U. P. SINGH and P. C. YADAV (*)

On hypernormal curves in the special Kawaguchi geometry (**)

1. Introduction

The properties of hypernormal and hyperasymptotic curves have been studied by Singh [4] (1) in locally Minkowskian Finsler subspace. On the parallel lines, we have defined and studied hypernormal curves of an even dimensional special Kawaguchi spaces of order two.

We shall in first instance outline some of the basic concepts and fundamental formulae which are mainly due to Kawaguchi [1, 2] and Yoshida [5].

Consider an n-dimensional special Kawaguchi space $\overset{(2)}{K}_n$ of order 2, in which the arc length of the curve $x^i = x^i(t)$ (2) is given by the integral

$$S = \int \{A_i(x, \acute{x})\, \acute{\acute{x}}^i + B(x, \acute{x})\}^{1/p}\, dt, \quad p \neq 0,\ 3/2 \tag{1.1}$$

where $\acute{x}^i = dx^i/dt$, $\acute{\acute{x}}^i = d^2x^i/dt^2$ and A_i, B being differentiable functions of x^i and $\acute{x}^i$.

From the Zermelo's conditions that the arc lenght of the curve remains unaltered by any transformation of the parameter t, we have

$$A_i\, \acute{x}^i = 0\ ,\quad A_{i(j)}\, \acute{x}^j\ (3) = (p-2)\, A_i\ ,\quad B_{(i)}\, \acute{x}^i = p\, B\ . \tag{1.2}$$

Since S given by (1.1) is a scalar, it follows that A_i is a vector.

(*) Department of Mathematics, University of Gorakhpur, India.

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(1) Number in brackets refer to the references at the end of the paper.

(2) Latin indices i, j, k run from 1 to n, Greek ones α, β, γ from 1 to m and μ, ν from m + 1 to n.

(3) The symbol (j) denotes the partial differentiaion with respect to $\acute{x}^j$.

Let a subspace $\overset{(2)}{K}_m$ of dimension m of $\overset{(2)}{K}_n$ be represented by the equations $x^i = x^i(u^\alpha)$ and the matrix of the projection factors $p^i_\alpha = \partial x^i / \partial u^\alpha$ has rank m. The fundamental tensors G_{ij} and $G_{\alpha\beta}$ of the special Kawaguchi space $\overset{(2)}{K}_n$ and its subspace $\overset{(2)}{K}_m$ are defined as (Yoshida [5])

$$G_{ij} \overset{\text{def}}{=} 2 A_{i(j)} - A_{j(i)} \ , \quad G_{\alpha\beta} \overset{\text{def}}{=} 2 a_{\alpha(\beta)} - a_{\beta(\alpha)} \tag{1.3}$$

and these are related by

$$G_{\alpha\beta} = G_{ij}\, p^i_\alpha\, p^j_\beta \, . \tag{1.4}$$

Assuming that n, m are both even and det (G_{ij}) does not vanish identically, we can easily show that

$$p^\alpha_i\, p^i_\beta = \delta^\alpha_\beta \quad \text{and} \quad G_{\beta\gamma}\, G^{\beta\alpha} = \delta^\alpha_\gamma \tag{1.5}$$

where $p^\alpha_i \overset{\text{def}}{=} G^{\alpha\beta}\, G_{ij}\, p^j_\beta$, $G^{\alpha\beta}$ being the tensor reciprocal to $G_{\alpha\beta}$.

The connections Γ^i of $\overset{(2)}{K}_n$ and Γ^α of $\overset{(2)}{K}_m$ are given by

$$2\,\Gamma^i = (2 A_{kj}\, x'^j - B_{(k)})\, G^{ki} \tag{1.6}$$

and

$$2\,\overset{\alpha}{\Gamma} = (2 a_{\beta\gamma}\, u'^\gamma - b_{(\beta)})\, G^{\beta\alpha} \tag{1.7}$$

where G^{ki} being the tensor reciprocal to G_{kj}, $A_{kj} = \partial A_k / \partial x^j$ and $a_{\beta\gamma} = \partial a_\beta / \partial u^\gamma$.

The covariant differential of a contravariant vector field $v^i(x, x')$ homogeneous of degree zero with respect to x'^i is given by Kawaguchi [2]

$$\delta v^i = d v^i + \Gamma^i_{jk}\, v^j\, d x^k \, . \tag{1.8}$$

where
$$\Gamma^i_{jk} = \partial^2 \Gamma^i / \partial x'^j\, \partial x'^k = \Gamma^i_{kj} \, .$$

Let v^α be a vector field in $\overset{(2)}{K}_m$ such that $v^i = p^i_\alpha\, v^\alpha$, then the induced differential $\tilde{\delta}\, v^\alpha\ (= p^\alpha_i\, \delta v^i)$ is given by

$$\tilde{\delta}\, v^\alpha = d v^\alpha + \tilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, d u^\gamma \tag{1.9}$$

where

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = p^\alpha_i\, (p^i_{\beta\gamma} + \Gamma^i_{jk}\, p^j_\beta\, p^k_\gamma) \ , \quad p^i_{\beta\gamma} = \partial p^i_\beta / \partial u^\gamma . \tag{1.10}$$

Moreover, it has been deduced that [5]

$$\overset{\circ}{H}{}^i_{\beta\alpha} = \overset{\circ}{D}_\beta\, p^i_\alpha = p^i_{\alpha\beta} + \Gamma^i_{jk}\, p^j_\beta\, p^k_\alpha - \tilde{\Gamma}^\gamma_{\alpha\beta}\, p^i_\gamma \, . \tag{1.11}$$

Assuming that the vectors $\underset{\mu}{n^i}$ of $\overset{(2)}{K}_n$ are normal to $\overset{(2)}{K}_m$, the quantities $\overset{o}{H}{}^i_{\beta\alpha}$ can be expressed as

$$\overset{o}{H}{}^i_{\beta\alpha} = \overset{\mu}{H}_{\beta\alpha} \underset{\mu}{n^i} \tag{1.12}$$

where

$$\overset{\mu}{H}_{\beta\alpha} = G^{\mu\nu} \underset{\nu}{n_i} \overset{o}{H}{}^i_{\beta\alpha} \tag{1.13}$$

and $G^{\mu\nu}$ will have the same meaning as defined by Yoshida [5].

Considering the tangent vector dx^i/ds, of a curve $C : x^i = x^i(s)$ of the subspace $\overset{(2)}{K}_m$, it has been shown that

$$\delta \dot{x}^i/d s = p^i_\alpha (\widetilde{\delta} \dot{u}^\alpha/d s) + \overset{o}{H}{}^i_{\beta\gamma} \dot{u}^\beta \dot{u}^\gamma, (\dot{x}^i = d x^i/d s \ , \ \dot{u}^\alpha = d u^\alpha/d s) . \tag{1.14}$$

If $p^\alpha (\overset{\text{def}}{=} \widetilde{\delta} \dot{u}^\alpha/d s) = 0$, then the curve will be called a geodesic of the subspace $\overset{(2)}{K}_m$ and it will be called an asymptotic line if

$$\overset{\mu}{H}_{\beta\gamma} \dot{u}^\beta \dot{u}^\gamma = 0 .$$

2. Hypernormal curves

Consider a congruence of curves in $\overset{(2)}{K}_n$ given by the vector field λ^i such that through each point of $\overset{(2)}{K}_n$ there passes exactly one curve of each congruence. At the points of the subspace, we write

$$\lambda^i = t^\alpha p^i_\alpha + \sum_\mu \Gamma_{(\mu)} \underset{\mu}{n^i} \qquad (\Gamma_{(\mu)} \neq 0) . \tag{2.1}$$

At a point of a curve C, the linear space spanned by the vectors $d x^i/d s \, (= p^i_\alpha \, d u^\alpha/d s)$ and $q^i \, (= \delta \, \dot{x}^i/d s)$ is called osculating geodesic surface of the curve. This is a subspace of $\overset{(2)}{K}_n$. We have, therefore, $(n - 2)$ normal vectors $\xi^i_{(\Phi)} (\Phi = 1, \ldots, n - 2)$ of this subspace satisfying the conditions:

$$G_{ij} (d x^i/d s) \xi^j_{(\Phi)} = 0 \ , \ G_{ij} q^i \xi^j_{(\Phi)} = 0 \quad \text{and} \quad G_{ij} \xi^i_{(\Phi)} \xi^j_{(\psi)} = 0 \text{ fov } \Phi \neq \psi . \tag{2.2}$$

The vectors $d x^i/d s$, q^i and $\xi^i_{(\Phi)}$ will form a set of n linearly independent vectors in $\overset{(2)}{K}_n$ and we can write

$$\lambda^i = x \, d x^i/d s + y \, q^i + \sum_{\Phi=1}^{n-2} A_{(\Phi)} \xi^i_{(\Phi)} , \tag{2.3}$$

where

$$x = G_{ij} (d x^i/d s) \lambda^j = G_{\alpha\beta} t^\alpha (d u^\beta/d s) , \tag{2.4}$$

$$y\,K^2 = G_{ij}\,q^i\,\lambda^j = G_{\alpha\beta}\,t^\alpha\,p^\beta + \sum_\mu H^\mu_{\alpha\beta}\,\dot u^\alpha\,\dot u^\beta\,\Gamma_{(\mu)} \tag{2.5}$$

and

$$K^2 \overset{\text{def}}{=} G_{ij}\,q^i\,q^j\,. \tag{2.6}$$

The vector $\alpha^i = x\,d\,x^i/d\,s + y\,q^i$ represents the component of λ^i in the osculating geodesic surface of C.

We define

Definition (2.1) - *A curve of* $\overset{(2)}{K}_n$, *for which* $x = 0$ *but* $y \neq 0$ *is called a hypernormal curve relative to* λ^i. *On the other hand it will be called a hyperasymptotic curve if* $y = 0$ *but* $x \neq 0$.

Therefore for a hypernormal curve we have

$$\lambda^i = y\,q^i + \sum_{\Phi=1}^{n-2} A_{(\Phi)}\,\xi^i_{(\Phi)}\,,\ y \neq 0\,,\ x = 0 \tag{2.7}$$

which gives

$$G_{\alpha\beta}\,t^\alpha\,(d\,u^\beta/d\,s) = 0 \tag{2.8}$$

and

$$G_{\alpha\beta}\,t^\alpha\,p^\beta + \sum_\mu H^\mu_{\alpha\beta}\,\dot u^\alpha\,\dot u^\beta\,\Gamma_{(\mu)} \neq 0\,. \tag{2.9}$$

Hence we derive the following properties:

Theorem (2.1) - *If the vector* λ^i *lies in a variety generated by* $n^i_{|\mu}$ *then any non-asympotic curve is a hypernormal curve relative to* λ^i *in an even dimensional special Kawaguchi space of order 2.*

Proof - Since λ^i lies in a variety generated by the $n^i_{|\mu}$ therefore, $t^\alpha = 0$ and the equation (2.8) is satisfied for an arbitrary du^α/ds. The equation (2.9) reduces to

$$\sum_\mu H^\mu_{\alpha\beta}\,\dot u^\alpha\,\dot u^\beta\,\Gamma_{(\mu)} \neq 0\,.$$

As $\Gamma_{(\mu)} \neq 0$, the above conditions will hold iff $H^\mu_{\alpha\beta}\dot u^\alpha\,\dot u^\beta \neq 0$ for at least one μ.

Further, in an even-dimensional special Kawaguchi space of order 2, the following conditions hold (Watanabe [3])

$$\delta A\,i\,(j) = 0\,,\ A\,i\,(j)\,(k) = 0 \ \text{ and } \ G\,i\,j\,(k) = 0\,. \tag{2.10}$$

Taking these under consideration, we find

$$\text{(a)}\quad \delta\,G\,i\,j\,/\,d\,s = 0\,,\qquad \text{(b)}\quad G_{\alpha\beta}\,p^\alpha\,(d\,u^\beta/d\,s) = 0\,. \tag{2.11}$$

THEOREM (2.2) - *If the subspace is a surface then the component of λ^i tangent to the surface is along the geodesic curvature vector $p^\alpha\ (=\tilde{\delta}\,\dot{u}^\alpha/ds)$ of the hypernormal curve.*

Proof - For a surface, we may write

$$t^\alpha = U\, du^\alpha/ds + V\, p^\alpha . \tag{2.12}$$

Multipyling the above equation by $G_{\alpha\beta}\, du^\beta/ds$, using the equations (2.8) and (2.11)b, we get

$$U = 0 . \tag{2.13}$$

In view of (2.12) and (2.13) we obtain

$$t^\alpha = V\, p^\alpha \tag{2.14}$$

This proves the Theorem.

3. UNION AND HYPERNORMAL CURVES

The union curve and union curvature vector of a curve relative to λ^i are given by Singh and Dubey [6]

$$\lambda^i = A\, p^i_\alpha\, (du^\alpha/ds) + B\, q^i \tag{3.1}$$

and

$$\eta^\alpha = p^\alpha - \left\{ \sum_\mu (H^\mu_{\beta\gamma}\, du^\beta/ds \cdot du^\gamma/ds)^2 / \sum_\mu \Gamma^2_{(\mu)} \right\}^{1/2} (t^\alpha - A\, du^\alpha/ds) . \tag{3.2}$$

Therefore, the condition $G_{ij}\lambda^i (dx^j/ds) = 0$ implies that $A = o$ and consequently we obtain

$$\lambda^i = B\, q^i \tag{3.4}$$

and

$$\eta^\alpha = p^\alpha - \left\{ \sum_\mu (H^\mu_{\alpha\gamma}\, du^\beta/ds \cdot du^\gamma/ds)^2 / \sum_\mu \Gamma^2_{(\mu)} \right\}^{1/2} t^\alpha . \tag{3.5}$$

The equation (3.4) yields.

THEOREM (3.1) - *A necessary and sufficient condition that the union curve relative to λ^i be the hypernormal curve relative to the same congruence is that λ^i is along the first curvature vector $q^i\ (=\delta\,\dot{x}^i/ds)$ of the curve in an even dimensional special Kawaguchi space of order 2.*

THEOREM (3.2) - *The result of Theorem (2.2) is true for an m-dimensional ($m > 2$, m even) subspace provied that the curve is also a union curve relative to λ^1.*

THEOREM (3.3) - *If the subspace of $\overset{(2)}{K}_n$ is a surface then any hypernormal geodesic is a union curve.*

Proof of Theorem (3.3) - In view of Theorem (2.2), the equation (3.5) and the fact that the union curve is characterised by $\gamma_i^x = 0$, we obtain the proof of the Theorem.

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