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Some theorems concerning Fourier series and its conjugate series on Abel type summability methods (**)

Introduction

In a recent paper, Borwein [1] constructed a new method of summability A_λ, $(\lambda > -1)$ for an infinite sequence $\{s_n\}$ and investigated some of its properties. One of the authors (Mishra [2, 3]) studied many of its properties for infinite series as well as for Fourier series. In the present paper, we investigate some theorems for this new summability method for Fourier series and the series conjugate to it.

1. Definitions

Let $\epsilon_n^\lambda = \binom{n+\lambda}{n}$ and $\{s_n\}$ be any sequence of numbers. If, for $\lambda > -1$,

$$(1.1) \qquad (1-x)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n$$

is convergent for all x in the open interval (0,1) and tends to a finite limit s as $x \to 1$ in (0,1), we say that the sequence $\{s_n\}$ is summable A_λ to the sum s and we write

$$s_n \to s\,(A_\lambda)\ .$$

(A_0) method is the ordinary Abel method (A).

2. A_λ - Kernels

Suppose that f (x) is integrable (L) in the interval $(-\pi, \pi)$ and periodic with the period 2π and let

$$(2.1) \qquad S[f] \sim \frac{a_0}{2} + \sum_{v=1}^{\infty} (a_v \cos v x + b_v \sin v x)$$

and

$$(2.2) \qquad \widetilde{S}[f] \sim \sum_{v=1}^{\infty} (a_v \sin v x - b_v \cos v x)$$

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are respectively the Fourier series and conjugate to the Fourier series of f (x) with the Fourier coefficients a_v, b_v. The A_λ means of S [f] is the function

$$(2.3) \qquad f_\lambda(r, x) = (1 - r)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda s_n(x) r^n \qquad (0 \leqslant r < 1),$$

$$= (1 - r)^{\lambda+1} \sum_{n=0}^{\infty} \left\{ \frac{1}{2} a_0 + \sum_{v=1}^{n} (a_v \cos v x + b_v \sin v x) \right\} \epsilon_n^\lambda r^n .$$

Since the Fourier coefficients a_v, b_v tend to zero, as $r \to 1$ in the open interval (0,1) the series converges absolutely and uniformly for $0 \leqslant r \leqslant 1 - \delta$, $\delta > 0$.
From (2.3), we see that

$$(2.4) \qquad f_\lambda(r, x) = (1 - r)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda r^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t - x)\, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) P_\lambda(r, t - x)\, dt ,$$

where Dirichlet kernel $D_n(t) = \frac{1}{2} + \sum_{v=1}^{n} \cos v t$

$$= \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}$$

and poisson type kernel

$$P_\lambda(r, t) = (1 - r)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda r^n D_n(t) .$$

Similarly the A_λ means of $\widetilde{S}[f]$ is given by

$$(2.5) \qquad \widetilde{f}_\lambda(r, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) Q_\lambda(r, t - x)\, dt ,$$

where

$$Q_\lambda(r, t) = (1 - r)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda r^n \widetilde{D}_n(t)$$

and

$$\widetilde{D}_n(t) = \sum_{v=1}^{n} \sin v t = \frac{\cos \frac{t}{2} - \cos\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}} .$$

Kernel $P_\lambda(\pi, t)$ satisfies the following conditions:

(A) $$\frac{1}{\pi}\int_{-\pi}^{\pi} P_\lambda(\pi, t)\, dt = 1 ;$$

(B) $$P_\lambda(\pi, t) \geqslant 0 , \qquad (-\pi \leqslant t \leqslant \pi) ;$$

(C) $$\mathrm{Max}_{\delta \le t \le \pi} | P_\lambda(\pi, t) | \to 0 , \qquad (0 < \delta < \pi) .$$

Proof of condition (A) is trivial. For condition (B), we have

$$P_\lambda(\pi, t) = (1-\pi)^{\lambda+1} \sum_{n=0}^{\infty} c_n^\lambda r^n \frac{\sin\left(n+\frac{1}{2}\right) t}{2 \sin \frac{t}{2}} .$$

(2.6) $$= (1-\pi)^{\lambda+1} \frac{1}{2 \sin \frac{t}{2}} \left[I_\psi \left\{ e^{\frac{it}{2}} (1 - \pi e^{it})^{-(\lambda+1)} \right\} \right]$$

$$= \frac{(1-\pi)^{\lambda+1} \sin \left\{ (\lambda+1) \tan^{-1} \frac{\pi \sin t}{1 - \pi \cos t} + \frac{t}{2} \right\}}{2 \sin \frac{t}{2} (1 + \pi^2 - 2\pi \cos t)^{\frac{\lambda+1}{2}}}$$

where I_ψ identifies the imaginary part of the expression under consideration. The right hand side in (2.6) is an even function and positive in the said interval. To satisfy condition (C), we have

$$^{*}\, P_\lambda(\pi, t) \leqslant \frac{\delta^{\lambda+1}}{t(\delta^2 + t^2)^{\frac{\lambda+1}{2}}}, \qquad (\delta = 1 - r),$$

$$\leqslant \frac{A\,\delta^{\lambda+1}}{t^{\lambda+2}} \leqslant \frac{A}{(n+1)^{\lambda+1} (\delta')^{\lambda+2}} \to 0 \quad \text{as } n \to \infty ,$$

where A is a positive number and $\delta' \leqslant t \leqslant \pi$.

Thus

$$\mathrm{Max}_{\delta' \le t \le \pi} | P_\lambda(\pi, t) | \to 0 .$$

(*) It is also easy to prove that

$$P_\lambda(\pi, t) \leqslant \frac{A}{\delta}, \quad (0 < t \leqslant \pi , \ 0 \leqslant \pi < 1) .$$

With the help of condition (A), we have

$$f_\lambda(\pi, x) - f(x) = \frac{2}{\pi}\int_0^\pi \varnothing_x(t)\, P_\lambda(\pi, t)\, dt$$

for

$$\varnothing_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

3. In this section, we have the following theorems

Theorem (*3.1*). At every point x_0 at which the limits $f(x_0+0)$ and $f(x_0-0)$ exist, we have

$$f_\lambda(\pi, x_0) \to \frac{1}{2}\{f(x_0+0) - f(x_0-0)\}.$$

In particular, $f_\lambda(\pi x_0) \to f(x_0)$ at every point of continuity of $f(x)$.

Theorem (*3.2*). The Fourier Series of a function $f(x) \in L(0, 2\pi)$ is summable $A\lambda$ almost everywhere to the sum $f(x)$.

Theorem (*3.3*). Suppose that

$$F(x) \sim \frac{A_0}{2} + \sum_{v=1}^{\infty}(A_v \cos vx + B_v \sin vx)$$

and that

$$\lim_{h\to+0}\frac{F(x_0+h) - F(x_0-h)}{2h} = D_1 F(x_0)$$

exists, finite or infinite. Then $S'[F]$ is summable A_λ at x_0 to the sum $D_1 F(x_0)$, ie.

(3.1) $$(1-\pi)^{\lambda+1}\sum_{v=1}^{\infty} v\, e_v^\lambda (B_v \cos vx_0 - A_v \sin vx_0)\, r^v$$

tends to $D_1 F(x_0)$ as $r \to 1$.

More generally, the limits of indetermination of (3.1) as $r \to 1$ are contained between $\underline{D}_1 F(x_0)$ and $\overline{D}_1 F(x_0)$, where $\underline{D}_1 F(x_0)$ and $\overline{D}_1 F(x_0)$ are respectively the lower and upper first symmetric derivatives of $F(x)$ at $x = x_0$ *.

Theorem (*3.4*). If $F'(x_0)$ exists and is finite, then $\frac{\partial F_\lambda(\pi, x)}{\partial x} \to F'(x_0)$ as (r, x) approaches $(1, x_0)$ non-tangentially **.

(*) For details, see ([*1*] p. 99).

(**) For definition, see N. K. Bary's A Treatise on Trigonometric series, English translation by M. F. Mullins, Pergamon Press, (1964), p. 15.

The proof of this theorem is almost parallel to that of theorem 7.6 (Zygmund [4], p. 100).

Theorem (*3.5*). Let $F(x)$ be the indefinite integral of an integrable and periodic $f(x)$. Then $S[f]$ is summable A_λ to the sum $D_1 F(x_0)$ at every point x_0 at which $D_1 F(x_0)$ exists, finite or infinite.

Theorem (*3.6*). Let $F(x)$ be the indefinite integral of an integrable and periodic $f(x)$, then at every point at which $F'(x_0) = f(x_0)$ exists and is finite (in particular amost every where), the Poission type integral $f_\lambda(r, x)$ of $f(x)$ tends to $f(x_0)$ as $r \to 1$ at the point x_0 non-tangentially.

Theorem (*3.7*). If $F(x)$ is a periodic and integrable function, the difference

$$\frac{\partial \widetilde{F}_\lambda(\pi, x)}{\partial x} - \left(-\frac{1}{\pi} \int_\delta^\pi \frac{F(x+t) + F(x-t) - 2F(x)}{4 \sin^2 \frac{t}{2}} dt \right) \tag{3.2}$$

tends to zero as $r \to 1$ for every x for which $F(x)$ is smooth ie. at which

$$F(x+t) + F(x-t) - 2F(x) = o(t) . \tag{3.3}$$

Theorem (*3.8*). * If $f(x)$ is integrable and $F(x)$ the indefinite integral of f then, for $1 - r = \delta$,

$$\widetilde{f}_\lambda(\pi, x) - \left(-\frac{1}{\pi} \int_\delta^\pi [f(x+t) - f(x-t)] \frac{1}{2} \cot \frac{t}{2} dt \right) \to 0 \text{ as } \pi \to 1 \tag{3.4}$$

at every point at which $F(x)$ is smooth, inparticular where $F(x)$ is continuous. If $f(x)$ is everywhere continuous, the convergence is uniform.

Proof of this theorem runs parallel to the Theorem (7.20) (Zygmund [*4*], Chapter III).

Theorem (*3.9*). If the conjugate series converges at a point where $\widetilde{f}(x)$ exists, then it is summable (A) to the sum $\widetilde{f}'(x)$.

The proof of this theorem is almost trivial.

The cases $\lambda = 0$ of Theorems (3.1)-(3.9) are given in (Zygmund [*4*] Chapter III). Theorems (3.1)-(3.6) are concerned with the Fourier series of $f(x)$ and Theorems (3.7)-(3.9) for the conjugate series to the Fourier series of $f(x)$.

(*) A similar theorem stands in ([*5*], Theorem 76) for $\lambda = 0$.

4. In this section we give the proofs

Proofs of Theorems (3.1) and (3.2) are trivial. To prove Theorem (3.3), we require the following lemma :

Lemma 1. If a positive kernel K_n satisfies condition (C), then for any sequence $\{x_n\} \to 0$, we have

$$m(x_0) \leqslant \lim\inf. F_\lambda(\pi, x) \leqslant \lim\sup. F_\lambda(\pi, x) \leqslant M(x_0),$$

where $m(x_0)$ and $M(x_0)$ are respectively minimum and maximum of $F(x)$ at the point x_0 and $F_\lambda(r, x)$ is the Abelmean of $S[F]$.

The proof of this lemma is analogous to that of Theorem 2.30 (Zygmund [*1*], p. 88).

Consider now the proof of Theorem (3.3). If $F_\lambda(r, x)$ is the poisson type integral of $F(x)$, we have

$$\left[\frac{\partial}{\partial x} F_\lambda(\pi, x)\right]_{x = x_0} = (1 - \pi)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda r^n \sum_{v=1}^{n} v (B_v \cos v x_0 - A_v \sin v x_0)$$

$$= (1 - \pi)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda r^n \frac{1}{\pi} \int_{-\pi}^{\pi} F(x_0 + t) \sum_{v=1}^{n} v \operatorname{siv} v t \, dt$$

$$= -(1 - \pi)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda r^n \frac{1}{\pi} \int_{\pi}^{\pi} \psi(t) \left[\frac{d}{dt}\left\{\frac{1}{2} + \sum_{v=1}^{n} \cos v t\right\}\right] dt$$

$$= -(1 - \pi)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda \pi^n \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t) \cdot \tag{4.1}$$

$$\cdot \left[\frac{\left(n + \frac{1}{2}\right) \cos\left(n + \frac{1}{2}\right) t \sin\frac{t}{2} - \frac{1}{2} \sin\left(n + \frac{1}{2}\right) t \cos\frac{t}{2}}{\sin^2\frac{t}{2}}\right] dt,$$

where

$$\psi(t) = \frac{1}{2} \{F(x_0 + t) - F(x_0 - t)\} .$$

For evaluation of integral, we have

$$\sum_{n=0}^{\infty} n\, e_n^{\lambda} \pi^n \cos\left(n+\frac{1}{2}\right) t = \pi(\lambda+1) \sum_{n=0}^{\infty} e_{n-1}^{\lambda+1} \pi^{n-1} \cos\left(n+\frac{1}{2}\right) t$$

$$= \pi(\lambda+1)\, R_{\psi}\left[e^{\frac{3it}{2}} (1-\pi e^{it})^{-(\lambda+2)}\right]$$

$$= \pi(\lambda+1)\, \pi_1^{-(\lambda+2)} \cos\left\{(\lambda+2)\theta + \frac{3t}{2}\right\},$$

where R_{ψ} stands for real part of the expression under consideration and

$$\pi_1 = \sqrt{(1+\pi^2 2\pi \cos t)}\ ,$$

$$\theta = \tan^{-1} \frac{\pi \sin t}{1-\pi\cos t}\ .$$

Similarly

$$\sum_{n=0}^{\infty} e_n^{\lambda} \pi^n \cos\left(n+\frac{1}{2}\right) t = \pi_1^{-(\lambda+1)} \cos\left\{(\lambda+1)\theta + \frac{t}{2}\right\}$$

and

$$\sum_{n=0}^{\infty} e_n^{\lambda} \pi^n \sin\left(n+\frac{1}{2}\right) t = \pi_1^{-(\lambda+1)} \sin\left\{(\lambda+1)\theta + \frac{t}{2}\right\}.$$

By these equations, (4.1) reduces to

$$\left[\frac{\partial}{\partial x} F_{\lambda}(\pi, x)\right]_{x=x_0} = -\frac{(1-\pi)^{\lambda+1}}{\pi_1^{\lambda+2}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\psi(t)}{\sin^2 \frac{t}{2}} \left[\pi(\lambda+1)\sin\frac{t}{2}\right.$$

$$\left.\cdot \cos\left\{(\lambda+2)\theta + \frac{3t}{2}\right\} - \frac{1}{2}\pi_1 \sin(\lambda+1)\theta\right] dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\, K_{\lambda}(\pi, t)\, dt\ , \tag{4.2}$$

where

$$g(t) = \frac{\psi(t)}{\sin t}\ , \text{ and}$$

$$K_{\lambda}(\pi, t) = -\frac{(1-\pi)^{\lambda+1}\cos\frac{t}{2}}{\pi_1^{\lambda+2}\sin\frac{t}{2}} \left[\pi(\lambda+1)\sin\frac{t}{2}\cos\left\{(\lambda+2)\theta + \frac{3t}{2}\right\}\right.$$

$$\left. - \frac{1}{2}\pi_1 \sin(\lambda+1)\theta\right]. \tag{4.3}$$

For ordinary Abel method, where $\lambda = 0$, we have

$$K_0(\pi, t) = K(\pi, t) = \frac{\pi(1-\pi^2)\sin^2 t}{\pi_1^4}. \tag{4.4}$$

Kernel $K_\lambda(\pi, t)$ of (4.2) satisfies conditions (A), (B) and (C) of Section 2. To prove $\frac{1}{\pi}\int_{-\pi}^{\pi} K_\lambda(\pi, t)\, dt = 1$, we suppose in particular that $F(x) = \sin x$, $x_0 = 0$. Hence $F_\lambda(\pi, x) = \sin x$ and (4.2) proves (A). Conditions (B) and (C) are trivial.

We divide now the proof into two parts:

Case 1: When $g(t)$ is integrable (L).

We see that minimum and maximum of $g(t)$ at the point $t = 0$ are nothing but $\underline{D}_1 F(x_0)$ and $\overline{D}_1 F(x_0)$ and kernel $K_\lambda(\pi, t)$ satisfies condition (C); applying Lemma, we have

$$\underline{D}_1 F(x_0) \leqslant \liminf \sum_{\nu=1}^{\infty} (1-\pi)^{\lambda+1}\, \nu\, (B_\nu \cos \nu x_0 - A_\nu \sin \nu x_0)\, \pi^\nu\, e_\nu^\lambda$$

$$\leqslant \limsup \sum_{\nu=1}^{\infty} (1-\pi)^{\lambda+1}\, \nu\, (B_\nu \cos \nu x_0 - A_\nu \sin \nu x_0)$$

$$\cdot\, r^\nu e_\nu^\lambda \leqslant \overline{D}_1 F(x_0) .$$

Or,

$$\lim_{\pi \to 1} \sum_{\nu=1}^{\infty} \nu\, (B_\nu \cos \nu x_0 - A_\nu \sin \nu x_0)\, (1-\pi)^{\lambda+1}\, \pi^\nu\, e_\nu^\lambda$$

$$= D_1 F(x_0) \qquad\qquad (g(t) \in L) ,$$

Which establishes the theorem for the case 1.

Proof of Case 2: When $g(t)$ is not integrable (L), the proof runs parallel to that of theorem 7.2 in (Zygmund [*1*] Chapter III) and therefore we omit it. This completes the proof of case 2.

Thus we have completely proved Theorem (3.3).

Proof of Theorem (3.5). Since $F(x) = \int_0^x f(t)\, dt$, let the constant term of $S[f]$ is zero, we have

$$S[f] = S'[F]$$

and then using Theorem (3.3) and Theorem (3.4) we have this theorem.

Proof of Theorem (3.6). Here

$$f_\lambda(\pi, x) = \frac{\partial}{\partial x} F_\lambda(\pi, x) \text{ as } (\pi, x) \to (1, x_0)$$

nontangentially. But Theorem (3.4) gives

$$\frac{\partial}{\partial x} F'_\lambda(\pi, x) \to F'(x_0)$$

ie.

$$f_\lambda(\pi, x) \to f(x_0)$$

as $(r, x) \to (1, x_0)$ nontangentially.

This completes the proof.

Proof of Theorem (3.7). For given $\widetilde{F}_\lambda(\pi, x)$, we have

$$\frac{\partial}{\partial x} \widetilde{F}_\lambda(\pi, x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \frac{\partial}{\partial x} Q_\lambda(\pi, t - x)\, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} \{F(x+t) + F(x-t) - 2F(x)\} Q'_\lambda(\pi, t)\, dt.$$

Where

$$Q'_\lambda(\pi, t) = \frac{d}{dt} \left\{ (1-\pi)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda \pi^n \frac{\cos \frac{t}{2} - \cos\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}} \right\}$$

$$= \frac{(1-\pi)^{\lambda+1}}{2 \sin^2 \frac{t}{2}} \sum_{n=0}^{\infty} \epsilon_n^\lambda \pi^n \left\{ -\frac{1}{2} + \frac{1}{2} \cos n t + n \sin\left(n + \frac{1}{2}\right) t \,.\, \sin \frac{t}{2} \right\}.$$

The fact $\sum_{n=0}^{\infty} \epsilon_n^\lambda \pi^n = (1-\pi)^{-(\lambda+1)}$, $(\lambda > -1)$, gives

$$\sum_{n=0}^{\infty} \epsilon_n^\lambda \pi^n \cos n t = R_\varphi\{(1 - \pi e^{it})^{-(\lambda+1)}\}$$

$$= \pi_1^{-(\lambda+1)} \cos(\lambda+1)\theta$$

and

$$\sum_{n=0}^{\infty} \epsilon_n^\lambda \pi^n n \sin\left(n + \frac{1}{2}\right) t = \pi(\lambda+1) I_\varphi\left[e^{\frac{3it}{2}} (1 - \pi e^{it})^{-(\lambda+2)}\right]$$

$$= \pi(\lambda+1)\pi_1^{-(\lambda+2)} \sin\left\{(\lambda+2)\theta + \frac{3t}{2}\right\}.$$

With the above relations, we have

$$(4.6) \qquad Q'_\lambda(\pi, t) = \frac{(1-\pi)^{\lambda+1}}{\pi_1^{\lambda+2}\, 4\sin^2\frac{t}{2}} \Big[\pi_1 \cos(\lambda+1)\theta - \frac{\pi_1^{\lambda+2}}{(1-\pi)^{\lambda+1}}$$

$$+ 2\pi(\lambda+1)\sin\Big\{(\lambda+2)\theta + \frac{3t}{2}\Big\}\sin\frac{t}{2}\Big] .$$

In particular

$$Q'_0(\pi, t) = Q'(\pi, t) = \frac{\pi\,[(1+\pi^2)\cos t - 2\pi]}{\pi_1^4}$$

and

$$Q'_\lambda(1, t) = Q'(1, t) = -\frac{1}{2(1-\cos t)}, \text{ (being independent of } \lambda).$$

We also have

$$| Q'_\lambda(\pi, t) | = O\left(\frac{1}{\delta^2}\right) .$$

By (3.3), $\xi(t) = o(t)$. Hence

$$(4.7) \qquad \frac{\partial}{\partial x} \widetilde{F}_\lambda(\pi, x) = \frac{1}{\pi}\int_0^\delta \xi(t)\, Q'_\lambda(\pi, t)\, dt + \frac{1}{\pi}\int_\delta^\pi \xi(t)\, Q'_\lambda(\pi, t)\, dt$$

$$= A + B ,$$

say ; where

$$(4.8) \qquad | A | = \frac{1}{\pi}\int_0^\delta | \xi(t) |\, O\left(\frac{1}{\delta^2}\right) dt = o(1)$$

and

$$(4.9) \qquad B = \frac{1}{\pi}\int_\delta^\pi \xi(t)\,\{Q'_\lambda(\pi, t) - Q'_\lambda(1, t)\}\, dt + \frac{1}{\pi}\int_\delta^\pi \xi(t)\, Q'_\lambda(1, t)\, dt$$

$$= B_1 + B_2 ,$$

say. Here

$$B_1 = \frac{1}{\pi}\int_\delta^\pi \xi(t)\,[Q'_\lambda(\pi, t) - Q'_\lambda(1, t)]\, dt$$

and

$$B_2 = \frac{1}{\pi}\int_\delta^\pi \xi(t)\, Q'_\lambda(1, t)\, dt .$$

It is easy to verify that

$$(4.10)\qquad Q'_\lambda(\pi, t) - Q'_\lambda(1, t) = \frac{(1-\pi)^{\lambda+1}}{4\pi_1^{\lambda+2}\sin^2\frac{t}{2}}\Big[\pi_1\cos(\lambda+1)\theta + 2\pi(\lambda+1)$$

$$\cdot \sin\frac{t}{2}\sin\Big\{(\lambda+2)\theta + \frac{3t}{2}\Big\}\Big]$$

$$= \frac{(1-\pi)^{\lambda+1}}{4\pi_1^{\lambda+2}\sin^2\frac{t}{2}}\Big[(1-\pi)\cos\lambda\theta - 2\sin\Big(\lambda\theta - \frac{t}{2}\Big)\sin\frac{t}{2}$$

$$+ 2\pi(\lambda+1)\sin\frac{t}{2}\sin\Big\{(\lambda+2)\theta + \frac{3t}{2}\Big\}\Big].$$

By virtue of

$$\cos\Big(2\theta + \frac{3t}{2}\Big) = \frac{\cos\frac{t}{2}}{\pi_1^2}\Big[4\cos^2\frac{t}{2} - 3 + \pi^2 - 2\pi\Big]$$

and

$$\sin\Big(2\theta + \frac{3t}{2}\Big) = \frac{\sin\frac{t}{2}}{\pi_1^2}\Big[4\cos^2\frac{t}{2} - (1+\pi)^2\Big],$$

it is easy to see that

$$\sin\Big\{(\lambda+2)\theta + \frac{3t}{2}\Big\} = O\Big(\frac{\delta}{t}\Big).$$

Similarly

$$\sin\Big(\lambda\theta - \frac{t}{2}\Big) = O\Big(\frac{\delta}{t}\Big).$$

Hence from (4.10), we have

$$Q'_\lambda(\pi, t) - Q'_\lambda(1, t) = O\Big(\frac{\delta^{\lambda+2}}{t^{\lambda+4}}\Big).$$

Thus,

$$|B_1| \leqslant \delta^{\lambda+2}\int_\delta^\pi o(t^{-\lambda-3})\,dt = o(1),$$

and from (4.7), (4.8) and (4.9), we have

$$\frac{\partial}{\partial x}\widetilde{F}_\lambda(\pi, x) - \Big(-\frac{1}{\pi}\int_\delta^\pi \frac{F(x+t) + F(x-t) - 2F(x)}{4\sin^2\frac{t}{2}}\,dt\Big) \longrightarrow 0$$

as $r \longrightarrow 1$. This completes the proof of Theorem (3.7).

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