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Analysis of a three-dimensional mathematical model of water circulation in a basin (**)

Analisi di un modello matematico tridimensionale relativo al moto di un liquido in un bacino

SUNTO. — Si considera un modello tridimensionale relativo al moto di un fluido viscoso incomprimibile in un bacino e si danno risultati di esistenza, unicità e dipendenza continua che provano che il modello stesso è ben posto. Il modello consta di disequazioni variazionali ottenute dalle equazioni di Navier-Stokes imponendo alle grandezze considerate limitazioni di ovvio significato fisico, in modo che il modello stesso risulti « fisicamente consistente ».

1. - INTRODUCTION AND GENERAL REMARKS ON MATHEMATICAL MODELS

A mathematical model of a physical problem consists, in general, of:

a) A system of equations (constitutive equations); these are, very often, partial differential equations, to which are associated:

b) Initial and (or) boundary value conditions;

c) Consistency conditions, under which equations *a*) have been deduced.

For example, in the classical model of the vibrating string, *a*) is represented by the D'Alembert equation, while *c*) imposes that the slope of the string at any point (which is proportional to the tension) is « small ».

In the study of a mathematical model, condition *c*) is generally overlooked and the model itself is associated only to *a*) and *b*); consequently, the eventual solutions of *a*), *b*) may not have any physical significance or, as we shall say, the model is not physically consistent. It may therefore be expected that, even if the original physical problem is well posed, the corresponding problem *a*), *b*) does not sometimes have the same property.

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Of the many problems related to the study of mathematical models in dynamics, two appear to be of special interest:

α) Whether the model is well posed and physically consistent, at least in a neighbourhood of the initial time $\bar{t}$;

β) Whether the model is well posed whenever it is physically consistent.

An affirmative reply to question α) can be given provided a local existence, uniqueness and continuous dependence theorem of the solution of *a*), *b*) is proved and it is shown that such a solution satisfies *c*) in a sufficiently small neighbourhood of $\bar{t}$.

Question β) can, on the other hand, be reformulated in the following way:

β') Is it possible to prove an existence, uniqueness and continuous dependence theorem of a solution of *a*), *b*) in the time interval $\bar{t} \leqslant t < t'$, where t' is the infimum of the values of t for which *c*) does not hold?

It is obvious that, if a global existence, uniqueness and continuous dependence theorem and a local regularity theorem held for the solutions of *a*), *b*), this would imply a positive reply to questions α) and β). On the other hand, if *a*) is non linear, no such global theorem, except in special cases, is known, the theorem itself having only a local character (for $\bar{t} \leqslant t < t''$, t'' sufficiently small).

A theorem of this type does not however exclude the possibility that the solution exist also for $t > t''$ and satisfy condition *c*); hence, while it can give an answer to question α), no information can be obtained from it regarding β), which must therefore be considered independently.

From a general point of view, we can say that β) deals with the intrinsic mathematical structure of the model, assuring us that, whenever it is reasonable to expect that the model represents a physical phenomenon, then the model is well posed; α), on the other hand, indicates whether under appropriate assumptions on the data the model is locally physically consistent.

2. - Motion in a closed domain: the Navier-Stokes model

As an example of what has been said in § 1, we shall now consider the motion of a viscous, incompressible fluid of unit density and viscosity μ in a given domain Ω, with boundary Γ constituted by a material surface.

The Navier-Stokes model corresponding to this problem is represented by the Navier-Stokes equations (constitutive equations).

$$(2.1) \qquad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \mu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} = 0, \end{cases}$$

by the initial and boundary conditions

$$(2.2)\qquad \begin{cases} \boldsymbol{u}(x,0)=\overline{\boldsymbol{u}}(x)\,, & (x=\{x_1,\dots,x_m\})\,,\ x\in\Omega\,,\\ \boldsymbol{u}(x,t)=0\,, & x\in\Gamma\,,\ 0\leqslant t\leqslant T \end{cases}$$

and by the consistency condition

$$(2.3)\qquad |\boldsymbol{u}|<M_2\,.$$

Relation (2.3) is deduced from the fact that, since the Navier-Stokes equations are non relativistic, the velocity $|\boldsymbol{u}|$ of the fluid must not approach the speed of light.

Let us now recall the following results regarding the problem considered above.

i) If $m=2$, there exists a global weak solution of (2.1), (2.2); moreover, this solution is unique, and depends continuously on the data [1].

ii) If $m=3$, there exists a weak global solution of (2.1), (2.2) [2].

iii) If $m=3$, such a solution is unique and depends continuously on the data in a sufficiently small neighbourhood of $t=0$, provided $\boldsymbol{f}$ and $\overline{\boldsymbol{u}}$ are « smooth » [3].

iv) Under the same assumptions of iii) and, in addition, assuming that Γ is « smooth », the solution satisfies (2.3) in a sufficiently small neighbourhood of $t=0$ and depends continuously on the data [4].

Hence, bearing in mind what has been said in § 1, in the 2-dimensional case question β) can, by i), be answered affirmatively and the same applies, by iii), to α), provided $\boldsymbol{f}, \overline{\boldsymbol{u}}$ are « smooth ». When $m=3$, the reply is, by ii), iii), affirmative to β) if $\boldsymbol{f}, \overline{\boldsymbol{u}}$ are « smooth »; if, in addition, Γ is « smooth », then also α) is verified, by iv).

3. - Free surface motion in a basin: the Navier-Stokes model

Some mathematical models connected with the motion of a viscous, incomprensible fluid in a basin, lake, river, canal, etc. have been studied, from the point of view of functional analysis, in some recent papers ([5], [6], [7], [8]).

The models considered (Liggett, Welander, Saint-Venant) were one- and two-dimensional in the space variables, the unknown functions being the shape of the free surface and appropriate mean velocities. It was shown that these models are well posed whenever they are physically consistent (question β)); this was done by associating to each model a system of variational inequalities which take into account, in addition to the constitutive equations, the consistency conditions and proving a global existence theorem for the solutions of such inequalities and a uniqueness and continuous dependence theorem whenever the solution satisfies c).

We now want to study a three-dimensional model, in which the constitutive equations are represented by the classical Navier-Stokes equations and which we shall therefore call the Navier-Stokes model.

Our aim is to prove that this model is well posed whenever it is physically consistent, i.e. to give a positive answer to equation β); as we shall see, it is however not possible to reply to question α).

Let us introduce the following notations.

A = open, bounded set of the (x, y) plane, with boundary satisfying locally a Lipschitz condition;

$$Q_\varphi \quad = \{(x, y) \in A, -1 < z < \varphi(x, y, t), 0 < t < T\} ;$$

$$\Omega_{\varphi,t} \quad = \{(x, y) \in A, -1 < z < \varphi(x, y, t)\} ;$$

$$A_{\bar{t}} \quad = \{(x, y) \in A, 0 < t < \bar{t}\} , \qquad A_T = A ;$$

$$\Gamma_{1,\varphi,t} = \partial\Omega_{\varphi,t} \cap (z = \varphi(x, y, t)) ;$$

$$\Gamma_2 \quad = \partial\Omega_{\varphi,t} - \Gamma_{1,\varphi,t} .$$

In what follows, $z = \varphi(x, y, t)$ will represent the equation of the free surface at the time t, $\Omega_{\varphi,t}$ the domain in which the motion takes place (which obviously depends on t), $\Gamma_{1,\varphi,t}$ the free surface, Γ_2 the bottom and sides of the basin.

Denoting by $\boldsymbol{u}$ the velocity of the fluid, by p its pressure, by μ the viscosity and by $\boldsymbol{f}$ the external forces, the motion in Q_φ of the fluid (assumed to be incompressible and of unit density) is governed by the classical Navier-Stokes equations

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \mu\, \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} + \nabla p = \boldsymbol{f} , \\ \operatorname{div} \boldsymbol{u} = 0 , \end{cases}$$

to which must be added the boundary conditions

$$\text{(3.2)} \qquad \boldsymbol{u}(x, y, z, t) = 0 \text{ when } (x, y, z) \in \Gamma_2,\ 0 < t < T,$$

$$\text{(3.3)} \qquad \frac{\partial \varphi}{\partial t} + u_1(x, y, \varphi(x, y, t), t) \frac{\partial \varphi}{\partial x} + u_2(x, y, \varphi(x, y, t), t) \frac{\partial y}{\partial \varphi} - $$
$$- u_3(x, y, \varphi(x, y, t), t) = 0 \qquad \text{when } (x, y) \in A ,\ 0 < t < T ,$$

$$\text{(3.4)} \qquad p(x, y, \varphi(x, y, t), t) = 0 \qquad \text{when } (x, y) \in A ,\ 0 < t < T ,$$

$$\text{(3.5)} \qquad \frac{\partial \boldsymbol{u}(x, y, \varphi(x, y, t), t)}{\partial \nu} = 0 \qquad \text{when } (x, y) \in A ,\ 0 < t < T ,$$

where u_i are the components of $\boldsymbol{u}$ and $\boldsymbol{\nu}$ denotes the normal to ∂A.

Relation (3.2) represents the classical « non slip » condition on the bottom and at the sides of the basin; (3.3) is obtained by assuming (as is customary in hydrodynamical problems with a free boundary) that the velocity of particles on the free boundary is tangent to the free boundary itself; (3.4) imposes that the pressure on the free boundary equals the atmospheric pressure (which we assume $= 0$); finally, (3.5) expresses the condition that the drag exercised by the atmosphere on the free surface is negligeable.

We assume, moreover, that, when $t = 0$, the following initial conditions hold:

$$\varphi(x, y, 0) = \bar{\varphi}(x, y) \qquad \text{when } (x, y) \in A, \tag{3.6}$$

$$\boldsymbol{u}(x, y, z, 0) = \bar{\boldsymbol{u}}(x, y, z) \qquad \text{when } (x, y, z) \in \Omega_{\bar{\varphi},0}. \tag{3.7}$$

Obviously, $\bar{\boldsymbol{u}} = 0$ on Γ_2 and $\bar{\varphi} > -1$, while we shall assume that $\bar{\varphi} = 0$ on ∂A.

Relations (3.1), (3.3) represent the constitutive equations of the Navier-Stokes model. Observe, on the other hand, that condition (3.3) can reasonably be imposed only if the free surface is sufficiently « smooth » and maintains a positive distance from the bottom; denoting by M_1, σ two appropriate positive constants, we shall therefore assume that

$$\begin{cases} |\varphi_{xx}| < M_1, & |\varphi_{yy}| < M_1, & |\varphi_{tt}| < M_1, \\ |\varphi_{xt}| < M_1, & |\varphi_{yt}| < M_1, & \varphi > -1 + \sigma. \end{cases} \tag{3.8}$$

Moreover, as already pointed out at § 2, the Navier-Stokes equations have a physical meaning provided the velocity is not too great (i.e. does not approach the speed of light); hence, we must assume that

$$|\boldsymbol{u}| < M_2. \tag{3.9}$$

Relations (3.8), (3.9) represent the consistency conditions of the Navier-Stokes model.

Observation 3.1: For the sake of simplicity, we have assumed the bottom of the basin to be represented by the surface $z = -1$; such a surface could however be substituted by any sufficiently smooth surface $z = \alpha(x, y)$, in which case the last of (3.8) would become $\varphi < \alpha + \sigma$.

Observation 3.2: From (3.2), (3.3) it follows that $\varphi_t = 0$ when $(x, y) \in \partial A$; consequently, since $\bar{\varphi} = 0$ on ∂A, $\varphi(x, y, t) = 0$ when $(x, y) \in \partial A$, $0 < t < T$.

4. - Weak formulation of the constitutive equations

Let $\boldsymbol{h}$ be a function $\in C^1(Q_\varphi)$, with $\operatorname{div} \boldsymbol{h} = 0$ and $\boldsymbol{h} = 0$ on $\Gamma_2 \times (0, T)$. Multiplying the first of (3.1) by $\boldsymbol{h}$, integrating over Q_φ, applying Green's formula and bearing in mind (3.4), (3.5), we obtain

$$\int\limits_{Q_\varphi}\left\{\frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{h}+\mu\left(\frac{\partial \boldsymbol{u}}{\partial x}\cdot\frac{\partial \boldsymbol{h}}{\partial x}+\frac{\partial \boldsymbol{u}}{\partial y}\cdot\frac{\partial \boldsymbol{h}}{\partial y}+\frac{\partial \boldsymbol{u}}{\partial z}\cdot\frac{\partial \boldsymbol{h}}{\partial z}\right)+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{h}-\boldsymbol{f}\cdot\boldsymbol{h}\right\}dQ_\varphi=0\,. \tag{4.1}$$

We now introduce a new set of variables, ξ, η, ζ, τ defined by

$$\xi=x\,,\qquad \eta=y\,,\qquad \zeta=\frac{z-\varphi(x,y,t)}{1+\varphi(x,y,t)}\,,\qquad \tau=t\,. \tag{4.2}$$

Relations (4.2) transform the sets defined at § 1 in the following way:

$$\begin{aligned}
&A\ \to A^*=A\,;\qquad \Lambda\to\Lambda^*=\Lambda\,;\\
&Q_\varphi\ \to Q^*=\{(\xi,\eta)\in A,\,-1<\zeta<0,\,0<\tau<T\}\,;\\
&\Omega_{\varphi,t}\ \to\Omega^*=\{(\xi,\eta)\in A,\,-1<\zeta<0\}\,;\\
&\Gamma_{1,\varphi,t}\to\Gamma_1^*=\partial\Omega^*\cap(\zeta=0)\,;\qquad \Gamma_2\to\Gamma_2^*=\partial\Omega^*-\Gamma_1^*=\Gamma_2\,,
\end{aligned}$$

the free surface being now represented by the plane $\zeta=0$.

Setting $\boldsymbol{v}^*(\xi,\eta,\zeta,\tau)=\boldsymbol{v}(\xi,\eta,(1+\varphi)\zeta+\varphi,\tau)$, equation (4.1) becomes, on the other hand,

$$\begin{aligned}
\int\limits_0^T\{(\boldsymbol{u}^{*\prime}(\tau),\boldsymbol{h}^*(\tau))_{L^2(\Omega^*)}+\mu a_\varphi(\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau))&+\\
+\,b_\varphi(\boldsymbol{u}^*(\tau),\boldsymbol{u}^*(\tau),\boldsymbol{h}^*(\tau))-(\boldsymbol{f}^*(\tau),\boldsymbol{h}^*(\tau))_{L^2(\Omega^*)}\}\,d\tau&=0
\end{aligned} \tag{4.3}$$

where we have set

$$(\boldsymbol{u}^*,\boldsymbol{v}^*)_{L^2(\Omega^*)}=\int\limits_{\Omega^*}\boldsymbol{u}^*\cdot\boldsymbol{v}^*(\varphi+1)\,d\xi\,d\eta\,d\zeta\,, \tag{4.4}$$

$$\begin{aligned}
a_\varphi(\boldsymbol{u}^*,\boldsymbol{v}^*)=\int\limits_{\Omega^*}\Big\{&\boldsymbol{u}_\xi^*\cdot\boldsymbol{v}_\xi^*+\boldsymbol{u}_\eta^*\cdot\boldsymbol{v}_\eta^*+\frac{1}{(\varphi+1)^2}(1+(\zeta+1)^2(\varphi_\xi^2+\varphi_\eta^2))\,\boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}_\zeta^*-\\
&-\frac{\zeta+1}{\varphi+1}[(\boldsymbol{u}_\xi^*\cdot\boldsymbol{v}_\zeta^*+\boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}_\xi^*)\varphi_\xi+(\boldsymbol{u}_\eta^*\cdot\boldsymbol{v}_\zeta^*+\boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}_\eta^*)\varphi_\eta]-\frac{\zeta+1}{\varphi+1}\varphi_\tau\boldsymbol{u}_\zeta^*\cdot\boldsymbol{v}^*\Big\}\cdot\\
&\cdot(\varphi+1)\,d\xi\,d\eta\,d\zeta\,,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
b_\varphi(\boldsymbol{u}^*,\boldsymbol{v}^*,\boldsymbol{h}^*)=\int\limits_{\Omega^*}\left\{\boldsymbol{u}^*\cdot\left(\frac{\partial}{\partial\xi}\boldsymbol{i}^*+\frac{\partial}{\partial\eta}\boldsymbol{j}^*+\frac{1}{\varphi+1}\frac{\partial}{\partial\zeta}\boldsymbol{k}^*\right)\right\}\boldsymbol{v}^*\cdot&\\
\cdot\boldsymbol{h}(\varphi+1)\,d\xi\,d\eta\,d\zeta\,,&
\end{aligned} \tag{4.6}$$

$\boldsymbol{i}^*,\boldsymbol{j}^*,\boldsymbol{k}^*$ being the unit vectors of the ξ, η, ζ axes.

Applying (4.2) to the second of (3.1) and to (3.3) and denoting by v_j^* the components of $\boldsymbol{v}^*$ in the new reference frame, we obtain, respectively

$$\frac{\partial u_1^*}{\partial \xi} + \frac{\partial u_2^*}{\partial \eta} + \frac{\partial u_3^*}{\partial \zeta} + \frac{\varphi_\xi}{\varphi + 1} u_1^* + \frac{\varphi_\eta}{\varphi + 1} u_2^* = \mathrm{div}_\varphi \boldsymbol{u}^* = 0\,, \tag{4.7}$$

$$\frac{\partial \varphi}{\partial \tau} - u_3^*(\xi, \eta, 0, \tau)(\varphi + 1) = 0\,. \tag{4.8}$$

Assuming now that $\boldsymbol{u}^* = 0$ on $\Gamma_2^* \times (0, T)$, $\boldsymbol{u}^*(\xi, \eta, \zeta, 0) = \bar{\boldsymbol{u}}^*(\xi, \eta, \zeta)$ in Ω^*, $\varphi(\xi, \eta, 0) = \bar{\varphi}(\xi, \eta)$ in A and that (4.3) holds $\forall \boldsymbol{h}^*(\tau) \in L^2(0, T; H^1(\Omega^*))$ with $\mathrm{div}_\varphi \boldsymbol{h}^* = 0$ and $\boldsymbol{h}^* = 0$ on $\Gamma_2 \times (0, T)$, equations (4.3), (4.7), (4.8) are equivalent to (3.1), ..., (3.7) and can be taken to represent the constitutive equations of the model.

It may be noted that the boundary conditions (3.4), (3.5) no longer appear explicity, as they have already been taken into account in the weak formulation of (4.3).

Observation 4.1: Assume that $\boldsymbol{u}, \boldsymbol{h}$ satisfy (4.1), (3.2), (3.3); we have then, $\forall t$,

$$\int_{\Omega_{\varphi,t}} (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{h}\, d\Omega = -\int_{\Omega_{\varphi,t}} (\boldsymbol{u} \cdot \nabla) \boldsymbol{h} \cdot \boldsymbol{u}\, d\Omega + \int_{\Gamma_{1,\varphi,t}} (\boldsymbol{u} \cdot \boldsymbol{\nu})(\boldsymbol{u} \cdot \boldsymbol{h})\, d\Gamma\,. \tag{4.9}$$

Hence, being, by (3.3), $\varphi_t = (1 + \varphi_x^2 + \varphi_y^2)^{\frac{1}{2}} (\boldsymbol{u} \cdot \boldsymbol{\nu})$, il follows from (4.9), setting $g_\varphi = \varphi_t (1 + \varphi_x^2 + \varphi_y^2)^{\frac{1}{2}}$,

$$\int_{\Gamma_{1,\varphi,t}} (\boldsymbol{u} \cdot \boldsymbol{\nu})(\boldsymbol{u} \cdot \boldsymbol{h})\, d\Gamma = \int_{\Gamma_{1,\varphi,t}} g_\varphi \boldsymbol{u} \cdot \boldsymbol{h}\, d\Gamma \tag{4.10}$$

and consequently

$$\int_{\Omega_{\varphi,t}} (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{h}\, d\Omega = -\int_{\Omega_{\varphi,t}} (\boldsymbol{u} \cdot \nabla) \boldsymbol{h} \cdot \boldsymbol{u}\, d\Omega + \int_{\Gamma_{1,\varphi,t}} g_\varphi \boldsymbol{u} \cdot \boldsymbol{h}\, d\Gamma\,. \tag{4.11}$$

Applying to (4.11) trasformation (4.2), we then obtain, by (4.6),

$$b_\varphi(\boldsymbol{u}^*, \boldsymbol{u}^*, \boldsymbol{h}) = -b_\varphi(\boldsymbol{u}^*, \boldsymbol{h}^*, \boldsymbol{u}^*) + (g_\varphi^* \boldsymbol{u}^*, \boldsymbol{h}^*)_{L^2(\Gamma_1^*)}\,. \tag{4.12}$$

We may therefore substitute, where convenient, the term $b_\varphi(\boldsymbol{u}^*, \boldsymbol{u}^*, \boldsymbol{h}^*)$ with the expression $-b_\varphi(\boldsymbol{u}^*, \boldsymbol{h}^*, \boldsymbol{u}^*) + (g_\varphi^* \boldsymbol{u}^*, \boldsymbol{h}^*)_{L^2(\Gamma_1^*)}$.

5. - Some functional spaces

In the following paragraphs we shall, for the sake of simplicity, denote by x, y, z, t (instead of ξ, η, ζ, τ) the new independent variables and drop the star symbol used in § 4 to indicate the sets and functions obtained by the application of transformation (4.2). This will not create any confusion, since we shall never have to revert to the original reference frame.

Let ψ be a function $\in C^1(\Lambda)$, $\mathcal{M} = \{\boldsymbol{v} \in C^\infty(\bar{\Omega}) : \boldsymbol{v} = 0 \text{ on } \Gamma_2\}$ and denote by $\tilde{H}_0^s$ (s integer $\geqslant 0$) the closure of $\mathcal{M}$ in $H^s(\Omega)$ and by V_ψ^s the space $\{\boldsymbol{v} \in H_0^s;\ \operatorname{div}_\psi \boldsymbol{v} = 0\}$; setting

$$(5.1) \qquad (\boldsymbol{u}, \boldsymbol{v})_{V_\psi^s} = (\boldsymbol{u}, \boldsymbol{v})_{H^s(\Omega)},$$

V_ψ^s are Hilbert spaces, the embedding of V_ψ^{s+1} in V_ψ^s being moreover completely continuous.

By the usual interpolation procedure between Hilbert spaces, it is then possible to define V_ψ^σ $\forall$ real $\sigma \geqslant 0$ in such a way that

$$(5.2) \qquad [V_\psi^\alpha, V_\psi^\beta]_\theta = V_\psi^{\alpha(1-\theta)+\beta\theta} \qquad (\alpha, \beta \geqslant 0,\ 0 \leqslant \theta \leqslant 1)$$

and $V_\psi^{\sigma_1}$ is dense and has a compact embedding in $V_\psi^{\sigma_2}$ if $\sigma_1 > \sigma_2 \geqslant 0$. Moreover, we shall denote by $V_\psi^{-\sigma}$ the dual space of V_ψ^σ, so that (5.2) holds $\forall$ real α, β.

In what follows, we shall have to consider the case in which the function ψ depends also on the parameter t; we shall therefore denote by $V_{\psi,\bar{t}}^\sigma$ the space corresponding to the function $\psi(x, y, \bar{t})$. For the sake of simplicity, however, wherever there may not be any possibility of confusion, we shall set $V_\psi = V_{\psi,t}^1$, $V_\psi' = (V_{\psi,t}^1)'$ and denote, $\forall$ fixed t, the duality between V_ψ' and V_ψ by the notation $\langle\, , \rangle$.

6. - Associated variational inequalities

As has already been pointed out, (4.3), (4.7), (4.8) represent the constitutive equations of the Navier-Stokes model, which however have physical meaning only if the consistency conditions (3.8), (3.9) are verified.

In order to take these last into account, we shall introduce some appropriate variational inequalities.

Let K_1, K_2, K_3 be three closed, convex sets defined by

$$(6.1) \qquad K_1 = \{\boldsymbol{v} \in H^1(\Omega) : |\boldsymbol{v}| \leqslant M_2 \text{ a.e. in } \Omega\},$$

$$(6.2) \qquad K_2 = \{\psi \in H^2(\Lambda) : |\psi_{xx}| \leqslant M_1,\ |\psi_{yy}| \leqslant M_1,\ |\psi_{tt}| \leqslant M_1,$$
$$|\psi_{xt}| \leqslant M_1,\ |\psi_{yt}| \leqslant M_1,\ |\psi_{tt}| \leqslant M_1,\ \psi \geqslant -1 + \sigma \text{ a.e. in } \Lambda\},$$

$$(6.3) \qquad K_3 = \{\boldsymbol{w} \in L^2(\Gamma_1 \times (0, T)) : |\boldsymbol{w}| \leqslant M_2 \text{ a.e. in } \Gamma_1 \times (0, T)\},$$

M_1, M_1, σ being the constants appearing in (3.8), (3.9), and consider the following variational inequalities, associated respectively to (4.3), (4.8)

$$(6.4)\qquad \tfrac{1}{2}\|\boldsymbol{u}(t)-\boldsymbol{h}(t)\|^2_{V^0_\varphi}-\tfrac{1}{2}\|\overline{\boldsymbol{u}}-\boldsymbol{h}(0)\|^2_{V^0_\varphi}+\int_0^t\{\langle\boldsymbol{h}',\boldsymbol{u}-\boldsymbol{h}\rangle+\mu a_\varphi(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})-$$

$$-b_\varphi(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h},\boldsymbol{u})+(g_\varphi\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})_{L^2(\Gamma_1)}-\langle\boldsymbol{f},\boldsymbol{u}-\boldsymbol{h}\rangle\}\,d\eta\leqslant 0\,,$$

$$(6.5)\qquad \int_{\Lambda_t}\left(\frac{\partial\varphi}{\partial\eta}-u_3(x,y,0,\eta)(\varphi+1)\right)(\varphi-l)\,d\Lambda\leqslant 0\,.$$

Bearing in mind the notations introduced at § 5, we shall then say that $\{\boldsymbol{u},\varphi\}$ *is a solution in* $[0,T]$ *of* (6.4), (6.5) *satisfying the initial conditions* (3.6), (3.7) *if*:

i) $\boldsymbol{u}(t)\in L^2(0,T;V_\varphi\cap K_1)$ *and satisfies* (6.4) *a.e. on* $(0,T)$ $\forall\boldsymbol{h}(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^1(0,T;V'_\varphi)$

ii) $\varphi(t)\in L^2(0,T;H^1_0(A))\cap K_2$, *satisfies*, $\forall t\in[0,T]$, (6.5) $\forall t\in K_2$ *and is such that* $\varphi(0)=\bar\varphi$.

Let us now recall the following well known property of the solutions of differential inequalities.

If $\{\boldsymbol{u},\varphi\}$ satisfy i), ii) and, for $t\in(0,t')$, do not touch the boundary of K_1 and K_2, then $\{\boldsymbol{u},\varphi\}$ are solutions, in an appropriate weak sense, of (4.3), (4.7), (4.8).

It is therefore natural to associate system (6.4), (6.5) *to the Navier-Stokes model*; in fact, by the property recalled above, if $\{\boldsymbol{u},\varphi\}$ is a solution of (6.4), (6.5), then it is also a solution of the physical problem considered, provided the Navier-Stokes model is physically consistent.

7. - Analysis of the Navier-Stokes model

According to what has been shown in § 6, the Navier-Stokes model is associated with the functions $\boldsymbol{u}$, φ which satisfy conditions i), ii). In the following three paragraphs we shall give the proof of an existence theorem, which we now state.

Theorem 1: *Assume that*

$$\boldsymbol{f}(t)\in L^2(0,T;V'_\varphi)\,,\qquad \bar\varphi\in H^1_0(A)\,,\qquad |\bar\varphi_{xx}|\leqslant M_1\,,$$

$$|\bar\varphi_{yy}|\leqslant M_1\,,\qquad \bar\varphi\geqslant -1+\sigma\ (\sigma>0)\,,\qquad \overline{\boldsymbol{u}}\in K_1\cap V^0_\varphi\,.$$

Then there exist $\{\boldsymbol{u},\varphi\}$ *satisfying* i), ii).

The problem of the uniqueness of the solution and, consequently, question β) considered in § 1, will be studied in § 11.

Regarding question α), on the other hand, no positive answer can be given. Observe, in fact, as is well known from the theory of Navier-Stokes equations, it is possible to prove the boundedness of weak solutions under appropriate smoothness assumptions on the data (for instance, boundary of class C^2). In our case, however, the boundary is constituted by the free surface and by the sides and bottom of the basin and all the points at which the free surface meets the sides are necessarily angle points. Hence the boundary can only be assumed of class C^0.

8. - Some auxiliary lemmas

We shall now prove some lemmas which are preliminary to the proof of Theorem 1.

Lemma 1: *Assume that*

$$\varphi \in K_2\,, \quad \bar{\boldsymbol{u}} \in V^0_{\varphi,0}\,, \quad \boldsymbol{f}(t) \in L^2(0, T;\, V'_\varphi) \qquad \big(\Rightarrow g_\varphi \in L^\infty(\Gamma_1 \times (0, T))\big)\,.$$

Then there exist a unique function

$$\boldsymbol{u}(t) \in L^2(0, T;\, V_\varphi) \cap H^1(0, T;\, V'_\varphi) \cap C^0(0, T;\, L^2(\Omega))$$

such that $\boldsymbol{u}(0) = \bar{\boldsymbol{u}}$ *and*

$$\int_0^t \{\langle \boldsymbol{u}', \boldsymbol{h}\rangle + \mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) + (g_\varphi \boldsymbol{u}, \boldsymbol{h}) - \langle \boldsymbol{f}, \boldsymbol{h}\rangle\}\, d\eta = 0 \tag{8.1}$$

$\forall \boldsymbol{h}(t) \in L^2(0, T;\, V_\varphi)$ *and* $\forall t \in [0, T]$.

Observe that, by applying in a straightforward way a well known existence theorem for linear abstract differential equations (see, for instance, [9], ch. 3, Th. 1.1), we can prove that there exists $\boldsymbol{u}(t) \in L^2(0, T;\, V_\varphi)$ satisfying rhe equation

$$\int_0^T \{-(\boldsymbol{u}, \boldsymbol{h}')_{L^2(\Omega)} + \mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) + (g_\varphi \boldsymbol{u}, \boldsymbol{h})_{L^2(\Gamma_1)} - \langle \boldsymbol{f}, \boldsymbol{h}\rangle\}\, dt - (\bar{\boldsymbol{u}}, \boldsymbol{h}(0))_{L^2(\Omega)} = 0 \tag{8.2}$$

$\forall \boldsymbol{h}(t) \in L^2(0, T;\, V_\varphi) \cap H^1(0, T;\, L^2(\Omega))$, $\boldsymbol{h}(T) = 0$.

In fact, by (4.5) and classical embedding and trace theorems, there exist three positive constants, $\lambda, \alpha_1, \alpha_2$, such that

$$\alpha_1 \|\boldsymbol{v}\|_{V_\varphi} \geqslant \mu a_\varphi(\boldsymbol{v}, \boldsymbol{v}) + (g_\varphi \boldsymbol{v}, \boldsymbol{v})_{L^2(\Gamma_1)} + \lambda \|\boldsymbol{v}\|^2_{L^2(\Omega)} \geqslant \alpha_2 \|\boldsymbol{v}\|^2_{V_\varphi} \qquad \forall \boldsymbol{v} \in V_\varphi\,. \tag{8.3}$$

On the other hand, by (8.2),

$$\int_0^T \langle \boldsymbol{u}', \boldsymbol{h}\rangle\, dt = \int_0^T \{-\mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) - (g_\varphi \boldsymbol{u}, \boldsymbol{h})_{L^2(\Gamma_1)} + \langle \boldsymbol{f}, \boldsymbol{h}\rangle\}\, dt \tag{8.4}$$

$\forall \boldsymbol{h}(t) \in L^2(0, T; V_\varphi) \cap H_0^1(0, T; L^2(\Omega))$ and

$$\left|\int_0^T \{-\mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) - (g_\varphi \boldsymbol{u}, \boldsymbol{h})_{L^2(\Gamma_1)} + \langle \boldsymbol{f}, \boldsymbol{h}\rangle\}\, dt\right| \leqslant c \|\boldsymbol{h}\|_{L^2(0,T;V_\varphi)} \tag{8.5}$$

$\forall \boldsymbol{h}(t) \in L^2(0, T; V_\varphi)$.

Hence

$$\int_0^T \{-\mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) - (g_\varphi \boldsymbol{u}, \boldsymbol{h})_{L^2(\Gamma_1)} + \langle \boldsymbol{f}, \boldsymbol{h}\rangle\}\, dt = \int_0^T \langle B\boldsymbol{u}, \boldsymbol{h}\rangle\, dt \tag{8.6}$$

with $B\boldsymbol{u} \in L^2(0, T; V_\varphi')$.

It follows therefore from (8.4) that $\boldsymbol{u}' = B\boldsymbol{u}$ in $L^2(0, T; V_\varphi') \cup H^{-1}(0, T; L^2(\Omega))$ and, consequently, $\boldsymbol{u}(t) \in H^1(0, T; V_\varphi')$.

The property that $\boldsymbol{u}(t) \in C^0(0, T; L^2(\Omega))$ can then be proved in a classical way (see, for instance [10]).

It is then obvious, by (8.2), that $\boldsymbol{u}(t)$ satisfies (8.1).

The uniqueness of the solution can be deduced directly from (8.1) setting $\boldsymbol{h} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, with $\boldsymbol{u}_1, \boldsymbol{u}_2$ solutions of (8.1).

LEMMA 2: *Suppose that the assumptions made in Lemma* 1 *are verified and let* P *be the operator defined by*

$$P\boldsymbol{v} = \boldsymbol{v} \quad \text{when } |\boldsymbol{v}| \leqslant M_2, \qquad P\boldsymbol{v} = M_2 \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \quad \text{when } |\boldsymbol{v}| > M_2. \tag{8.7}$$

There exists then, $\forall \varepsilon > 0$, *a unique function*

$$\boldsymbol{u}(t) \in L^2(0, T; V_\varphi) \cap H^1(0, T; V_\varphi') \cap C^0(0, T; L^2(\Omega))$$

such that $\boldsymbol{u}(0) = \bar{\boldsymbol{u}}$ *and*

$$\int_0^t \Big\{\langle \boldsymbol{u}', \boldsymbol{h}\rangle + \mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) + (g_\varphi \boldsymbol{u}, \boldsymbol{h})_{L^2(\Gamma_1)} + \frac{1}{\varepsilon}(\boldsymbol{u} - P\boldsymbol{u}, \boldsymbol{h})_{L^2(\Omega)} - \langle \boldsymbol{f}, \boldsymbol{h}\rangle\Big\}\, d\eta = 0, \tag{8.8}$$

$\forall \boldsymbol{h}(t) \in L^2(0, T; V_\varphi)$ *and* $\forall t \in [0, T]$.

Consider, in fact, the transformation, from $H^{\frac{1}{2}}(0, T; V^{\frac{1}{2}}_{\varphi,t})$ to itself, $\boldsymbol{v} \to \boldsymbol{u} = S_{\varepsilon,\lambda}(\boldsymbol{v})$, where $\boldsymbol{u}$ is the solution, given by Lemma 1 of the equation

$$(8.9)\qquad \int_0^t \Big\{\langle \boldsymbol{u}', \boldsymbol{h}\rangle + \mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) + \lambda(g_\varphi \boldsymbol{u}, \boldsymbol{h})_{L^2(\Gamma_1)} + \\ + \frac{\lambda}{\varepsilon}(\boldsymbol{v} - P\boldsymbol{v}, \boldsymbol{h})_{L^2(\Omega)} - \lambda\langle \boldsymbol{f}, \boldsymbol{h}\rangle\Big\} d\eta = 0\,,$$

with $\boldsymbol{u}(0) = \lambda\bar{\boldsymbol{u}}$, λ being a real parameter $\in [0, 1]$.

Let $\boldsymbol{u}$ be an eventual solution of $\boldsymbol{u} = S_{\varepsilon,\lambda}(\boldsymbol{u})$, i.e. such that

$$(8.10)\qquad \int_0^t \Big\{\langle \boldsymbol{u}', \boldsymbol{h}\rangle + \mu a_\varphi(\boldsymbol{u}, \boldsymbol{h}) + \lambda(g_\varphi \boldsymbol{u}, \boldsymbol{h})_{L^2(\Gamma_1)} + \\ + \frac{\lambda}{\varepsilon}(\boldsymbol{u} - P\boldsymbol{u}, \boldsymbol{h})_{L^2(\Omega)} - \langle \boldsymbol{f}, \boldsymbol{h}\rangle\Big\} d\eta = 0$$

with $\boldsymbol{u}(0) = \lambda\bar{\boldsymbol{u}}$. Setting $\boldsymbol{h} = \boldsymbol{u}$, we obtain, bearing in mind (8.3),

$$(8.11)\qquad \|\boldsymbol{u}\|_{C^0(0,T;L^2(\Omega))\cap L^2(0,T;V_\varphi)\cap H^1(0,T;V'_\varphi)} \leqslant N_1\,,$$

with N_1 independent of $\lambda \in [0, 1]$.

Denoting, moreover, by $\{\boldsymbol{v}_n\}$ a sequence such that $\boldsymbol{v}_n \to \boldsymbol{v}$ in the weak topology of $H^{\frac{1}{2}}(0, T; V^{\frac{1}{2}}_{\varphi,t})$, let $\{\boldsymbol{u}_n\}$ be the sequence defined by $\boldsymbol{u}_n = S_{\varepsilon,\lambda}(\boldsymbol{v}_n)$. We have, analogously to (8.11),

$$(8.12)\qquad \|\boldsymbol{u}_n\|_{C^0(0,T;L^2(\Omega))\cap L^2(0,T;V_\varphi)\cap H^1(0,T;V'_\varphi)} \leqslant N_1 \quad \text{(independent of } n).$$

Hence, it is possible to select from $\{\boldsymbol{u}_n\}$ a subsequence (again denoted by $\{\boldsymbol{u}_n\}$) such that

$$(8.13)\qquad \lim_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{u}$$

in the weak topology of $L^2(0, T; V_\varphi) \cap H^1(0, T; V'_\varphi)$, the weak * topology of $L^\infty(0, T; L^2(\Omega))$ and, since the embedding of $L^2(0, T; V_\varphi) \cap H^1(0, T; V'_\varphi)$ into $H^{\frac{1}{2}}(0, T; V^{\frac{1}{2}}_{\varphi,t})$ is completely continuous, in the strong topology of $H^{\frac{1}{2}}(0, T; V^{\frac{1}{2}}_{\varphi,t})$.

Writing (8.9) for the functions $\boldsymbol{u}_n$, $\boldsymbol{v}_n$ and letting $n \to \infty$, we then obtain that the limit functions $\boldsymbol{u}$, $\boldsymbol{v}$ satisfy (8.9); by the uniqueness theorem proved in Lemma 1, the whole sequence $\{\boldsymbol{u}_n\}$ must then necessarily converge to $\boldsymbol{u}$. It follows that $S_{\varepsilon,\lambda}$ is, $\forall\varepsilon > 0$ and $\forall\lambda \in [0, 1]$ completely continuous, while the eventual fixed points satisfy (8.11) and, obviously, $S_{\varepsilon,0}(\boldsymbol{v}) = 0$.

By the Leray-Schauder fixed point theorem, there exists then $\boldsymbol{u}^*$ such that $\boldsymbol{u}^* = S_{\varepsilon,0}(\boldsymbol{u}^*)$; the function $\boldsymbol{u}^*$ thus determined is obviously a solution of (8.8).

The uniqueness of this solution can be proved as in Lemma 1, observing that the operator $I - P$ is monotone.

LEMMA 3: *Let $\boldsymbol{u}_\varepsilon$ be the solution given by Lemma* 2. *Denoting by N_3, N_4 quantities independent of* ε, *we have*, $\forall\varepsilon > 0$, $s < \frac{1}{2}$

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(0,T;\,V_\varphi)\cap L^\infty(0,T;\,L^2(\Omega))} \leqslant N_3\,, \qquad \|\boldsymbol{u}_\varepsilon\|_{H^s(0,T;\,V_\varphi^{s-2})} \leqslant N_4\,. \tag{8.14}$$

Setting, in fact, in (8.8), $\boldsymbol{u} = \boldsymbol{h} = \boldsymbol{u}_\varepsilon$, we obtain

$$\frac{1}{2}\,\|\boldsymbol{u}_\varepsilon(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\,\|\overline{\boldsymbol{u}}\|^2_{L^2(\Omega)} + \tag{8.15}$$
$$+\int_0^t \Big\{\mu a_\varphi(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon) + (g_\varphi \boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon)_{L^2(\Gamma_1)} + \frac{1}{\varepsilon}(\boldsymbol{u}_\varepsilon - P\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon)_{L^2(\Omega)} - \langle \boldsymbol{f}, \boldsymbol{u}_\varepsilon\rangle\Big\}\, d\eta = 0\,,$$

from which the first of (8.14) follows directly. Moreover,

$$\frac{1}{\varepsilon}\int_0^T (\boldsymbol{u}_\varepsilon - P\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon)_{L^2(\Omega)}\, dt \leqslant N_5\,. \tag{8.16}$$

On the other hand, by (8.16) and the definition of P,

$$\frac{M_2}{\varepsilon}\int_Q |\boldsymbol{u}_\varepsilon - P\boldsymbol{u}_\varepsilon|\, dQ \leqslant \frac{1}{\varepsilon}\int_Q |\boldsymbol{u}_\varepsilon - P\boldsymbol{u}_\varepsilon|\,|\boldsymbol{u}_\varepsilon|\, dQ = \tag{8.17}$$
$$= \frac{1}{\varepsilon}\int_0^T (\boldsymbol{u}_\varepsilon - P\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon)_{L^2(\Omega)}\, dt \leqslant N_5\,.$$

Hence, by (8.8) and the first of (8.14) and well known embedding theorems,

$$\left|\int_0^T \langle \boldsymbol{u}'_\varepsilon, \boldsymbol{h}\rangle\, dt\right| \leqslant N_6 \|\boldsymbol{h}\|_{H^{1-s}(0,T;\,V_\varphi^{2-s})} \tag{8.18}$$

and, consequently, $\|\boldsymbol{u}'_\varepsilon\|_{H^{s-1}(0,T;\,V_\varphi^{s-2})} \leqslant N_6$, i.e. the second of (8.14).

LEMMA 4: *Assume that* $\varphi \in K_2$, $\overline{\boldsymbol{u}} \in V^0_{\varphi,0} \cap K_1$, $\boldsymbol{f}(t) \in L^2(0, T;\, V'_\varphi)$. *There exists then a unique function* $\boldsymbol{u}(t) \in L^2(0, T;\, V_\varphi \cap K_1) \cap H^s(0, T;\, V_\varphi^{s-2})$ $(s < \frac{1}{2})$ *which satisfies*, $\forall \boldsymbol{h}(t) \in L^2(0, T;\, V_\varphi \cap K_1) \cap H^1(0, T;\, V'_\varphi)$, *a.e. on* $(0, T)$, *the inequality*

$$\tfrac{1}{2}\|\boldsymbol{u}(t) - \boldsymbol{h}(t)\|_{V^0_\varphi} - \tfrac{1}{2}\|\overline{\boldsymbol{u}} - \boldsymbol{h}(0)\|_{V^0_\varphi} + \int_0^t \{\langle \boldsymbol{h}', \boldsymbol{u} - \boldsymbol{h}\rangle + \tag{8.19}$$
$$+ \mu a_\varphi(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h}) + (g_\varphi \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h})_{L^2(\Gamma_1)} - \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{h}\rangle\}\, d\eta \leqslant 0\,.$$

The existence of the solution can be proved by a standard procedure (see, for instance, [10], ch. 3, Th. 6.2) utilizing the results contained in Lemmas 2 and 3.

The uniqueness is obtained by a standard procedure in the theory of parabolic inequalities (see, for instance [11]).

9. - Some auxiliary theorems

Let us prove some theorems which will be utilized in the next paragraph for the proof of Theorem 1.

Theorem 2: *Suppose that the assumptions made in Lemma 4 are verified. There exists then a unique function* $\boldsymbol{u}(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^s(0,T;V_\varphi^{s-2})$ $(s<\frac12)$ $(\Rightarrow\in H^\sigma(0,T;V_\varphi^{\frac14+\sigma}),\ (\sigma<\frac1{12}))$, *which satisfies, a.e. in* $(0,T)$, *the inequality*

$$(9.1)\qquad \tfrac12\|\boldsymbol{u}(t)-\boldsymbol{h}(t)\|^2_{V^0_\varphi}-\tfrac12\|\overline{\boldsymbol{u}}-\boldsymbol{h}(0)\|^2_{V^0_\varphi}+\int_0^t\{\langle\boldsymbol{h}',\boldsymbol{u}-\boldsymbol{h}\rangle+$$

$$+\mu a_\varphi(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})+b_\varphi(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h},\boldsymbol{u})+(g_\varphi\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})_{L^2(\Gamma_1)}-\langle\boldsymbol{f},\boldsymbol{u}-\boldsymbol{h}\rangle\}\,d\eta\leqslant0\,.$$

$\forall\boldsymbol{h}(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^1(0,T;V'_\varphi)$.

We shall divide the proof in three parts.

a) Let $\boldsymbol{v}\in L^4(Q)$ and G_δ be a smoothing operator such that $G_\delta\boldsymbol{v}\in H^{1,4}(Q)$, $G_\delta\boldsymbol{v}\xrightarrow[\delta\to0]{}\boldsymbol{v}$ strongly in $L^4(Q)$. We begin by proving that there exists, $\forall\delta>0$, a function

$$\boldsymbol{u}_\delta\in L^2(0,T;V_\varphi\cap K_1)\cap H^s(0,T;V_\varphi^{s-2})\,,$$

which satisfies, a.e. in $(0,T)$, the inequality

$$(9.2)\qquad \tfrac12\|\boldsymbol{u}_\delta(t)-\boldsymbol{h}(t)\|^2_{V^0_\varphi}-\tfrac12\|\overline{\boldsymbol{u}}-\boldsymbol{h}(0)\|^2_{V^0_\varphi}+\int_0^t\{\langle\boldsymbol{h}',\boldsymbol{u}-\boldsymbol{h}\rangle+\mu a_\varphi(\boldsymbol{u}_\delta,\boldsymbol{u}_\delta-\boldsymbol{h})+$$

$$+b_\varphi(G_\delta\boldsymbol{u}_\delta,\boldsymbol{u}_\delta-\boldsymbol{h},G_\delta\boldsymbol{u}_\delta)+(g_\varphi\boldsymbol{u}_\delta,\boldsymbol{u}_\delta-\boldsymbol{h})-\langle\boldsymbol{f},\boldsymbol{u}_\delta-\boldsymbol{h}\rangle\}\,d\eta\leqslant0$$

$\forall\boldsymbol{h}(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^1(0,T;V'_\varphi)$.

Consider, in fact, $\forall$ fixed δ, the transformation $\boldsymbol{v}\to S_\delta\boldsymbol{v}=\boldsymbol{u}$ from $L^4(Q)$ to itself, where $\boldsymbol{u}$ is the solution of the inequality

$$(9.3)\qquad \tfrac12\|\boldsymbol{u}(t)-\boldsymbol{h}(t)\|^2_{V^0_\varphi}-\tfrac12\|\overline{\boldsymbol{u}}-\boldsymbol{h}(0)\|^2_{V^0_\varphi}+\int_0^t\{\langle\boldsymbol{h}',\boldsymbol{u}-\boldsymbol{h}\rangle+$$

$$+\mu a_\varphi(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})+b_\varphi(G_\delta\boldsymbol{v},\boldsymbol{u}-\boldsymbol{h},G_\delta\boldsymbol{v})+(g_\varphi\boldsymbol{u},\boldsymbol{u}-\boldsymbol{h})_{L^2(\Gamma_1)}-\langle\boldsymbol{f},\boldsymbol{u}-\boldsymbol{h}\rangle\}\,d\eta\leqslant0\,,$$

Since

$$(9.4)\qquad \left|\int_0^T b_\varphi(G_\delta\boldsymbol{v},\boldsymbol{u}-\boldsymbol{h},G_\delta\boldsymbol{v})\,dt\right|\leqslant C_1\|G_\delta\boldsymbol{v}\|^2_{L^4(\Omega)}\|\boldsymbol{u}-\boldsymbol{h}\|_{L^2(0,T;V_\varphi)}\,,$$

it follows from Lemma 4 that such a solution is uniquely determined; moreover, $\boldsymbol{u}(t) \in L^2(0, T; V_\varphi \cap K_1) \cap H^s(0, T; V_\varphi^{s-2})$ and satisfies (9.3) $\forall \boldsymbol{h}(t) \in L^2 \cdot (0, T; V_\varphi \cap K_1) \cap H^1(0, T; V'_\varphi)$, while, since $|\boldsymbol{u}| \leqslant M_2$ a.e., S_δ transforms in itself every sphere of $L^4(Q)$ with sufficiently large radius.

Let $\{\boldsymbol{v}_j\}$ be an $L^4(Q)$-weakly convergent sequence; then, by the definition of G_δ,

(9.5)
$$\lim_{n\to\infty} G_\delta \boldsymbol{v}_j = G_\delta \boldsymbol{v}$$

weakly in $H^{1,4}(Q)$ and strongly in $L^4(Q)$. Setting $\boldsymbol{u}_j = S_\delta \boldsymbol{v}_j$, the sequence $\{\boldsymbol{u}_j\}$ is, by (9.5) and Lemma 4, uniformly bounded in $L^2(0, T; V_\varphi \cap K_1) \cap H^s \cdot (0, T; V_\varphi^{s-2})$; by well known interpolation, embedding and trace theorems, it is then possible to select from $\{\boldsymbol{u}_j\}$ a subsequence (again denoted by $\{\boldsymbol{u}_j\}$) such that

(9.6)
$$\lim_{j\to\infty} \boldsymbol{u}_j(t) = \boldsymbol{u}(t)$$

in the weak topology of $L^2(0, T; V_\varphi) \cap H^s(0, T; V_\varphi^{s-2})$, the weak * topology of $L^\infty(Q)$ and the strong topology of $H^\sigma(0, T; V_\varphi^{\frac{1}{2}+\sigma})$ $(\sigma < \frac{1}{12})$, hence also

(9.7)
$$\lim_{j\to\infty} \gamma_1 \boldsymbol{u}_j(t) = \gamma_1 \boldsymbol{u}(t)$$

in the strong topology of $L^2(0, T; L^2(\Gamma_1))$.

Observe moreover that, since $|\boldsymbol{u}_j|$, $|\boldsymbol{u}| \leqslant M_2$ a.e. in Q, it follows from (9.6) that

(9.8)
$$\lim_{j\to\infty} \boldsymbol{u}_j = \boldsymbol{u}$$

in the strong topology of $L^4(Q)$.

Writing (9.3) for $\boldsymbol{u}_j$, $\boldsymbol{v}_j$ letting $j \to \infty$ and bearing in mind (9.5), (9.6), (9.7), (9.8), it follows then, by the semicontinuity of the weak limit, that $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$ satisfy (9.3). Since however, by Lemma 4, the solution $\boldsymbol{u}(t)$ is unique, we conclude that the whole sequence $\{\boldsymbol{u}_j\}$ converges to $\boldsymbol{u}$; hence S_δ is, $\forall \delta > 0$, completely continuous from $L^4(Q)$ to itself.

By the Tychonov fixed point theorem, there exists then $\boldsymbol{u}_\delta$ such that $\boldsymbol{u}_\delta = S_\delta \boldsymbol{u}_\delta$; this function is obviously a solution of (9.2).

b) We now prove that, when $\delta \to 0$, the sequence $\{\boldsymbol{u}_\delta\}$ defined in *a*) converges to a solution of (9.1).

By the same procedure followed in *a*) (bearing in mind that $|\boldsymbol{u}_\delta| \leqslant M_2$ a.e.) it can be shown that the sequence $\{\boldsymbol{u}_\delta(t)\}$ is uniformly bounded in $L^2(0, T; V_\varphi \cap K_1) \cap H^s(0, T; V_\varphi^{s-2})$; hence (cfr. (9.6), (9.8))

(9.9)
$$\lim_{\delta\to 0} \boldsymbol{u}_\delta(t) = \boldsymbol{u}(t)$$

in the weak topology of $L^2(0, T; V_\varphi) \cap H^s(0, T; V_\varphi^{s-2})$, the weak * topology of $L^\infty(Q)$ and the strong topology of $H^\sigma(0, T; V_\varphi^{\frac{1}{2}+\sigma})\,(\sigma < \frac{1}{12})$ and of $L^4(Q)$; moreover

$$\lim_{\delta\to 0} \gamma_1 \boldsymbol{u}_\delta(t) = \gamma_1 \boldsymbol{u}(t) \tag{9.10}$$

in the strong topology of $L^2(0, T; L^2(\Gamma_1))$.

Letting $\delta \to 0$ in (9.2) we obtain then, by (9.9), (9.10) and the semicontinuity of the weak limit, relation (9.1). The existence of a solution is therefore proved.

c) The uniqueness of the solution can be proved by exactly the same procedure mentinoed in Lemma 4

THEOREM 3: *Let $\{\varphi_n\}$ be a sequence of functions $\in K_2$ and such that $\varphi_n \to \varphi$ strongly in $C^1(\bar{\Lambda})$ and let $\{\boldsymbol{u}_n\}$ be the sequence of the corresponding solutions of* (9.1), *whose existence and uniqueness has been proved in Theorem* 2. *Then, denoting by $\boldsymbol{u}$ the solution corresponding to φ* (which obviously $\in K_2$).

$$\lim_{n\to\infty} \boldsymbol{u}_n(t) = \boldsymbol{u}(t) \tag{9.11}$$

in the weak topology of $L^2(0, T; H^1(\Omega))$, the weak topology of $L^\infty(Q)$ and the strong topology of $H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))\,(\sigma < \frac{1}{12})$ and of $L^4(Q)$.*

Observe, to begin with, that, by Theorem 2 and the assumption that $\varphi_n \in K_2$,

$$\|\boldsymbol{u}_n\|_{L^\infty(0,T;\,L^\infty(\Omega))\,\cap\, L^2(0,T;\,V_{\varphi_n})\,\cap\, H^s(0,T;\,V_{\varphi_n}^{s-2})} \leq N_6 \tag{9.12}$$

with N_6 independent of n. Hence, since $V_{\varphi_n} \subset H^1(\Omega)$, and

$$L^2(0, T; V_{\varphi_n}) \cap H^s(0, T; V_{\varphi_n}^{s-2}) \subset H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))\,(\sigma < \tfrac{1}{12}), \tag{9.13}$$

with embedding constants independent of n, (9.11) is, following the proof given in Theorem 2, satisfied for an appropriate subsequence of $\{\boldsymbol{u}_n\}$. Moreover, by the assumptions made on $\{\varphi_n\}$,

$$\lim_{n\to\infty} \operatorname{div}_{\varphi_n} \boldsymbol{u}_n = \operatorname{div}_\varphi \boldsymbol{u} \qquad \text{weakly in } L^2(Q), \tag{9.14}$$

so that $\boldsymbol{u}(t) \in L^2(0, T; V_\varphi \cap K_1)$.

We must now show that $\boldsymbol{u}(t)$ is a solution of (9.1).

Setting $\mathring{K}_1 = \{\boldsymbol{v} \in H^1(\Omega) : |\boldsymbol{v}| < M_2 \text{ a.e.}\}$, let $\boldsymbol{h}(t)$ be an arbitrary function

$\in L^2(0, T; V_\varphi \cap \mathring{K}_1)$ and $\{\boldsymbol{h}_n(t)\}$ a sequence such that

$$(9.15)\qquad \boldsymbol{h}_n(t) \in L^2(0, T; V_{\varphi_n} \cap K_1) \cap H^1(0, T; V'_{\varphi_n}), \qquad \lim_{n\to\infty} \boldsymbol{h}_n(t) = \boldsymbol{h}(t)$$

in the strong topology of $L^2(0, T; H^1(\Omega))$ (1).

The functions $\boldsymbol{u}_n$ satisfy, by definition, the inequality

$$(9.16)\qquad \tfrac{1}{2}\|\boldsymbol{u}_n(t)-\boldsymbol{h}_n(t)\|^2_{V^0_\varphi} - \tfrac{1}{2}\|\bar{\boldsymbol{u}}-\boldsymbol{h}_n(0)\|^2_{V^0_\varphi} + \int_0^t \{\langle \boldsymbol{h}'_n, \boldsymbol{u}_n - \boldsymbol{h}_n\rangle + \mu a_{\varphi_n}(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n) -$$
$$- b_{\varphi_n}(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n, \boldsymbol{u}_n) + (g_{\varphi_n}\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n)_{L^2(\Gamma_1)} - \langle \boldsymbol{f}, \boldsymbol{u}_n - \boldsymbol{h}_n\rangle\}\, d\eta \leqslant 0 .$$

Observe now that, by the assumptions made, $g_{\varphi_n} \to g_\varphi$ strongly in $C^0(\bar{\Lambda})$; moreover

$$(9.17)\qquad a_{\varphi_n}(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n) - a_\varphi(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h}) = a_{\varphi_n}(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n) -$$
$$- a_\varphi(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n) + a_\varphi(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n) - a_\varphi(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h})$$

and consequently, by (9.11) and the semicontinuity of the weak limit (since $a_\varphi(\boldsymbol{v}, \boldsymbol{v})$ is equivalent, $\forall \boldsymbol{v} \in \tilde{H}^1_0$, to $\|\boldsymbol{v}\|^2_{H^1(\Omega)}$)

$$(9.18)\qquad \lim_{n\to\infty} \int_0^t a_{\varphi_n}(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n)\, d\eta \leqslant \int_0^t a_\varphi(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h})\, d\eta .$$

On the other hand, by (4.12), (9.11),

$$(9.19)\qquad \lim_{n\to\infty} \int_0^t b_{\varphi_n}(\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{h}_n, \boldsymbol{u}_n)\, d\eta =$$
$$= \lim_{n\to\infty} \int_0^t \{- b_{\varphi_n}(\boldsymbol{u}_n, \boldsymbol{h}_n, \boldsymbol{u}_n) + (g_{\varphi_n}\boldsymbol{u}_n, \boldsymbol{u}_n)_{L^2(\Gamma_1)}\}\, d\eta =$$
$$= \int_0^t \{- b_\varphi(\boldsymbol{u}, \boldsymbol{h}, \boldsymbol{u}) + (g_\varphi \boldsymbol{u}, \boldsymbol{u})_{L^2(\Gamma_1)}\}\, d\eta = \int_0^t b_\varphi(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h}, \boldsymbol{u})\, d\eta .$$

(1) $\boldsymbol{h}_n$ can, for instance, be defined in the following way. Let

$$\tilde{\Omega} = \left\{(x, y) \in A,\ -1 < z < \sup_{x,y,t,n} \frac{\varphi_n - \varphi}{1 + \varphi}\right\}$$

and $\tilde{\boldsymbol{h}}$ be an extension of $\boldsymbol{h}$ to $\tilde{\Omega} \times (0, T)$ such that $\tilde{\boldsymbol{h}}(t) \in L^2(0, T; H^1(\tilde{\Omega}))$, with $|\tilde{\boldsymbol{h}}| \leqslant M_2$, $\operatorname{div}_\varphi \tilde{\boldsymbol{h}} = 0$, $\tilde{\boldsymbol{h}} = 0$ on $\Gamma_2 \times (0, T)$; this extension is obviously possible (being $|\boldsymbol{h}| < M_2$) for n sufficiently large, since $\varphi_n \to \varphi$ uniformly. We can then set

$$\boldsymbol{h}_n(x, y, z, t) = \tilde{\boldsymbol{h}}\left(x, y, \frac{1 + \varphi_n}{1 + \varphi} z + \frac{\varphi_n - \varphi}{1 + \varphi}, t\right).$$

Letting now $n \to \infty$ in (9.16) we have, by (9.11), (9.15), (9.18), (9.19),

$$(9.20) \quad \tfrac{1}{2}\|\boldsymbol{u}(t) - \boldsymbol{h}(t)\|^2_{V^0_\varphi} - \tfrac{1}{2}\|\bar{\boldsymbol{u}} - \boldsymbol{h}(0)\|^2_{V^0_\varphi} + \int_0^t \{\langle \boldsymbol{h}', \boldsymbol{u} - \boldsymbol{h}\rangle + $$

$$+ \mu a_\varphi(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h}) - b_\varphi(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h}, \boldsymbol{u}) + (g_\varphi \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{h})_{L^2(\Gamma_1)} - \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{h}\rangle\}\, d\eta \leqslant 0$$

a.e. in $(0, T)$, $\forall \boldsymbol{h}(t) \in L^2(0, T; V_\varphi \cap \mathring{K}_1) \cap H^1(0, T; V'_\varphi)$.

Since the space of such functions is dense in $L^2(0, T; V_\varphi \cap K_1) \cap \cap H^1(0, T; V'_\varphi)$ we conclude that $\boldsymbol{u}$ is the unique solution of (9.1) corresponding to φ. Hence, (9.12) holds for the whole sequence $\{\boldsymbol{u}_n\}$ and the theorem is proved.

THEOREM 4: *Assume that* $\boldsymbol{v} \in L^\infty(\Gamma_1 \times (0, T))$, $\bar{\varphi} \in H^1_0(A)$,

$$|\bar{\varphi}_{xx}| \leqslant M_1, \qquad |\bar{\varphi}_{yy}| \leqslant M_1, \qquad \bar{\varphi} \geqslant -1 + \sigma \ (\sigma > 0).$$

There exists then a unique function $\varphi(t) \in L^2(0, T; H^1_0(A)) \cap K_2$ *such that* $\varphi(0) = \bar{\varphi}$ *and satisfying,* $\forall l \in K_2$ *and* $t \in [0, T]$, *the inequality*

$$(9.21) \qquad \int_{A_t} \left(\frac{\partial \varphi}{\partial \eta} - v_3(x, y, \eta)(\varphi + 1)\right)(\varphi - l)\, dA \leqslant 0\,.$$

Consider, at first, $\forall \varepsilon > 0$, the inequality

$$(9.22) \qquad \int_{A_t} \left(\frac{\partial \varphi}{\partial \eta} - \varepsilon\, \Delta\varphi - v_3(x, y, \eta)(\varphi + 1)\right)(\varphi - l)\, dA \leqslant 0\,.$$

It is well known (see, for instance, [11], ch. 6, Th. 6.2) that (9.22) admits, $\forall \varepsilon > 0$, a solution $\varphi_\varepsilon(t) \in L^2(0, T; H^1_0(A)) \cap K_2$, $\forall l \in K_2$ and $t \in [0, T]$, such that $\varphi_\varepsilon(0) = \bar{\varphi}$.

Letting $\varepsilon \to 0$ it is then obviously possible, since $\varphi_\varepsilon \in K_2$, to select from $\{\varphi_\varepsilon\}$ a subsequence which converges to a solution of (9.21).

The uniqueness of this solution can easily be proved by setting, in (9.21) $\varphi = \varphi_1$, $l = \varphi_2$ and $\varphi = \varphi_2$, $l = \varphi_1$ and adding.

THEOREM 5: *Let* $\{\boldsymbol{v}_n\}$ *be a sequence of functions* $\in K_3$ *such that* $\boldsymbol{v}_n \to \boldsymbol{v}$ *strongly in* $L^2(\Gamma_1 \times (0, T))$ *and let* $\{\varphi_n\}$ *be the sequence of the corresponding solutions of* (9.21), *whose existence and uniqueness has been proved in Theorem* 4. *Then, denoting by* φ *the solution corresponding to* $\boldsymbol{v}$

$$(9.23) \qquad \lim_{n \to \infty} \varphi_n = \varphi$$

in the strong topology of $C^1(\bar{A})$.

Observe that since, by Theorem 4, $\varphi_n \in K_2$, (9.23) certainly holds for an appropriate subsequence of $\{\varphi_n\}$.

On the other hand

$$\int_{\Lambda_t} \left(\frac{\partial \varphi_n}{\partial t} - v_{3,n}(\varphi_n + 1)\right)(\varphi_n - l)\, d\Lambda \leqslant 0 \tag{9.24}$$

and, letting $n \to \infty$, it is obvious that φ satisfies (9.21). Since, however, the solution is unique, the whole sequence must tend to φ. This proves our theorem.

10. - Proof of Theorem 1

Let us introduce three transformations $S_{i,\lambda}$ ($i = 1, 2, 3$, λ real parameter $\in [0, 1]$) defined in the following way.

a_1) $\boldsymbol{z} = S_{1,\lambda}(\boldsymbol{v}) = P\gamma_1(\lambda \boldsymbol{v})$, where P, γ_1 are respectively the operators « projection on K_3 » and « trace on $\Gamma_1 \times (0, T)$ »;

a_2) $\varphi = S_{2,\lambda}(\boldsymbol{z})$, with φ satisfying condition ii) of § 6, having substituted, in (6.5), $\boldsymbol{z}(x, y, t)$ to $\boldsymbol{u}(x, y, 0, t)$ and the initial condition $\varphi(0) = \bar{\varphi}$ with $\varphi(0) = \lambda\bar{\varphi}$;

a_3) $\boldsymbol{u} = S_{3,\lambda}(\varphi)$, with $\boldsymbol{u}$ satisfying condition i) of § 6, having substituted $\boldsymbol{f}$, $\bar{\boldsymbol{u}}$ respectively with $\lambda \boldsymbol{f}$, $\lambda\bar{\boldsymbol{u}}$.

Observe now that $S_{1,\lambda}$ is, by well known trace and embedding theorems, completely continuous from $H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))$ $(\sigma < \frac{1}{12})$, to K_3, with $S_{1,0}(\boldsymbol{v})=0$.

On the other hand, by Theorems 2 and 3, $S_{3,\lambda}$ is a transformation from K_2 to $L^2(0, T; V_\varphi \cap K_1) \cap H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))$, and is continuous from $C^1(\bar{\Lambda})$ to $H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))$, with $S_{3,0}(\varphi) = 0$ (by the uniqueness theorem).

Finally, by Theorems 4 and 5, $S_{2,\lambda}$ transforms K_3 into $L^2(0, T; H_0^1(\Lambda)) \cap$ $\cap\, K_2$ and is continuous from $L^2(\Gamma_1 \times (0, T))$ to $C^1(\bar{\Lambda})$ with $S_{2,0}(\boldsymbol{z}) = 0$ (by the uniqueness theorem).

We can then conclude that the transformation $S_\lambda = \prod_{i=1,2,3} S_{i,\lambda}$ is completely continuous from $H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))$ $(\sigma < \frac{1}{12})$ to itself $\forall \lambda \in [0, 1]$ and is such that $S_0(\boldsymbol{v}) = 0$ $\forall \boldsymbol{v} \in H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))$ $(\sigma < \frac{1}{12})$.

Assume now that we have fixed the data $\boldsymbol{f}$, $\bar{\boldsymbol{u}}$, $\bar{\varphi}$; by Theorem 2 we have, $\forall \varphi \in K_2$,

$$\|\boldsymbol{u}\|_{H^\sigma(0,T;\, H^{\frac{1}{2}+\sigma}(\Omega))} \leqslant M_3 \tag{10.1}$$

and, consequently, all functions $\boldsymbol{u}=S_\lambda(\boldsymbol{v})$ satisfy (10.1) $\forall \boldsymbol{v} \in H^\sigma(0, T; H^{\frac{1}{2}+\sigma}(\Omega))$ with M_3 independent of $\lambda \in [0, 1]$.

It follows therefore, by the Leray-Schauder fixed point theorem, that there exists $\boldsymbol{u}^*$ such that $\boldsymbol{u}^* = S_1(\boldsymbol{u}^*)$, i.e. by the definitions given, such that (cfr.

(6.4), (6.5))

$$
(10.2)\quad \left|\begin{aligned}
&\tfrac{1}{2}\|\boldsymbol{u}^*(t)-\boldsymbol{h}(t)\|^2_{V^0_\varphi}-\tfrac{1}{2}\|\overline{\boldsymbol{u}}-\boldsymbol{h}(0)\|^2_{V^0_\varphi}+\int_0^t\{\langle\boldsymbol{h}',\boldsymbol{u}^*-\boldsymbol{h}\rangle+\\
&\qquad+\mu a_\varphi(\boldsymbol{u}^*,\boldsymbol{u}^*-\boldsymbol{h})-b_\varphi(\boldsymbol{u}^*,\boldsymbol{u}^*-\boldsymbol{h},\boldsymbol{u}^*)+\\
&\qquad+(g_\varphi\boldsymbol{u}^*,\boldsymbol{u}^*-\boldsymbol{h})_{L^2(\Gamma_1)}-\langle\boldsymbol{f},\boldsymbol{u}^*-\boldsymbol{h}\rangle\}\,d\eta\leqslant 0\,,\\
&\int_{\Lambda_t}\left(\frac{\partial\varphi}{\partial\eta}-Pu_3^*(x,y,0,\eta)(\varphi+1)\right)(\varphi-l)\,d\Lambda\leqslant 0\,,
\end{aligned}\right.
$$

$\forall\boldsymbol{h}(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^1(0,T;V'_\varphi)$, $l\in K_2$, a.e. in $(0,T)$, with

$$\boldsymbol{u}^*(t)\in L^2(0,T;V_\varphi\cap K_1)\cap H^s(0,T;V_\varphi^{s-2})\cap H^\sigma(0,T;V_\varphi^{\frac{1}{2}+\sigma})\ \ (s<\tfrac{1}{2},\ \sigma<\tfrac{1}{12}),$$

$$\varphi(t)\in L^2(0,T;H^1_0(\Lambda))\cap K_2\,,\qquad \varphi(0)=\bar\varphi\,.$$

Since, however $|\boldsymbol{u}^*|\leqslant M_2$ a.e., it follows that $|\gamma_1\boldsymbol{u}^*|\leqslant M_2$ and, consequently, $P\boldsymbol{u}^*(x,y,0,t)=\boldsymbol{u}^*(x,y,0,t)$.

Hence, $\{\boldsymbol{u}^*,\varphi\}$ satisfy conditions i), ii) of § 6 and the existence theorem is proved.

11. - The problem of uniqueness

We now assume that the solution given in Theorem 1 satisfies (3.8) a.e. in $Q'=\Omega\times(0,t')$ and is such that

$$(11.1)\qquad |\boldsymbol{u}|\leqslant M_2'<M_2 \qquad \text{a.e. in } Q'\,.$$

It can then easily by proved (see, for instance, [12]) that $\{\boldsymbol{u},\varphi\}$ satisfy a.e. in $(0,t')$, the equations

$$(11.2)\qquad \int_0^t\{\langle\boldsymbol{u}',\boldsymbol{h}\rangle+\mu a_\varphi(\boldsymbol{u},\boldsymbol{h})+b_\varphi(\boldsymbol{u},\boldsymbol{u},\boldsymbol{h})+\langle\boldsymbol{f},\boldsymbol{h}\rangle\}\,d\eta=0\,,$$

$$(11.3)\qquad \frac{\partial\varphi}{\partial t}-(\gamma_1u_3)(\varphi+1)=0\,,$$

$\forall\boldsymbol{h}(t)\in L^2(0,t';V_\varphi)$.

Bearing in mind what was said in § 1, in order to be able to give a positive reply to question β), we should prove a uniqueness and continuous dependence theorem for the solution of (11.2), (11.3), with $\boldsymbol{u}(0)=\overline{\boldsymbol{u}}$, $\varphi(0)=\bar\varphi$.

Unfortunately, however, such a theorem is not known, so that question β) remains, for the Navier-Stokes model, open.

Uniqueness and continuous dependence theorems can, on the other hand, be given if equation (11.3) is modified slightly. More precisely, let G be a continuous « smoothing » operator from $L^2(A)$ to $H^2(A)$ (for example a double average on x, y or Green's operator relative to the Laplace operator in A) and substitute (11.3) by

$$\frac{\partial \varphi}{\partial t} - (G\gamma_1 u_3)(\varphi + 1) = 0 \,. \tag{11.4}$$

Observe that, from a physical point of view, this substitution seems acceptable. In fact, as explained in § 3, (11.3) imposes the condition that the velocity of the fluid particles on the free surface is tangent to the surface itself; such a condition excludes therefore the existence of superficial turbulence and may not be strictly applicable to practical cases. Formulation (11.4) obviously avoids this difficulty by substituting « avarage » values of the velocity on the free surface to local ones.

One further physical advantage of (11.4) over (11.3) is that its solutions $\varphi(x, y, t)$ do not necessarily vanish when $(x, y) \in \partial A$; this condition, as indicated in Observation 3.2, is on the other hand necessary for the solutions of (11.3) and is not generally verified in practical cases.

It can be easily seen that all the results obtained in the preceding §§ with regard to the Navier-Stokes model, associated to inequalities (6.4), (6.5), can be repeated without change to the « modified » Navier-Stokes model, associated to (6.4) and the inequality, corresponding to (11.4),

$$\int\limits_{A_t} \left(\frac{\partial \varphi}{\partial \eta} - G u_3(x, y, 0, \eta)(\varphi + 1)\right)(\varphi - l)\, d\eta \leqslant 0 \,. \tag{11.5}$$

In particular, it is possible to prove that *the inequalities* (6.4), (11.5) *admit a global solution, in the sense indicated in* § 6.

Let us now prove the following uniqueness and continuous dependence theorem.

THEOREM 6: *Assume that* $\{\boldsymbol{u}, \varphi\}$ *is a solution of* (11.2), (11.4) *satisfying* (3.8), (11.1), *with* $\boldsymbol{u}(0) = \boldsymbol{u}$, $\varphi(0) = \bar{\varphi}$.
Then $\{\boldsymbol{u}, \varphi\}$ *is unique and depends continuously on the data.*

Assume, in fact, that there exist two solutions, $\{\boldsymbol{u}, \varphi\}$ and $\{\boldsymbol{v}, \psi\}$ satisfying (3.8), (11.1) and the same initial and boundary conditions and set $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, $\chi = \varphi - \psi$.

Let us consider the function $\boldsymbol{k}^{(1)} = \boldsymbol{w} - \boldsymbol{\sigma}^{(1)}$, $\boldsymbol{\sigma}^{(1)}$ being defined by

$$\left\{\begin{aligned} &\sigma_1^{(1)} = \sigma_2^{(1)} = 0 \,, \\ &\sigma_3^{(1)} = \left(\frac{\psi_x - \varphi_x}{\psi + 1} + \frac{\psi_x(\varphi - \psi)}{(\varphi + 1)(\psi + 1)}\right) \int_{-1}^{z} v_1\, d\zeta \,+ \\ &\qquad + \left(\frac{\psi_y - \varphi_y}{\psi + 1} + \frac{\psi_y(\varphi - \psi)}{(\varphi + 1)(\psi + 1)}\right) \int_{-1}^{z} v_2\, d\zeta \,. \end{aligned}\right. \tag{11.6}$$

A direct calculation shows that

$$\text{(11.7)} \qquad \operatorname{div}_\varphi \boldsymbol{\sigma}^{(1)} = \frac{\partial \sigma_3^{(1)}}{\partial z} = -\operatorname{div}_\varphi \boldsymbol{v} = \operatorname{div}_\psi \boldsymbol{v} - \operatorname{div}_\varphi \boldsymbol{v}$$

and, consequently,

$$\text{(11.8)} \qquad \operatorname{div}_\varphi \boldsymbol{k}^{(1)} ; = \operatorname{div}_\varphi \boldsymbol{u} - \operatorname{div}_\varphi \boldsymbol{v} - \operatorname{div}_\varphi \boldsymbol{\sigma}^{(1)} = 0 \,.$$

In an analogous way, setting

$$\text{(11.9)} \qquad \left\{ \begin{aligned} & \sigma_1^{(2)} = \sigma_2^{(2)} = 0 \,, \\ & \sigma_3^{(2)} = \left(\frac{\psi_x - \varphi_x}{\psi + 1} + \frac{\psi_x(\varphi - \psi)}{(\varphi + 1)(\psi + 1)} \right) \int_{-1}^{z} u_1 \, d\zeta + \\ & \qquad\qquad + \left(\frac{\psi_y - \varphi_y}{\psi + 1} + \frac{\psi_y(\varphi - \psi)}{(\varphi + 1)(\psi + 1)} \right) \int_{-1}^{z} u_2 \, d\zeta \,, \end{aligned} \right.$$

we have

$$\text{(11.10)} \qquad \operatorname{div}_\psi \boldsymbol{k}^{(2)} = 0 \,.$$

Let us now write (11.2), (11.4) for $\{\boldsymbol{u}, \varphi\}$ and $\{\boldsymbol{v}, \psi\}$ setting respectively $\boldsymbol{h} = \boldsymbol{k}^{(1)}$ and $\boldsymbol{h} = \boldsymbol{k}^{(2)}$ and considering the scalar product, in $H^2(A)$, of (11.3) with χ.

Adding then the corresponding equations, we obtain

$$\text{(11.11)} \qquad \left\{ \begin{aligned} & \int_0^t \Big\{ \frac{1}{2} \frac{d}{d\eta} \| \boldsymbol{w}(\eta) \|^2_{L^2(\Omega)} - \langle \boldsymbol{u}', \boldsymbol{\sigma}^{(1)} \rangle + \langle \boldsymbol{v}', \boldsymbol{\sigma}^{(2)} \rangle + \mu [a_\varphi(\boldsymbol{w}, \boldsymbol{w}) - a_\varphi(\boldsymbol{u}, \boldsymbol{\sigma}^{(1)}) + \\ & \quad + a_\varphi(\boldsymbol{v}, \boldsymbol{w}) - a_\psi(\boldsymbol{v}, \boldsymbol{w}) + a_\psi(\boldsymbol{v}, \boldsymbol{\sigma}^{(2)})] + b_\varphi(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) - b_\varphi(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\sigma}^{(1)}) - \\ & \quad - b_\psi(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{w}) + b_\psi(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\sigma}^{(2)}) + (g_\varphi \boldsymbol{u}, \boldsymbol{w})_{L^2(\Gamma_1)} - (g_\varphi \boldsymbol{u}, \boldsymbol{\sigma}^{(1)})_{L^2(\Gamma^1)} - \\ & \qquad\qquad - (g_\psi \boldsymbol{v}, \boldsymbol{w})_{L^2(\Gamma_1)} + (g_\psi \boldsymbol{v}, \boldsymbol{\sigma}^{(2)})_{L^2(\Gamma_1)} + \langle \boldsymbol{f}, \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \rangle \Big\} d\eta = 0, \\ & \left(\frac{\partial \chi}{\partial t} - (G\gamma_1 u_3)\chi - (G\gamma_1 w_3)(\psi + 1), \chi \right)_{H^2(A)} = 0 \,. \end{aligned} \right.$$

On the other hand

$$\text{(11.12)} \qquad -\langle \boldsymbol{u}', \boldsymbol{\sigma}^{(1)} \rangle + \langle \boldsymbol{v}', \boldsymbol{\sigma}^{(2)} \rangle = -\langle \boldsymbol{u}', \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \rangle - \langle \boldsymbol{w}', \boldsymbol{\sigma}^{(2)} \rangle$$

and, by the assumptions made,

$$\text{(11.13)} \qquad \left| \int_0^t \langle \boldsymbol{u}', \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \rangle \, d\eta \right| \leqslant \| \boldsymbol{u}' \|_{L^2(0,t;\, V'_\varphi)} \| \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \|_{L^2(0,t;\, V_\varphi)} \leqslant$$

$$\leqslant c_1 \| \chi \|_{L^2(0,t;\, H^2(A))} \| \boldsymbol{w} \|_{L^2(0,t;\, V_\varphi)} \,,$$

$$(11.14)\qquad \left|\int_0^t \langle \boldsymbol{w}', \boldsymbol{\sigma}^{(2)}\rangle\, d\eta\right| \leqslant \left|(\boldsymbol{w}(t), \boldsymbol{\sigma}^{(2)}(t))_{L^2(\Omega)}\right| + \left|\int_0^t \langle \boldsymbol{w}, \boldsymbol{\sigma}^{(2)\prime}\rangle\, d\eta\right| \leqslant$$
$$\leqslant \|\boldsymbol{w}(t)\|_{L^2(\Omega)} \|\boldsymbol{\sigma}^{(2)}(t)\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{L^2(0,t;\,V_\varphi)} \|\boldsymbol{\sigma}^{(2)\prime}\|_{L^2(0,t;\,V'_\varphi)} \leqslant$$
$$\leqslant c_2 \|\boldsymbol{w}(t)\|_{L^2(A)} \|\chi(t)\|_{H^1(A)} + c_3 \|\boldsymbol{w}\|_{L^2(0,t;\,V_\varphi)} \|\boldsymbol{\sigma}^{(2)\prime}\|_{L^2(0,t;\,V'_\varphi)} .$$

Observe however that, by (3.8), (11.1) and bearing in mind that φ, ψ satisfy (1.3)

$$(11.15)\qquad \|\boldsymbol{\sigma}^{(2)\prime}\|_{L^2(0,t;\,V'_\varphi)} \leqslant c_4 \|\chi'\|_{L^2(0,t;\,L^2(A))} + c_5 \|\chi\|_{L^\infty(0,t;\,H^{1,\infty}(A))} \|\boldsymbol{u}'\|_{L^2(0,t;\,V'_\varphi)} \leqslant$$
$$\leqslant c_6 \|\chi\|_{L^2(0,t;\,L^2(A))} + c_7 \|\chi\|_{L^2(0,t;\,H^1(A))} .$$

Hence,

$$(11.16)\qquad \left|\int_0^t \{-\langle \boldsymbol{u}', \boldsymbol{\sigma}^{(1)}\rangle + \langle \boldsymbol{v}', \boldsymbol{\sigma}^{(2)}\rangle\}\, d\eta\right| \leqslant c_1 \|\chi\|_{L^2(0,t;\,H^2(A))} \|\boldsymbol{w}\|_{L^2(0,t;\,V_\varphi)} +$$
$$+ c_2 \|\boldsymbol{w}(t)\|_{L^2(\Omega)} \|\chi(t)\|_{H^1(A)} + \|\boldsymbol{w}\|_{L^2(0,t;\,V_\varphi)} \big(c_8 \|\chi\|_{L^2(0,t;\,L^2(A))} + c_9 \|\chi\|_{L^2(0,t;\,H^1(A))}\big) .$$

Treating the other terms that appear in the first of (11.13) in a similar way, we obtain, by a straightforward calculation,

$$(11.17)\qquad \tfrac{1}{2} \|\boldsymbol{w}(t)\|^2_{L^2(\Omega)} + \mu \int_0^t a_\varphi(\boldsymbol{w}, \boldsymbol{w})\, d\eta \leqslant$$
$$\leqslant c_2 \|\boldsymbol{w}(t)\|_{L^2(\Omega)} \|\chi(t)\|_{H^1(A)} + c_{10} \int_0^t \|\boldsymbol{w}\|_{V_\varphi} \|\chi\|_{H^2(A)}\, d\eta .$$

On the other hand, from the second of (11.11) it follows that

$$(11.18)\qquad \tfrac{1}{2} \|\chi(t)\|^2_{H^2(A)} \leqslant \int_0^t \big(c_{11} \|\chi\|^2_{H^2(A)} + c_{12} \|\chi\|_{H^2(A)} \|\boldsymbol{w}\|_{V_\varphi}\big)\, d\eta .$$

Multiplying (11.18) by an appropriately large number $\varrho > 0$ and adding it to (11.17), we obtain then, bearing in mind (8.2)

$$(11.19)\qquad \|\boldsymbol{w}(t)\|^2_{L^2(\Omega)} + \|\chi(t)\|^2_{H^2(A)} + \alpha \int_0^t \|\boldsymbol{w}\|^2_{V_\varphi}\, d\eta \leqslant$$
$$\leqslant \int_0^t \big(c_{13} \|\boldsymbol{w}\|^2_{L^2(\Omega)} + c_{14} \|\chi\|^2_{H^2(A)}\big)\, d\eta \qquad (\alpha > 0) ,$$

and, consequently, $\boldsymbol{w} = \chi = 0$.

The uniqueness theorem is therefore proved.

In a similar way it can be shown that the solution depends continuously on the initial data $\bar{\boldsymbol{u}}$, $\bar{\varphi}$ and the known term $\boldsymbol{f}$. Hence, we can conclude that the « modified » Navier-Stokes model, associated to inequalities (6.4), (11.5), is well posed whenever it is physically consistent, i.e. question β) has a positive answer. As already pointed out in § 3, no reply can, on the other hand, be given to question α). This is due to the fact that the only smoothness assumption that can be made on Γ is that it satisfies a Lipschitz condition (since the points at which the free surface meets the sides of the basin are obviously angle points); this assumption is not however sufficient to guarantee that the solution $\boldsymbol{u}$ satisfies (3.9).

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