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GUIDO LUPACCIOLU (*)

A Rouché's Type Theorem in Several Complex Variables (**)

Un teorema del tipo di Rouché in più variabili complesse

SUNTO. — Si stabilisce una estensione n-dimensionale del teorema di Rouché. Come applicazione se ne deduce una rapida dimostrazione del teorema di Bézout.

Let Ω be a relatively compact open domain of $\mathbf{C}^n$ (or of any n-dimensional Stein manifold) with connected C^1 boundary $\partial\Omega = \Gamma$. Let $f_1, \dots, f_n, g_1, \dots, g_n$ be complex-valued functions holomorphic on Ω and C^1 on $\bar{\Omega}$, and consider the systems of equations

$$f_1(z) = 0, \dots, f_n(z) = 0\,, \tag{1}$$

$$g_1(z) = 0, \dots, g_n(z) = 0\,. \tag{2}$$

In this paper we shall prove the following theorem:

THEOREM 1: *Set*

$$\Gamma_\alpha = \{z \in \Gamma : |f_\alpha(z) - g_\alpha(z)| < |g_\alpha(z)|\}\,, \qquad \alpha = 1, \dots, n\,,$$

and assume that

$$\Gamma = \bigcup_{\alpha=1}^{n} \Gamma_\alpha\,.$$

Then the systems (1) *and* (2) *have finite numbers* N_1 *and* N_2 *of solutions all contained in* Ω *(each solution being counted with its multiplicity) and* $N_1 = N_2$.

Moreover, if $\bigcap_{\alpha=1}^{n} \Gamma_\alpha = 0$, *then* $0 = N_1 = N_2$.

(*) Indirizzo dell'Autore: Istituto Matematico «G. Castelnuovo» dell'Università degli Studi «La Sapienza», Roma.

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This theorem is an extension to n complex variables of the classical Rouché's theorem for holomorphic functions of one complex variable (cf. Ahlfors [1], p. 153).

A simpler extension is given by the following assertion:

THEOREM 2: *Assume that*

$$\sum_{\alpha=1}^{n} |f_\alpha(\zeta) - g_\alpha(\zeta)|^2 < \sum_{\alpha=1}^{n} |g_\alpha(\zeta)|^2$$

for every $\zeta \in \Gamma$. *Then the systems* (1) *and* (2) *have finite numbers* N_1 *and* N_2 *of solutions all contained in* Ω *and* $N_1 = N_2$.

The proof of the latter is quite immediate (cf. n. 2, footnote (¹)).

As an application of Theorem 1 we will show that it allows one to give a simple proof of Bézout's theorem, parallel to the proof of the fundamental theorem of Algebra as a corollary of Rouché's theorem for one complex variable (cf. Alexandroff-Hopf [2], p. 469). On the contrary Theorem 2 is not sufficient for this purpose (cf. n. 3, Remark 3).

1. - PROOF OF THEOREM 1

Set $V_\alpha = \{\zeta \in \bar{\Omega}: f_\alpha(\zeta) = 0\}$, $W_\alpha = \{\zeta \in \bar{\Omega}: g_\alpha(\zeta) = 0\}$, $\alpha = 1, \dots, n$. Then $\Gamma_\alpha \subset \Gamma \setminus V_\alpha$, $\Gamma_\alpha \subset \Gamma \setminus W_\alpha$, and hence $\Gamma = \bigcup_{\alpha=1}^{n} \Gamma_\alpha \subset \Gamma \setminus \bigcap_{\alpha=1}^{n} V_\alpha$, $\Gamma \subset \Gamma \setminus \bigcap_{\alpha=1}^{n} W_\alpha$, which means that $\bigcap_{\alpha=1}^{n} V_\alpha$ and $\bigcap_{\alpha=1}^{n} W_\alpha$ do not intersect $\Gamma = \partial\Omega$. It follows that $\bigcap_{\alpha=1}^{n} V_\alpha$, $\bigcap_{\alpha=1}^{n} W_\alpha$ are compact analytic subvarieties of Ω and, as such, finite subsets of Ω (cf. Gunning-Rossi [6], p. 106). This shows that the solutions of the systems (1) and (2) are at most finitely many and contained in Ω. Hence also N_1 and N_2, the sums of the multeplicities of these solutions, are finite numbers.

We recall that the multeplicity μ of a solution $\zeta^0 \in \Omega$ of the system (1) can be understood, in the most elementary way, as follows: $\mu = 1$ if and only if ζ^0 is a simple solution, that is the differentials $df_1, \dots, df_n$ are independent at ζ^0; in general, for almost all $a \in \mathbf{C}^n$ sufficiently close to the origin, the system $f_1(\zeta) = a_1, \dots, f_n(\zeta) = a_n$ has precisely μ simple solutions within a small neighbourhood of ζ^0 (cf. Milnor [8], Appendix B).

Next, consider the Cauchy kernel

$$\omega = (2\pi i)^{-n} \frac{d\zeta_1}{\zeta_1} \wedge \dots \wedge \frac{d\zeta_n}{\zeta_n}$$

and the maps $f = (f_1, \dots, f_n)$, $g = (g_1, \dots, g_n)$: $\bar{\Omega} \to \mathbf{C}^n$. We shall derive Theorem 1 from the following lemma, which is indeed the main point in the proof and can deserve some interest for itself (cf. n. 2, Remark 1).

LEMMA: *There exist C^1 singular n-cycles with integral coefficients γ_n^1, γ_n^2 of Ω such that*

$$\operatorname{Supp}(\gamma_n^1) \subset \Gamma \setminus \bigcup_{n=1}^{n} V_n, \qquad N_1 = \int_{\gamma_n^1} f^* \omega,$$

$$\operatorname{Supp}(\gamma_n^2) \subset \Gamma \setminus \bigcup_{n=1}^{n} W_n, \qquad N_2 = \int_{\gamma_n^2} g^* \omega.$$

Moreover, if $\bigcap_{n=1}^{n} \Gamma_n \neq 0$, we may assume

$$\gamma_n^1 = \gamma_n^2 = \gamma_n, \qquad \operatorname{Supp}(\gamma_n) \subset \bigcap_{n=1}^{n} \Gamma_n;$$

while, if $\bigcap_{n=1}^{n} \Gamma_n = 0$, then

$$\gamma_n^1 \sim 0 \quad \text{in } \Gamma \setminus \bigcup_{n=1}^{n} V_n, \qquad \gamma_n^2 \sim 0 \quad \text{in } \Gamma \setminus \bigcup_{n=1}^{n} W_n.$$

The proof of this lemma will be given later. Plainly, it already disposes of the case when $\bigcap_{n=1}^{n} \Gamma_n = 0$. Otherwise, it enables one to extend the usual proof of Rouché's theorem for one complex variable, based on the principle of the argument.

As a matter of fact, let us consider the C^1 maps

$$F_h = \left(g_1, \dots, g_{h-1}, \frac{f_h}{g_h}, f_{h+1}, \dots, f_n\right) : \bigcap_{n=1}^{n} \Gamma_n \to \mathbf{C}^n,$$

$h = 1, \dots, n$. Then we have on $\bigcap_{n=1}^{n} \Gamma_n$:

$$f^* \omega - g^* \omega = \sum_{h=1}^{n} F_h^* \omega,$$

so that we are led to show that

$$\int_{\gamma_n} F_h^* \omega = \int_{F_h \circ (\gamma_n)} \omega = 0, \qquad h = 1, \dots, n.$$

Setting

$$A_h = \{z \in \mathbf{C}^n : z_1 \neq 0, \dots, z_{h-1} \neq 0, |z_h - 1| < 1, z_{h+1} \neq 0, \dots, z_n \neq 0\},$$

$h = 1, \dots, n$, it is clear that $F_h\left(\bigcap_{n=1}^{n} \Gamma_n\right) \subset A_h$, and hence

$$\operatorname{Supp}(F_{h*}(\gamma_n)) = F_h(\operatorname{Supp}(\gamma_n)) \subset A_h.$$

Moreover the singular homology group with integral coefficients $H_n(A_k)$ is zero, since A_k has the homotopy type of a $(n-1)$-dimensional torus. It follows that

$$F_{kk}(\gamma_n) \sim 0 \quad \text{in} \quad \mathbf{C}^n \setminus \{z \in \mathbf{C}^n : z_1 \dots z_n = 0\},$$

$k = 1, \dots, n$, which concludes the proof of Theorem 1.

2. - Proof of the lemma

We shall use the following facts:

i) Given an exact sequence

$$0 \to K_0 \to K_1 \to \dots \to K_n \to 0$$

of chain complexes (cochain complexes), there exists a canonical homomorphism of graded modules

$$\varepsilon : H(K_n) \to H(K_0)$$

of degree $-(n-1)$ (degree $n-1$). Moreover ε is natural, in the sense that, if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0 & \longrightarrow & K_1 & \longrightarrow & \dots & \longrightarrow & K_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_n} & & \\ 0 & \longrightarrow & K'_0 & \longrightarrow & K'_1 & \longrightarrow & \dots & \longrightarrow & K'_n & \longrightarrow & 0 \end{array}$$

is a commutative diagram of chain maps (cochain maps) with exact rows, then the diagram

$$\begin{array}{ccc} H(K_n) & \xrightarrow{\varepsilon} & H(K_0) \\ \downarrow{\scriptstyle \varphi_{n*}} & & \downarrow{\scriptstyle \varphi_{0*}} \\ H(K'_n) & \xrightarrow{\varepsilon} & H(K'_0) \end{array}$$

is also commutative.

This is a straightforward generalization of the parallel well-known fact concerning a short exact sequence. As in that case, ε will be termed *connecting homomorphism*.

ii) Consider the Martinelli kernel

$$k = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi i)^n} \frac{d\zeta_1 \wedge \dots \wedge d\zeta_n}{(|\zeta_1|^2 + \dots + |\zeta_n|^2)^n} \wedge \sum_{k=1}^{n} (-1)^{k-1} \bar{\zeta}_k \, d\bar{\zeta}_1 \wedge \dots \widehat{d\bar{\zeta}_k} \dots \wedge d\bar{\zeta}_n,$$

and assume that Ω is oriented by the form $(i/2)^n \bigwedge_{k=1}^{n} d\zeta_k \wedge d\bar{\zeta}_k$ and Γ as the

boundary of Ω. Then

$$N_1 = \int_{\Gamma} f^* k, \qquad N_2 = \int_{\Gamma} g^* k$$

(cf. Severi [10], p. 192) (¹).

iii) Consider the generalized Mayer-Vietoris exact sequence (cf. Bott-Tu [3]) for complex-valued C^∞ differential forms relative to the open covering of $\mathbf{C}^n \setminus \{O\}$ made by the sets $U_\alpha = \{z \in \mathbf{C}^n : z_\alpha \neq 0\}$, $\alpha = 1, \ldots, n$. If

$$\partial^* : H^*_{DR}\Big(\bigcap_{\alpha=1}^{n} U_\alpha\Big) \to H^{*+(n-1)}_{DR}(\mathbf{C}^n \setminus \{O\})$$

is the induced connecting homomorphism, then

$$\partial^*([\omega]) = [k] .$$

This fact is essentially known: it is indeed a straightforward consequence of a result found in Harvey [7] (Theorem 2.2) (²).

That being stated, let

$$\partial^1_* : H_*(\Gamma) \to H_{*-(n-1)}\Big(\Gamma \setminus \bigcup_{\alpha=1}^{n} V_\alpha\Big),$$

$$\partial^*_1 : H^*_{DR}\Big(\Gamma \setminus \bigcup_{\alpha=1}^{n} V_\alpha\Big) \to H^{*+(n-1)}_{DR}(\Gamma),$$

be the connecting homomorphisms induced by the Mayer-Vietoris sequences relative to the open covering $\{\Gamma \setminus V_1, \ldots, \Gamma \setminus V_n\}$ of Γ, respectively for C^1 singular chains with integral coefficients and for complex-valued regular differential forms (cf. Whitney [12]) (³). Denote by σ_Γ the canonical generator of $H_{2n-1}(\Gamma)$ and choose a representative γ^1_α of the class $\partial^1_*(\sigma_\Gamma) \in H_n\Big(\Gamma \setminus \bigcup_{\alpha=1}^{n} V_\alpha\Big)$. Since plainly ∂^*_1 and ∂^1_* are dual with respect to the Kronecker index $\langle -, - \rangle$,

(¹) It follows that

$$N_1 = \int_{f_*(\Gamma)} k, \qquad N_2 = \int_{g_*(\Gamma)} k .$$

Now, suppose that f and g satisfy

$$\operatorname{dist}(f(z), g(z)) < \operatorname{dist}(g(z), O),$$

for every $z \in \Gamma$. The $(2n-1)$-cycles $f_*(\Gamma)$, $g_*(\Gamma)$ are then homotopic in $\mathbf{C}^n \setminus \{O\}$ (cf. Alexandroff-Hopf [2], p. 459), and hence the above integrals are equal. This proves Theorem 2.

(²) Cf. also Griffiths-Harris [5], Chapter 5.

(³) Since Γ is assumed to be only C^1, we may not consider C^∞ forms on Γ.

it follows from i), ii) and iii) that

$$\int_{\gamma_*^1} f^*\omega = \langle [f^*\omega], \iota_*^1(o_\Gamma)\rangle = \langle \iota_1^*([f^*\omega]), o_{\Gamma'}\rangle =$$

$$= \langle f^*\iota^*([\omega]), o_{\Gamma'}\rangle = \langle [f^*k], o_{\Gamma'}\rangle = \int_{\Gamma'} f^*k = N_1 .$$

In the same way one shows that

$$\int_{\gamma_*^2} g^*\omega = N_2 ,$$

where γ_*^2 is a representative of the class $\iota_*^2(o_\Gamma) \in H_n\left(\Gamma \setminus \bigcup_{\alpha=1}^{n} W_\alpha\right)$ (with obvious meaning of ι_*^2).

Next, consider the connecting homomorphism

$$\iota_*: H_*(\Gamma) \to H_{*-(n-1)}\left(\bigcap_{\alpha=1}^{n} \Gamma_\alpha\right)$$

induced by the Mayer-Vietoris sequence for C^1 singular chains with integral coefficients relative to the open covering $\{\Gamma_1, \ldots, \Gamma_n\}$ of Γ. Then the inclusions $\Gamma_\alpha \subset \Gamma \setminus V_\alpha$, $\Gamma_\alpha \subset \Gamma \setminus W_\alpha$, $\alpha = 1, \ldots, n$ induce commutative diagrams

$$\begin{tikzcd} H_*(\Gamma) \arrow[r, "\iota_*"] \arrow[dr, "\iota_*^1"'] & H_{*-(n-1)}\left(\bigcap_{\alpha=1}^{n} \Gamma_\alpha\right) \arrow[d, "i_*^1"] \\ & H_{*-(n-1)}\left(\Gamma \setminus \bigcup_{\alpha=1}^{n} V_\alpha\right) \end{tikzcd} , \qquad \begin{tikzcd} H_*(\Gamma) \arrow[r, "\iota_*"] \arrow[dr, "\iota_*^2"'] & H_{*-(n-1)}\left(\bigcap_{\alpha=1}^{n} \Gamma_\alpha\right) \arrow[d, "i_*^2"] \\ & H_{*-(n-1)}\left(\Gamma \setminus \bigcup_{\alpha=1}^{n} W_\alpha\right) \end{tikzcd} .$$

It follows that, if $\bigcap_{\alpha=1}^{n} \Gamma_\alpha \neq 0$, any representative γ_* of the class $\iota_*(o_\Gamma) \in H_n\left(\bigcap_{\alpha=1}^{n} \Gamma_\alpha\right)$ represents also the classes $\iota_*^1(o_\Gamma)$ and $\iota_*^2(o_\Gamma)$, so that we may take $\gamma_*^1 = \gamma_*^2 = \gamma_*$. On the contrary, if $\bigcap_{\alpha=1}^{n} \Gamma_\alpha = 0$, then the homomorphisms ι_*^1, ι_*^2 are both zero, and consequently $\gamma_*^1 \sim 0$ in $\Gamma \setminus \bigcup_{\alpha=1}^{n} V_\alpha$, $\gamma_*^2 \sim 0$ in $\Gamma \setminus \bigcup_{\alpha=1}^{n} W_\alpha$.

The proof of the lemma is then completed.

Remark 1: The lemma just proved can be generalized to the case that $f^*\omega$, $g^*\omega$ are replaced by two n-forms φ, ψ such that φ is C^1 on $\Omega \setminus \bigcup_{\alpha=1}^{n} V_\alpha$ and holomorphic on $\Omega \setminus \bigcup_{\alpha=1}^{n} V_\alpha$, ψ is C^1 on $\Omega \setminus \bigcup_{\alpha=1}^{n} W_\alpha$ and holomorphic on $\Omega \setminus \bigcup_{\alpha=1}^{n} W_\alpha$, and N_1, N_2 are replaced by $(2\pi i)^n \sum_{m \in \bigcap_{\alpha=1}^{n} \Gamma_\alpha} \mathrm{Res}_m \varphi$, $(2\pi i)^n \sum_{m \in \bigcap_{\alpha=1}^{n} W_\alpha} \mathrm{Res}_m \psi$, respectively.

The proof is essentially the same, using Dolbeault's representatives η_φ, η_ψ in place of f^*k, g^*k (cf. Griffiths-Harris [5], pp. 649-656).

3. - Bézout's theorem

Now, let $f_1, \dots, f_n : \mathbf{C}^n \to \mathbf{C}$ be polynomials of respective degrees $\nu_1, \dots, \nu_n$ all positive. Denote by $g_1, \dots, g_n$ the homogeneous parts of highest degrees of $f_1, \dots, f_n$, respectively. Then we will prove:

Theorem 3: *Assume that the homogeneous system $g_1(z) = 0, \dots, g_n(z) = 0$ has only the trivial solution $O = (0, \dots, 0)$. Then the system $f_1(z) = 0, \dots, f_n(z) = 0$ has precisely $\nu_1 \dots \nu_n$ solutions, each one counted with its multeplicity.*

This is the affine rephrasement of Bézout's theorem for n divisors of $\mathbf{P}^n(\mathbf{C})$ meeting in a finite set of points, after removing from $\mathbf{P}^n(\mathbf{C})$ a hyperplane L that does not contain any of these points (L may be assumed to be the hyperplane «at infinity» of $\mathbf{C}^n$).

We are going to show that Theorem 1 may be applied to the polynomial systems $f_1(z) = 0, \dots, f_n(z) = 0$ and $g_1(z) = 0, \dots, g_n(z) = 0$ assuming as Ω an open ball with center the origin and radius r large enough. After that we'll need only to prove that O has multeplicity $\mu = \nu_1 \dots \nu_n$ as the unique solution of the homogeneous system $g_1(z) = 0, \dots, g_n(z) = 0$.

For every positive real number r, let $S(r)$ be the $(2n-1)$-sphere of $\mathbf{C}^n$ with center O and radius r. Choose a positive real number ε such that

$$S(1) \cap \{z \in \mathbf{C}^n : |g_1(z)| < \varepsilon, \dots, |g_n(z)| < \varepsilon\} = 0$$

and set

$$S_\alpha(1) = \{z \in S(1) : |g_\alpha(z)| > \varepsilon\},$$

$\alpha = 1, \dots, n$. Then $S(1) = \bigcup_{\alpha=1}^{n} S_\alpha(1)$. For every $r \geq 1$, let $S_\alpha(r)$ be the transfotm of $S_\alpha(1)$ by the homothety $z \mapsto rz$. Then, by the homogeneity of the g_α's,

$$S_\alpha(r) = \{z \in S(r) : |g_\alpha(z)| > r^{\nu_\alpha}\varepsilon\},$$

$\alpha = 1, \dots, n$, and of course $S(r) = \bigcup_{\alpha=1}^{n} S_\alpha(r)$. Next, if we set

$$M_\alpha(r) = \max_{S(r)} |f_\alpha - g_\alpha|, \qquad \alpha = 1, \dots, n,$$

since $\deg(f_\alpha - g_\alpha) < \nu_\alpha - 1$, it is clear that $M_\alpha(r)/r^{\nu_\alpha}$ approaches zero for large r. Hence we can find some $r_\varepsilon \geq 1$ such that $M_\alpha(r) < r^{\nu_\alpha}\varepsilon$ for $r \geq r_\varepsilon$, $\alpha = 1, \dots, n$. It follows that for $r \geq r_\varepsilon$ and $z \in S_\alpha(r)$ we have

$$|f_\alpha(z) - g_\alpha(z)| < r^{\nu_\alpha}\varepsilon < |g_\alpha(z)|,$$

$\alpha = 1, \dots, n$, which yields the desired result.

Now, to show that $\mu = \nu_1 \dots \nu_n$, assume that $g_1, \dots, g_n$ are the general homogeneous polynomials of degrees $\nu_1, \dots, \nu_n$ with undetermined complex coefficients a_j. Then, as a consequence of Hilbert's Nullstellensatz, there exists a set $\{D(a_j)\}$ of homogeneous polynomials in those coefficients, such that the system $g_1(z) = 0, \dots, g_n(z) = 0$ has only the trivial solution if and only if the corresponding a_j's do not satisfy at least one of the equations $D(a_j) = 0$ (cf. van der Waerden [11], vol. 2, p. 158) (4). It follows that μ is an integer-valued function of the a_j's defined on a connected open subset of $\mathbf{C}^N$, where N is the number of the a_j's. This function is continuous, as being equal to the integral $\int_{S(1)} g^* k$ (cf. n. 2, ii)), and therefore is constant. Then taking $g_\alpha = z_\alpha^{\nu_\alpha}$, $\alpha = 1, \dots, n$ obviously gives $\mu = \nu_1 \dots \nu_n$ (5).

Remark 2: In the case $n = 2$ a proof of Bézout's theorem based on the Cauchy kernel has first been given by Caccioppoli [4]. Though the argument used there is rather intricated and not suited for an easy extension to general n, we have followed here the conceptual lines of Caccioppoli's proof.

Remark 3: Theorem 2 is not suitable to prove that the polynomial systems $f_1(z) = 0, \dots, f_n(z) = 0$ and $g_1(z) = 0, \dots, g_n(z) = 0$ have the same number of solutions in a large domain containing O. For example, if $n = 2$, $f_1 = z_1^2 + z_2$, $f_2 = 1 + z_2$, then

$$|g_1(z)|^2 + |g_2(z)|^2 - |f_1(z) - g_1(z)|^2 - |f_2(z) - g_2(z)|^2 = |z_1|^4 - 1 ,$$

so that the condition of Theorem 2 is not fulfilled on the boundary of any domain containing O and the solutions $(1, -1)$, $(-1, -1)$ of the system $f_1(z) = 0$, $f_2(z) = 0$.

(4) It is known that the set $\{D(a_j)\}$ can be replaced by a single « resultant » $R(a_j)$ (cf. Segre [9], p. 26), but the above elementary result is all we need.

(5) Another way to derive this equality is comparing the residues $\mathrm{Res}_0(g^*\omega) = (2\pi i)^{-n}\mu$ and $\mathrm{Res}_0(\omega) = (2\pi i)^{-n}$ by means of the « transformation law » (cf. Griffiths-Harris [5], p. 657), taking into account the identities $\nu_\alpha g_\alpha = \sum_{\beta=1}^{n} (\partial g_\alpha/\partial z_\beta) z_\beta$, $\alpha = 1, \dots, n$.

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