1] where also it has been proven that $PA(\Omega) = A(\Omega)$ implies $PA(\Sigma) = A(\Sigma)$ for every tangent half space Σ. Suppose by translation of the coordinate system, that $\partial\Sigma$ is homogeneous (with normal $N = (n_1, \ldots, n_n)$).

(*a*) It is not restrictive to assume N parallel to the plane of the $x_k, \ldots, x_n$ variables. Otherwise consider the projection $\tilde{N}$ of N on such plane. If $\tilde{N} = 0$ then Γ^* is tangent to Σ; or else let, e.g., $n_k \neq 0$.

Setting $\tilde{\Sigma} = \{x : \langle x, \tilde{N} \rangle < 0\}$ consider the mappings:

$$G: \ \Sigma \to \tilde{\Sigma}, \qquad G: \ x \to \left(\ldots x_{k-1}, x_k + \sum_{1}^{k-1}{}_i \frac{x_i n_i}{n_k}, x_{k+1}, \ldots, x_n \right)$$

and

$$G^*: \ A(\tilde{\Sigma}) \to A(\Sigma), \qquad G^*: f \mapsto f \circ G \, .$$

G^* is a surjective ismorphism which commutes with P (since P is independent of the first $k-1$ variables). Thus $PA(\Sigma) = A(\Sigma)$ if and only if $PA(\tilde{\Sigma}) = A(\tilde{\Sigma})$ and, being Γ^* in the $x_k, \ldots, x_n$ plane, $\Sigma \cap \pm \Gamma^* = \emptyset$ if and only if $\tilde{\Sigma} \cap \pm \Gamma^* = \emptyset$.

(*b*) Consider the symmetry of $\mathbb{R}^n$ with the plane of the first $n-1$ variables as fixed plane: $S: x \mapsto (\ldots, x_{n-1}, -x_n)$; set $\bar{\Sigma} = S(\Sigma)$.

The mapping $S^*: A(\bar{\Sigma}) \to A(\Sigma)$ is again a surjective isomorphism which commutes with P and so again $PA(\Sigma) = A(\Sigma)$ if and only if $PA(\bar{\Sigma}) = A(\bar{\Sigma})$. Besides since the (Mayer Vietoris) sequence:

$$0 \to A(\Sigma \cup \bar{\Sigma}) \to A(\Sigma) \oplus A(\bar{\Sigma}) \to A(\Sigma \cap \bar{\Sigma}) \to 0$$

is exact, then every $f \in A(\Sigma \cap \bar{\Sigma})$ can be written as a sum $f_1 + f_2$ with $f_1 \in A(\Sigma)$ and $f_2 \in A(\bar{\Sigma})$. As we can find global solutions $u_1 \in A(\Sigma)$ and $u_2 \in A(\bar{\Sigma})$ of every equation $Pu_i = f_i$, $i = 1, 2$, it follows: $PA(\Sigma \cap \bar{\Sigma}) = A(\Sigma \cap \bar{\Sigma})$.

(*c*) Let $x \in \partial\Sigma$, $x_n > 0$ (otherwise N is parallel to the axis of the last coordinate) and suppose that the interior of the segment $x \mapsto S(x)$ is contained in $\Sigma \cap \bar{\Sigma}$ (or else consider the segment $-x \mapsto S(-x)$). Let K be a compact interval in the interior of such segment; so $K \subset\subset \Sigma \cap \bar{\Sigma}$ and K is parallel to the x_n axis. Since $PA(\Sigma \cap \bar{\Sigma}) = A(\Sigma \cap \bar{\Sigma})$ it follows by Theorem 6.7 of [3] that $\Sigma \cap \bar{\Sigma}$ must contain a compact set K' whose projection on the $x_k, \ldots, x_{n-1}$ plane contains the ball B of diameter equal to the length of K and with center at the projection of K. Calling π such projection we have:

$$B \subset \pi K' \subset \Sigma \cap \bar{\Sigma} \subset \Sigma \quad \text{since } \pi \text{ projects } \Sigma \cap \bar{\Sigma} \text{ into itself}.$$

That obviously implies $\Sigma \cap + \Gamma^* = \emptyset$; then the theorem follows from the arbitrariness of the tangent half space Σ.

(d) It remains to be proven that we can assume without loss of generality that P be in Sylvester's canonical form. Let B be the matrix of the real linear coordinate change for which $P' = P \circ B^*$ is in canonical form; let $v' = B^{*-1}(v)$ and $\Omega' = B(\Omega)$. If $PA(\Omega) = A(\Omega)$ then $P'A(\Omega') = A(\Omega')$ and so $\forall x' \in \partial\Omega'$ $x' + \Gamma^*(P', v') \cap \Omega' = \emptyset$ or else $x' - \Gamma^*(P', v') \cap \Omega' = \emptyset$. Since $\Gamma^*(P', v') = B(\Gamma^*(P, v))$ the theorem follows.

3. - Locally hyperbolic operators.

In this section we'll deal only with locally hyperbolic operators trying to give some inversion of Theorem 2.5.

Definition 3.1. *P's germ at $\xi \in \mathbb{R}^n$ is said to be locally hyperbolic if its localization can be normalized in the ζ_n direction in order that:*

$$P(\zeta', \zeta_n) = 0\,, \quad \zeta - \xi \text{ small}\,, \quad \zeta' \in \mathbb{R}^{n-1} \quad \text{implies} \quad \zeta_n \in \mathbb{R}$$

where we denote by ζ' the first $n-1$ variables.

Definition 3.2. *P is said to be locally hyperbolic if every germ of P at real non null points is such.*

With every locally hyperbolic operator P we can associate (continuous) vector fields $\xi \mapsto \pm v(\xi)$ from $\mathbb{R}^n - \{0\}$ into $\mathbb{R}^n - \{0\}$ homogeneous of degree 0 mapping ξ into the opposite versors of the directions in which P's germ at ξ is locally hyperbolic.

We were persuaded that in the locally hyperbolic case, the Ph. L. on $V(P)$ at ξ and the Ph. L. on $V(P_\xi)$ at 0 (and so the global on $V(P_\xi)$ by subsequent Lemma 3.1) were equivalent (apart from arbitrarily small dilatation of the compact set K' which appears in (1) of Definition 1.1); namely we believed that an open convex set Ω admits the local Ph. L. on $V(P)$ at ξ if and only if it admits the global one on $V(P_\xi)$. When $\Omega = \mathbb{R}^n$ this is trivially true; in fact in our subsequent Lemma 3.2 we prove (following an observation of Hörmander) that $\mathbb{R}^n$ admits the Ph. L. on $V(P)$ at ξ whenever P's germ at ξ is locally hyperbolic (and so $PA(\mathbb{R}^n) = A(\mathbb{R}^n)$ if P is locally hyperbolic at every non null real zero). However we'll see that this is true only in the case $\Omega = \mathbb{R}^n$.

First observe that, P_ξ being independent of the variable on the ξ-line, the Ph. L. on $V(P_\xi)$ at ξ is the same as that at 0. More

LEMMA 3.1. *The global Ph. L. on $V(P_\xi)$ relative to K, K' is equivalent to the local one on $V(P_\xi)$ at 0 relative to the same K, K' (and with a new constant δ').*

PROOF. Naturally the only interesting case is when ξ is zero of P; we can also suppose ξ unit. By Lemma 1.3 the global Ph. L. relative to K, K', (δ) implies the local one at ξ relative to K, K', $(\delta', \frac{1}{2})$ and so the local one at 0 by the observation preceding the lemma.

Conversely let the Ph. L. at 0 relative to K, K', (δ, r_0) hold. Take φ w.p.s.h. on $V(P_\xi)$ there verifying:

$$\varphi(\zeta) \leqslant H_K(\operatorname{Im}\zeta) + \delta|\zeta|\,; \qquad \varphi(\zeta) \leqslant 0\,, \quad \zeta \in \mathbb{R}^n.$$

If $P_\xi(\zeta) = 0$ set:

$$\psi(\eta) = \frac{r}{4|\zeta|}\,\varphi\left(\frac{4|\zeta|}{r}\,\eta\right), \qquad P_\xi(\eta) = 0\,, \quad r \leqslant r_0$$

and consider the restriction of ψ to $\{P_\xi(\eta) = 0,\ \eta \in B(0, r)\}$.

It verifies there:

$$\psi(\eta) \leqslant H_K(\operatorname{Im}\eta) + \delta r\,; \qquad \psi(\eta) \leqslant 0\,, \quad \eta \in \mathbb{R}^n$$

and so also:

$$\psi(\eta) \leqslant H_{K'}(\operatorname{Im}\eta)\,, \qquad \eta \in B\left(0, \frac{r}{2}\right).$$

Using the last estimation for $\eta = (r/4|\zeta|)\,\zeta$ we then conclude:

$$\varphi(\zeta) \leqslant H_{K'}(\operatorname{Im}\zeta)\,.$$

If P's localization at ξ is normalized in the ζ_n direction then for small r: $P(\zeta) = 0$, $|\zeta' - \xi'| < r$, $\zeta_n - \xi_n$ small, implies $|\zeta_n - \xi_n| < k|\zeta' - \xi'|$ for some constant k (Lemma 2.1).

LEMMA 3.2. *Let P's germ at ξ be locally hyperbolic with respect to the ζ_n direction (and so with localization normalized in such direction). If φ is w.p.s.h. on a neighbourhood of ξ on $V(P)$ and verifies the estimates:*

$$\varphi(\zeta) \leqslant Cr\,, \qquad P(\zeta) = 0\,, \quad |\zeta' - \xi'| < r\,, \quad |\zeta_n - \xi_n| \text{ small}\,;$$

$$\varphi(\zeta) \leqslant 0\,, \qquad P(\zeta) = 0\,, \quad |\zeta' - \xi'| < r\,, \quad \zeta_n - \xi_n \text{ small}\,, \ \zeta \in \mathbb{R}^n$$

it consequently also verifies:

$$\varphi(\zeta) \leqslant C' |\mathrm{Im}\, \zeta'| , \qquad P(\zeta) = 0 , \quad |\zeta' - \xi'| < r' , \quad \zeta_n - \xi_n \text{ small}$$

where $C' = (8/\pi)(n-1)C$, $r' = r/2(n-1)^{\frac{1}{2}}$ *if* n *is the dimension of* $\mathbb{R}^n$.

PROOF. Define:

$$\psi(\zeta') = \sup\,(\varphi(\zeta', \zeta_n) \colon P(\zeta', \zeta_n) = 0,\ |\zeta_n - \xi_n| < k|\zeta' - \xi'|) ,$$
$$\zeta' \in \mathbb{C}^{n-1} , \quad |\zeta' - \xi'| < r .$$

This function is p.s.h. for $|\zeta' - \xi'| < r$ because the number of the zeros ζ_n of $P(\zeta', \zeta_n)$ with $|\zeta' - \xi'| < r$, $|\zeta_n - \xi_n| < k|\zeta' - \xi'|$ doesn't depend on ζ' (it is everywhere equal to deg P_ξ).

Supposing ξ real (otherwise the lemma is trivially fulfilled) we have:

$$\psi(\zeta') \leqslant Cr , \quad |\zeta' - \xi'| < r ; \qquad \psi(\zeta') \leqslant 0 , \quad |\zeta' - \xi'| < r , \ \zeta' \in \mathbb{R}^{n-1}$$

where the second inequality arises from the local hyperbolicity of P at ξ. So for the classical Ph. L. principle on $\mathbb{C}^{n-1}$:

$$\psi(\zeta') \leqslant \frac{8}{\pi}(n-1) C |\mathrm{Im}\, \zeta'| = C' |\mathrm{Im}\, \zeta'| , \qquad |\zeta' - \xi'| < \frac{r}{2(n-1)^{\frac{1}{2}}} = r'$$

from which the lemma follows considering how the function ψ has been defined.

By the previous lemma and using Theorems 2.1, 2.2, we have in the case when P is locally hyperbolic at the real (non null) zero ξ with respect to the ζ_n direction:

THEOREM 3.1. *Assume that the Ph. L. on* $V(P)$ *at* ξ *relative to* K, K', (δ, r_0) *holds. For every* ε *there exists* $r(\varepsilon)$ *s.t.* $\forall r \leqslant r(\varepsilon)$:

$$\varphi(\zeta) \leqslant H_{K'}(\mathrm{Im}\, \zeta) + \varepsilon r , \qquad |\zeta'| < \frac{r}{4}$$

if φ *is w.p.s.h. on* $\{P_\xi(\zeta) = 0,\ |\zeta'| < r\}$ *there verifying*:

$$\varphi(\zeta) \leqslant H_K(\mathrm{Im}\, \zeta) + \frac{\delta}{2} r ; \qquad \varphi(\zeta) \leqslant 0 , \qquad \zeta \in \mathbb{R}^n .$$

Conversely if the Ph. L. on $V(P_\xi)$ *at* 0 *relative to* K, K', (δ, r_0) *holds then for every* ε *there is* $r(\varepsilon)$ *s.t.* $\forall r \leqslant r(\varepsilon)$:

$$\varphi(\zeta) \leqslant H_{K'}(\mathrm{Im}\, \zeta) + \varepsilon r , \qquad P(\zeta) = 0 , \quad |\zeta' - \xi'| < \frac{r}{4} , \quad \zeta_n - \xi_n \text{ small}$$

if φ is w.p.s.h. on $\{P(\zeta)=0,\ |\zeta'-\xi'|<r,\ \zeta_n-\xi_n$ *small*$\}$ *there verifying*:

$$\varphi(\zeta)\leqslant H_K(\operatorname{Im}\zeta)+\frac{\delta}{2}r\,;\qquad \varphi(\zeta)\leqslant 0\,,\quad \zeta\in\mathbb{R}^n\,.$$

Because of Lemma 3.1 we can draw from the previous theorem (letting $\varepsilon\to 0$) the following statement which is a better version, in the locally hyperbolic case, of Lemma 2.6.

COROLLARY 3.1. *The Ph. L. on $V(P)$ at ξ relative to K, K' implies the (global) Ph. L. on $V(P_\xi)$ with the same K, K'.*

The corollary doesn't admit inversion since for $\varepsilon\to 0$ then also $r\to 0$ ($r=$ radius of the neighbourhood of ξ on which we have the Ph. L. estimations); so the conic neighbourhood in $\mathbb{C}^n$ of the ray through ξ over which the local estimations can be extended by homogeneity consequently decreases (see Lemma 1.3). Clearly that doesn't happen in the case of a neighbourhood of the origin.

First of all let us consider a polynomial P with real coefficients and with regular germ at a real non null zero ξ (i.e. grad $P(\xi)\neq 0$). If then the germ has localization normalized in the ζ_n direction (namely if $(\partial P/\partial x_n)(\xi)\neq 0$), since: $P(\xi+\zeta)=\langle \operatorname{grad}P(\xi),\zeta\rangle+o(|\zeta|)$ it then follows

$$P(\xi+\zeta)=0\,,\ \zeta\ \text{small}\,,\ \zeta'\ \text{real}\quad \text{implies}\quad \zeta_n\ \text{real}\,.$$

Thus every real polynomial is locally hyperbolic at all real simple characteristics. In this regular case the Ph. L. on $V(P)$ at ξ and that on $V(P_\xi)$ at 0 are equivalent (apart from an arbitrarily small expansion of K'). Let ξ be a real non null zero of P, $K\subset K'$ be compact convex sets; K'' be an arbitrary compact convex neighbourhood of K'.

THEOREM 3.2. *Assume that P is real and simply characteristic at ξ. The global Ph. L. on $V(P_\xi)$ relative to K, K' implies the local one on $V(P)$ relative to K, K'' and conversely the local Ph. L. with K, K' implies the global one with K, K''.*

PROOF. Suppose $(\partial P/\partial x_n)(\xi)\neq 0$; by Lemma 1.1 there is a constant k s.t. just one zero $\mu(\zeta')$ of $P(\xi+(\zeta',\zeta_n))$ is $<k|\zeta'|$ while the other ones are bounded away from 0 when ζ' is small. Thus μ is an analytic function of ζ' on a neighbourhood of 0. Denoting by $\mu^0(\zeta')$ the unique zero of $P_\xi(\zeta',\zeta_n)$ then by Lemma 2.3

$$|\mu(\zeta')-\mu^0(\zeta')|<\varepsilon|\zeta'|\,,\qquad |\zeta'|<r\ \ (=r(\varepsilon))\,.$$

Define the following p.s.h. function:

$$\tau(\zeta')=|\operatorname{Im}(\mu(\zeta')-\mu^0(\zeta'))|\,,\qquad |\zeta'|<r\,.$$

We have for $|\zeta'|<r$:

$$\tau(\zeta')\leqslant\varepsilon|\zeta'|\,; \qquad \tau(\zeta')\leqslant 0\,, \quad \zeta'\in\mathbb{R}^n\,,$$

where the second inequality is consequence of P's local hyperbolicity at ξ and of P_ξ's hyperbolicity. Thus from the classical Ph. L. principle:

$$\tau(\zeta')\leqslant\varepsilon'\,|\mathrm{Im}\,\zeta'| \quad \text{if} \quad |\zeta'|<r' \text{ where } \varepsilon'=\frac{8}{\pi}(n-1)\,\varepsilon\,,\ r'=\frac{r}{2(n-1)^{\frac{1}{2}}}\,.$$

Assume that the Ph. L. on $V(P_\xi)$ relative to K, K', (δ) holds; so the local at 0 with the same K, K', and with a new constants δ' also holds (Lemma 3.1). Let c be the radius of a ball which covers K and K'. Let φ be w.p.s.h. on $\{P(\zeta)=0,\ |\zeta'-\xi'|<r$ (r small), $\zeta_n-\xi_n$ small$\}$ there verifying:

$$\varphi(\zeta)\leqslant H_K(\mathrm{Im}\,\zeta)+\frac{\delta'}{2}\,r\,; \qquad \varphi(\zeta)\leqslant 0\,, \quad \zeta\in\mathbb{R}^n\,.$$

Define the following map on $\{P_\xi(\zeta)=0,\ |\zeta'|<r\}$:

$$\psi(\zeta',\mu^0(\zeta'))=\varphi\big(\xi+(\zeta',\mu(\zeta'))\big)\,.$$

We have:

$$\psi(\zeta',\mu^0(\zeta'))\leqslant H_K(\mathrm{Im}\,(\zeta',\mu^0(\zeta')))+c\varepsilon r+\frac{\delta'}{2}\,r\leqslant H_K(\mathrm{Im}\,(\zeta',\mu^0(\zeta')))+\delta' r.$$

if ε is small and if $r\leqslant r(\varepsilon)$.

Besides: $\psi(\zeta',\mu^0(\zeta'))\leqslant 0$ if $(\zeta',\mu^0(\zeta'))$ is real since then $(\zeta',\mu(\zeta'))$ also is because of local hyperbolicity. It follows

$$\psi(\zeta',\mu^0(\zeta'))\leqslant H_{K'}(\mathrm{Im}\,(\zeta',\mu^0(\zeta')))\,, \qquad |\zeta'|<\frac{r}{2}\,.$$

So by the definition of ψ:

$$\varphi(\xi+(\zeta',\mu(\zeta')))\leqslant H_{K'}(\mathrm{Im}\,(\zeta',\mu(\zeta')))+c\varepsilon'\,|\mathrm{Im}\,\zeta'|\leqslant H_{K''}(\mathrm{Im}\,(\zeta',\mu(\zeta')))\,, \qquad |\zeta'|<\frac{r}{2}$$

if ε' is so small that $K'_{c\varepsilon'}\subset K''$ (and if $r\leqslant r(\varepsilon)$). The vice versa is at all analogous.

We can paraphrase the theorem saying that if P (real) is simply characteristic at the real non null zero ξ then an open convex set Ω admits the local Ph. L. on $V(P)$ at ξ if and only if it admits the global one on $V(P_\xi)$. And so if P is simply characteristic (at every real zero $\xi\neq 0$) then $PA(\Omega)=A(\Omega)$ if and only if $P_\xi A(\Omega)=A(\Omega)$ for every ξ. It follows

THEOREM 3.3. *Let P be real simply characteristic. Then $PA(\Omega)=A(\Omega)$ for every open convex set Ω.*

PROOF. It is enough to prove that $P_\xi A(\Omega) = A(\Omega)$, $\forall \xi$. Now for every ξ P_ξ is a real linear form and so the cone $\Gamma^*_\xi = \Gamma^*(P_\xi, \operatorname{grad} P(\xi))$ is the bicharacteristic ray through $\operatorname{grad} P(\xi)$. Since Ω is convex then for every $x \in \partial\Omega$ either $x + \Gamma^* \cap \Omega = \emptyset$ or $x - \Gamma^* \cap \Omega = \emptyset$; so we conclude by the result of [5].

We want to explain how it is possible to reduce to the hypotheses of the previous theorem, all polynomials in $\mathbb{R}^n$, $n \leqslant 3$ which are analytically solvable on the whole of $\mathbb{R}^n$. Thus analytic solvability on open convex subsets of $\mathbb{R}^n$, $n \leqslant 3$ becomes equivalent to solvability on $\mathbb{R}^n$ (which is in any way fulfilled when $n = 2$). First let $n = 2$; then P is a product of linear terms which can be proportional to real or complex forms. Therefore these factors are hyperbolic or elliptic according to the two cases listed above.

Finally any convex set admits the Ph.L. principle on the complex lines of the zeros of the previous forms, in the first case because e.g. of the argument used in Theorem 3.3 (or better directly because of the classical Ph.L. on $\mathbb{C}$) and in the second case trivially because of the ellipticity of the form. (This remark was already made in [1]).

For the case $n = 3$ we observe, by repeating the proof of Proposition 6.2 of [3], that Ω admits the Ph.L. principle on a germ of analytic set if and only if it admits the Ph.L. on each irreducible component. Then we claim that in the hypothesis $PA(\mathbb{R}^3) = A(\mathbb{R}^3)$, any germ of $V(P)$ at real non null characteristics ξ, decomposes into the union of locally hyperbolic (5) non singular irreducible components or equivalently that any germ of P factorizes into the product of real simply characteristic irreducible factors (apart from analytic factors non vanishing at ξ). This would allow us to achieve the proof by the same use of Theorem 3.2 as in Theorem 3.3. In fact observe that the hypothesis $PA(\mathbb{R}^3) = A(\mathbb{R}^3)$ assures that any irreducible component of the germ of $V(P)$ at ξ is the complexification of its real part (see the proof of Theorem 6.3 of [3]); this obviously implies that any local non unit factor of P at ξ can be chosen real. If now ξ is not a regular characteristic point of an irreducible component then all points on the homogeneous straight line through ξ are not regular, due to the homogeneity of P. This disagrees with the fact that the real singular parts of the irreducible components of the germ of $V(P)$ at ξ must have real codimension greater or equal 3 (Proposition 27, Section 17 of [1]). We note the conclusion as a corollary.

COROLLARY 3.2. *If $n = 2$ then for any P and any convex open set Ω in $\mathbb{R}^2$ we have $PA(\Omega) = A(\Omega)$.*

If $n = 3$ and $PA(\mathbb{R}^3) = A(\mathbb{R}^3)$ then $PA(\Omega) = A(\Omega)$ for any open convex subset Ω of $\mathbb{R}^3$.

All previous statements hinge on Theorem 3.2 which only holds under the essential assumption that P has regular irreducible germs. In general indeed if P's germ at ξ is not regular, even if locally hyperbolic, we can't affirm that

(5) It is clear from Definition 3.1 that local hyperbolicity is a geometric property of germs of analytic sets rather than of analytic functions.

an open convex set admits the Ph. L. on $V(P)$ at ξ if it admits the Ph. L. on $V(P_\xi)$. In fact consider the polynomial in four variables (already presented in Introduction):

$$P(\zeta) = \zeta_4^2\zeta_3^2 - \zeta_4^2\zeta_1^2 - \zeta_2^4 .$$

P's germ at the real zero $\xi^0 = (0, \dots, 1)$ is obviously locally hyperbolic with respect to the x_3 axis. In the following theorem we prove that when K is the unit ball of the x_3 axis then the Ph. L. on $V(P)$ at ξ^0 relative to K, K' imposes strong estimates « from below » for K'.

THEOREM 3.4. *Let P, ξ^0, K be as above. Let the Ph. L. on $V(P)$ at ξ^0 relative to $K, K', (\delta, r_0)$ hold. Then K' satisfies the following conditions*:

(i) *The range of the projection $K' \ni x \mapsto x_2$ contains the sphere with center at the origin and of radius ε for some ε depending on δ and r_0.*

(ii) *The range of the projection $K' \ni x \mapsto x_1$ contains the unit sphere.*

PROOF. Consider the mapping:

$$\varphi(\zeta) = \left|\operatorname{Im}\left(\zeta_1^2 + \frac{\zeta_2^4}{\zeta_4^2}\right)^{\frac{1}{2}}\right| , \qquad \zeta \in B(\xi^0, r_0)$$

which is obviously p.s.h. on $B(\xi^0, r_0)$.

If $P(\zeta) = 0$, $\zeta \in B(\xi^0, r_0)$ we have:

$$\varphi(\zeta) = |\operatorname{Im}\zeta_3| = H_K(\operatorname{Im}\zeta) .$$

From the continuity it follows:

$$\varphi(\zeta + \xi) \leqslant H_K(\operatorname{Im}\zeta) + \delta r_0 ; \qquad \varphi(\zeta + \xi) \leqslant 0 , \qquad \zeta \in \mathbb{R}^n$$

if $P(\zeta) = 0$, $\zeta \in B(\xi^0, r_0)$, $\xi \in \mathbb{R}^n$, $|\xi| < \varepsilon$ $(= \varepsilon(\delta r_0))$.

From the hypothesis we then get, changing ζ with $\xi^0 + \zeta$:

$$(1) \qquad \varphi(\xi^0 + \zeta + \xi) \leqslant H_{K'}(\operatorname{Im}\zeta) , \qquad P(\xi^0 + \zeta) = 0 , \qquad \zeta \in B\left(0, \frac{r_0}{2}\right), \quad |\xi| < \varepsilon .$$

(*a*) Let $\operatorname{Im}\zeta_2$ be an arbitrary number and set $\zeta_2 = i \operatorname{Im}\zeta_2$; let ζ_1 be solution of the equation $\zeta_1^2 + \zeta_2^4 = 0$ namely $\zeta_1 = \pm i (\operatorname{Im}\zeta_2)^2$. Set:

$$\zeta_t = (t^2\zeta_1, t\zeta_2, 0, 0) , \qquad \xi = (0, \varepsilon, 0, 0) .$$

From (1) we must have:

$$\varphi(\xi^0 + \zeta_t + \xi) \leqslant t H_{K'}(\operatorname{Im}(t\zeta_1, \zeta_2, 0, 0))$$

if t is so small that $|\zeta_t| < r_0/2$. Observe then that:

$$\begin{aligned}\varphi(\xi^0 + \zeta_t + \xi) &= |\mathrm{Im}\,((t^2\zeta_1)^2 + (t\zeta_2 + \varepsilon)^4)^{\frac{1}{2}}| = \\ &= |\mathrm{Im}\,(\varepsilon^4 + i4t\varepsilon^3 \,\mathrm{Im}\,\zeta_2 - 6t^2\varepsilon^2\,\mathrm{Im}\,\zeta_2^2 - i4t^3\varepsilon\,\mathrm{Im}\,\zeta_2^3)^{\frac{1}{2}}| = \\ &= \left|\varepsilon^2 \sin\frac{1}{2}\operatorname{arctg}\frac{4t\varepsilon^3\,\mathrm{Im}\,\zeta_2}{\varepsilon^4}\right| + o(t) = 2t\varepsilon|\mathrm{Im}\,\zeta_2| + o(t)\,, \qquad t\to 0\,.\end{aligned}$$

So the following inequality must hold:

$$2t\varepsilon|\mathrm{Im}\,\zeta_2| + o(t) \leqslant tH_{K'}\left(\mathrm{Im}\frac{\zeta_t}{t}\right), \qquad t \to 0\,.$$

Dividing by t and passing to the limit for $t\to 0$

$$2\varepsilon|\mathrm{Im}\,\zeta_2| \leqslant H_{K'}(0, \mathrm{Im}\,\zeta_2, 0, 0) \qquad \text{which proves (i)}\,.$$

(*b*) Let $\mathrm{Im}\,\zeta_1$ be arbitrary and set $\zeta_1 = i\,\mathrm{Im}\,\zeta_1$; let $\zeta_2 = +\,|\mathrm{Im}\,\zeta_1|^{\frac{1}{2}}$ from which $\zeta_1^2 + \zeta_2^4 = 0$. Set $\zeta_t = (t^2\zeta_1, t\zeta_2, 0, 0)$, $\xi = (\varepsilon, 0, 0, 0)$. Then we have

$$\begin{aligned}\varphi(\xi^0 + \zeta_t + \xi) &= |\mathrm{Im}\,((t^2\zeta_1 + \varepsilon)^2 + t^4\zeta_2^4)^{\frac{1}{2}}| = \\ &= |\mathrm{Im}\,(\varepsilon^2 + i2t^2\varepsilon\,\mathrm{Im}\,\zeta_1)^{\frac{1}{2}}| = t^2\,|\mathrm{Im}\,\zeta_1| + o(t^2)\,, \qquad t\to 0\,.\end{aligned}$$

Thus by (1) the following inequality is fulfilled:

$$t^2\,|\mathrm{Im}\,\zeta_1| + o(t^2) \leqslant t^2 H_{K'}\left(\mathrm{Im}\frac{\zeta_t}{t^2}\right), \qquad t\to 0\,,$$

Again dividing by t^2 and at the limit for $t\to 0$ (also remembering that ζ_2 is real) we have:

$$|\mathrm{Im}\,\zeta_1| \leqslant H_{K'}(\mathrm{Im}\,\zeta_1, 0, 0, 0)\,,$$

which obviously implies (ii).

From the previous theorem we can draw the following conclusions.

(I) The compact set K' must have a non null « depth » in the direction of the x_2 axis. Consider then the real half space $\Omega = \{x\in R^4\colon x_2 < 0\}$. If $K \subset\subset \Omega$ is parallel to the x_3 axis, of unit length, and $K\to\partial\Omega$, then in order to find $K' \subset\subset \Omega$ for which the Ph. L. implication holds we must let $\delta r_0 \to 0$. We'll make use of this observation in proving that the half space Ω admits the Ph. L. principle on $V(P)$ at ξ^0 (see [6]).

(II) *The main consequence, however, is that the real half space* $\Omega = \{x\in R^4\colon x_1 < 0\}$ *doesn't admit the Ph. L. on* $V(P)$ *at* ξ^0. In fact if K is a segment of length 1 parallel to the x_3 axis and contained in Ω then Ω would contain a K'

whose projection on the x_1 axis contains the segment of length 1 at with center at the projection of K. This is obviously impossible when $K \to \partial\Omega$.

On the other hand every open convex set, and consequently the half space Ω also, admits the global Ph. L. on the variety $\{P_{\xi^0}(\zeta)\ (=\zeta_3^2-\zeta_1^2)=0\}$; in fact P_{ξ^0} is the product of two real linear forms and so use Theorem 3.3. This phenomenon seems to depend on the fact that P_{ξ^0} is reducible without P's germ at ξ^0 being so.

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