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On the method of successive approximations for functional differential equations of retarded type (**)

1. - Introduction

In this paper we are dealing with functional differential equations of retarded type:

$$x'(t) = f(t, x_t) \tag{1}$$

in a Banach space. We shall prove by the method of successive approximations an existence and uniqueness theorem for (1), analogous to the uniqueness theorem of Bompiani for ordinary differential equations (see Bompiani [1], Walter [4]). We shall also study the global existence, boundedness and stability of solutions of (1) by estimating their distance from and dependence on initial mappings by *minimal* solutions of suitable comparison problems.

2. - Notations and preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and let $\tau > 0$ be given. Denote by C the Banach space of all continuous mappings φ from $[-\tau, 0]$ into X with norm given by

$$\|\varphi\|_0 = \max_{-\tau \leq s \leq 0} \|\varphi(s)\| .$$

Let I be a real interval with t_0 as the left end point and let A be an open subset of C.

Given a continuous mapping f from $I \times A$ into X, consider the functional differential equation (1) where x' denotes the strong derivative of a mapping x, and x_t the mapping $s \to x(t+s)$; $-\tau \leq s \leq 0$. In view of the definition (see for ex. Ladas-Lakshmikantham [2] p. 185) a mapping x is a solution of (1) on an

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(**) Memoria presentata dall'Accademico E. Bompiani il 4-4-1974.

interval $[t_0 - \tau, t_0 + \alpha)$; $\alpha > 0$, with the given initial mapping $\varphi_0 \in A$, i.e. with $x_{t_0} = \varphi_0$, if and only if x is a fixed point of the operator T defined in the set

$$B = \{x : [t_0 - \tau, t_0 + \alpha) \to X \mid x_t \in A \quad \text{for} \quad t_0 \le t < t_0 + \alpha\}$$

by

$$(2.1) \qquad T\,x(t) = \begin{cases} \varphi_0(t - t_0), & t_0 - \tau \le t \le t_0 \\ \varphi_0(0) + \int_{t_0}^{t} f(s, x_s)\,ds, & t_0 < t < t_0 + \alpha . \end{cases}$$

In the following we use the notation $T\,x_t$ for $(Tx)_t$.

Let $\bar{T}$ be another operator defined in B by (2.1) with $\bar{\varphi}_0 \in A$ as the initial mapping. If $x, \bar{x} \in B$ and $t \in [t_0, t_0 + \alpha)$ we have

$$\| T\,x_t - \bar{T}\,\bar{x}_t \|_0 = \max_{-\tau \le z \le 0} \| T\,x(t+z) - \bar{T}\,\bar{x}(t+z) \|$$

$$\le \| \varphi_0 - \bar{\varphi}_0 \|_0 + \max_{-\tau \le z \le 0} \int_{t_0}^{\max\{t+z, t_0\}} \| f(s, x_s) - f(s, \bar{x}_s) \|\,ds$$

$$= \| \varphi_0 - \bar{\varphi}_0 \|_0 + \int_{t_0}^{t} \| f(s, x_s) - f(s, \bar{x}_s) \|\,ds ,$$

so that

$$(2.2) \qquad \| T\,x_t - \bar{T}\,\bar{x}_t \|_0 \le \| \varphi_0 - \bar{\varphi}_0 \|_0 + \int_{t_0}^{t} \| f(s, x_s) - f(s, \bar{x}_s) \|\,ds$$

whenever $x, \bar{x} \in B$ and $t_0 \le t < t_0 + \alpha$.

Analogous reasoning shows that, if the mapping $\varphi^0 : [t_0 - \tau, \infty) \to X$ is defined by

$$(2.3) \qquad \varphi^0_{t_0} = \varphi_0 \quad \text{and} \quad \varphi^0(t) = \varphi_0(0) \quad \text{for} \quad t > t_0 ,$$

we have for all $x \in B$ and $t \in [t_0, t_0 + \alpha)$

$$(2.4) \qquad \| T\,x_t - \varphi^0_t \|_0 \le \int_{t_0}^{t} \| f(s, x_s) \|\,ds .$$

Let N denote the set of all natural numbers $1, 2, \ldots$ and R_+ the set of all nonnegative reals. Denote moreover by E the set of all such continuous functions g from $I \times R_+$ into R_+ that $g(t, r)$ is nondecreasing in r for each $t \in I$.

The following result shall be applied for deriving various estimates.

LEMMA 2.1. — Let $g \in E$ be such that the initial value problem

(2.5) $$v'(t) = g(t, v(t)), \quad v(t_0) = r_0$$

has for each $r_0 \in R_+$ a solution on I. Then for each continuous function $u_0 : I \to R_+$ and for each $r_0 \geq 0$ the initial value problem

(2.6) $$u'(t) = u_0(t) + g(t, u(t)), \quad u(t_0) = r_0$$

has a solution on I, and the minimal solution u of (2.6) is obtained as the limit of the sequence (u_n) defined inductively by

$$u_1(t) = r_0 + \int_{t_0}^{t} u_0(s)\, ds, \quad t \in I$$

and

$$u_{n+1}(t) = u_1(t) + \int_{t_0}^{t} g(s, u_n(s))\, ds, \quad t \in I, n \in N.$$

PROOF. — Let $J = [t_0, t_0 + a]$; $a > 0$ be a closed interval contained in I. The sequence (u_n) is nondecreasing on J, i.e.

$$u_n(t) \leq u_{n+1}(t) \quad \text{for all} \quad t \in J \text{ and } n \in N.$$

Moreover, it is bounded above by any solution of (2.5) with $r_0 = u_1(t_0 + a)$. Thus the sequence (u_n) converges on J. By the hypotheses given for g it is easy to show that the convergence is uniform on J and that the limit mapping u is a solution of (2.6). By induction one can verify that, given any solution $\bar{u}$ of (2.6) on J, then

$$u_n(t) \leq \bar{u}(t)$$

for all $t \in J$ and $n \in N$, which proves the minimality of u.

3. - ON THE EXISTENCE AND UNIQUENESS

Applying the notations and results of previous section we shall prove

THEOREM 3.1. — Let f be a continuous mapping from $I \times A$ into X. Suppose that for all $(t, \varphi), (t, \bar{\varphi}) \in I \times A$

(3.1) $$\| f(t, \varphi) - f(t, \bar{\varphi}) \| \leq g(t, \| \varphi - \bar{\varphi} \|_0)$$

where $g \in E$ such that the initial value problem (2.5) has for each $r_0 > 0$ a solution on I, and $v(t) \equiv 0$ is the only solution of (2.5) for $r_0 = 0$. Then for each $\varphi_0 \in A$ there is $\alpha > 0$ such that the functional differential equation (1) has on $[t_0 - \tau, t_0 + \alpha)$ a unique solution with φ_0 as the initial mapping.

PROOF. (¹) — Let $\varphi_0 \in A$ be given. To find the number α, choose $a > 0$ such that the interval $J = [t_0, t_0 + a]$ is contained in I, and that $\varphi_t^0 \in A$ for all $t \in J$. The set $\{\varphi_t^0 \mid t \in J\}$ is a compact subset of A and the complement A^c of A in C is closed, so that

$$d = \inf \{ \| \varphi - \varphi_t^0 \|_0 \mid \varphi \in A^c, t \in I \}$$

is positive (or ∞ when $A = C$). Define

$$\alpha = \sup \{ z \in [0, a] \mid u(t_0 + z) < d \} \tag{3.2}$$

where u is the minimal solution of the initial value problem

$$u'(t) = \| f(t, \varphi_t^0) \| + g(t, u(t)), \quad u(t_0) = 0 \; . \tag{3.3}$$

By Lemma 2.1 this solution u is obtained as the uniform limit of the nondecreasing sequence (u_n) defined on J inductively by

$$u_1(t) = \int_{t_0}^{t} \| f(s, \varphi_s^0) \| \, d s, \quad t \in J$$

and

$$u_{n+1}(t) = u_1(t) + \int_{t_0}^{t} g(s, u_n(s)) \, d s, \quad t \in J, n \in N \; .$$

With α given by (3.2) we shall now show that the relations

$$\begin{cases} x^1(t) = \varphi^0(t), & t_0 - \tau \leq t < t_0 + \alpha \\ x^{n+1} = T x^n, & n \in N \end{cases} \tag{3.4}$$

define inductively a sequence (x^n) which converges on $[t_0 - \tau, t_0 + \alpha)$ to a fixed point of T. For $n = 1$ and $t \in [t_0, t_0 + \alpha)$ we have

$$\| x_t^n - \varphi_t^0 \|_0 \leq u_n(t) \; . \tag{3.5}$$

The constant α is so chosen that, given any $n \in N$ for which x^n exists and satisfies the inequality (3.5) for $t \in [t_0, t_0 + \alpha)$, then $x_t^n \in A$ for all $t \in [t_0, t_0 + \alpha)$, whence $x^n \in B$. From (2.4) and (3.1) we deduce that

$$\| T x_t^n - \varphi_t^0 \|_0 \leq \int_{t_0}^{t} \| f(s, \varphi_s^0) \| \, d s + \int_{t_0}^{t} g(s, \| x_s^n - \varphi_s^0 \|_0) \, d s$$

whenever $t_0 \leq t < t_0 + \alpha$. This inequality together with (3.5) and the definitions of the sequences (x^n) and (u_n) implies that

$$\| x_t^{n+1} - \varphi_t^0 \|_0 \leq u_{n+1}(t) \text{ for all } t \in [t_0, t_0 + \alpha) \; .$$

(¹) With S. SEIKKALA.

This shows by induction that for each $n \in N$ x^n exists, belongs to B and satisfies the inequality (3.5) for $t \in [t_0, t_0 + \alpha)$.

To prove the convergence of the sequence (x^n) on $[t_0 - \tau, t_0 + \alpha)$ define a sequence (v_n) inductively by

$$v_1 = u = \text{the minimal solution of (3.3)}$$

and

$$v_{n+1}(t) = \int_{t_0}^{t} g(s, v_n(s))\, ds, \quad t \in J, n \in N. \tag{3.6}$$

From (3.5) it then follows that the inequality

$$\| x_t^{n+m} - x_t^m \|_0 \leq v_m(t) \tag{3.7}$$

holds for $m = 1$, $n \in N$ and $t \in [t_0, t_0 + \alpha)$. From (2.2) with $\bar{\varphi}_0 = \varphi_0$ and from (3.1) we deduce that

$$\| T x_t^{n+m} - T x_t^m \|_0 \leq \int_{t_0}^{t} g(s, \| x_s^{n+m} - x_s^m \|_0)\, ds$$

whenever $m, n \in N$ and $t \in [t_0, t_0 + \alpha)$. Using this inequality one can then easily verify by induction that (3.7) holds for all $m, n \in N$ and $t \in [t_0, t_0 + \alpha)$. Particularly we have

$$\| x^{n+m}(t+s) - x^m(t+s) \| \leq \| x_t^{n+m} - x_t^m \|_0 \leq v_m(t_0 + \alpha) \tag{3.8}$$

for all $n, m \in N$, $s \in [-\tau, 0]$ and $t \in [t_0, t_0 + \alpha)$.

The sequence (v_n) of nondecreasing functions v_n is nonincreasing and bounded below by zero-function, whence it converges on J uniformly, and the limit function v is a solution of the initial value problem (2.5) with $r_0 = 0$. By a hypothesis $v(t) \equiv 0$, so that by (3.8) the sequence (x^n) converges uniformly on $[t_0 - \tau, t_0 + \alpha)$. Thus the limit mapping x is continuous. From (3.5) we deduce as $n \to \infty$, that for all $t \in [t_0, t_0 + \alpha)$

$$\| x_t - \varphi_t^0 \|_0 \leq u(t). \tag{3.9}$$

This, together with the definition (3.2) of α and the monotonicity of u implies that $x_t \in A$ for all $t \in [t_0, t_0 + \alpha)$, whence $x \in B$.

Using the continuity of f and the definitions of T and (x^n) it is easy to verify that x is a fixed point of T, and hence a solution of (1) on $[t_0 - \tau, t_0 + \alpha)$ with φ_0 as the initial mapping.

To prove the uniqueness suppose that $\bar{x}$ is another fixed point of T and let $b \in (0, \alpha)$ be given. Applying (2.2) with $\bar{\varphi}_0 = \varphi_0$ and (3.1) one can verify by induction that for all $n \in N$ and $t \in [t_0, t_0 + b]$

$$\| x_t - \bar{x}_t \|_0 \leq v_n(t) \tag{3.10}$$

where (v_n) is a sequence defined by (3.6) for $n \in N$ and $t \in [t_0, t_0 + b]$; v_1 being any solution of the initial value problem (2.5) with

$$r_0 = \sup \{ \| x_t - \bar{x}_t \|_0 \mid t_0 \leq t \leq t_0 + b \} .$$

As above, the sequence (v_n) is nondecreasing and converges on $[t_0, t_0 + b]$ to zero function so that by (3.10) $x_t = \bar{x}_t$ for all $t \in [t_0, t_0 + b]$. This holds for all $b \in (0, \alpha)$, which proves the uniqueness.

REMARKS. — The method used to prove the convergence of the sequence (x^n) and the uniqueness of the solution of (1) has been applied in WALTER [4] in the context of ordinary differential equations and Volterra integral equations. Another proof of uniqueness, without the monotonicity hypothesis for g, is given in LADAS-LAKSHMIKANTHAM [2] p. 188.

From the inequality (3.7) it follows as $n \to \infty$

$$\| x_t - x_t^m \|_0 \leq v_m(t) , \quad m \in N , t \in [t_0, t_0 + \alpha) , \tag{3.11}$$

which estimates the rapidity of the convergence $x_t^m \to x_t$.

Assume that the set A is of the form

$$A = \{ \varphi \in C \mid \mathrm{I\,m}\,\varphi \subset U \}$$

where U is an open proper subset of X. Then the constant α in Theorem 3.1 can be replaced by

$$\alpha = \sup \{ z \in R_+ \mid t_0 + z \in I \text{ and } u(t_0 + z) < d(\mathrm{I\,m}\,\varphi_0 , U^c) \} \tag{3.12}$$

where u is the minimal solution of the initial value problem (3.3) and $d(\mathrm{I\,m}\,\varphi_0, U^c)$ denotes the distance of $\mathrm{I\,m}\,\varphi_0$ from the complement U^c of U in X.

In case $A = C$ we obtain

COROLLARY 3.1 — Assume that the mapping $f: I \times C \to X$ satisfies the hypotheses of Theorem 3.1 with $A = C$. Then for each $\varphi_0 \in C$ the functional differential equation (1) has a unique solution with φ_0 as the initial mapping, on $[t_0 - \tau, t_0 + \alpha)$ where α = length of I.

4. - ON THE BOUNDEDNESS

By reasoning analogous to that used in the proof of the inequality (3.9) (p. 7) one can verify

THEOREM 4.1. — Assume that f is a mapping from $I \times A$ into X satisfying the hypotheses of Theorem 3.1. Let $\varphi_0 \in A$ be given and let φ^0 denote the mapping defined by (2.3) (p. 4). Suppose moreover that for all $(t, \varphi) \in I \times A$

$$\| f(t, \varphi) \| \leq h(t, \| \varphi - \varphi_t^0 \|_0) \tag{4.1}$$

where $h \in E$ is such that the initial value problem

$$u'(t) = h(t, u(t)), \quad u(t_0) = 0 \tag{4.2}$$

has a solution on I. Then there is $\alpha > 0$ such that for all $t \in [t_0, t_0 + \alpha)$

$$\| x_t - \varphi_t^0 \|_0 \leq u(t) \tag{4.3}$$

where x is the solution of (1) with $x_{t_0} = \varphi_0$ and u is the minimal solution of (4.2).
The inequality (3.9) is a special case of (4.3) with

$$h(t, r) = \| f(t, \varphi_t^0) \| + g(t, r), \quad (t, r) \in I \times R_+ .$$

If f is bounded on $I \times A$ we can take $h(t, r) = M = \sup_{I \times A} \| f(t, \varphi) \|$, in which case $u(t) = M \mid t - t_0 \mid$ is the unique solution of (4.2).
As a consequence of the inequality (4.3) we obtain

Corollary 4.1 — Assume that the hypotheses of Theorem 4.1 hold with $I = [t_0, b); t_0 < b \leq \infty$, and $A = C$, and that the minimal solution u of (4.2) is bounded on I. Then the solution x of the functional differential equation (1) with φ_0 as the initial mapping is bounded on $[t_0 - \tau, b)$, and $\lim_{t \to b_-} x(t)$ exists in X.

Proof. — In this case the inequality (4.3) holds for all $t \in I$, which together with the boundedness of u implies the boundedness of x. From (4.1) and (4.3) we deduce that

$$\| f(t, x_t) \| \leq h(t, u(t))$$

for all $t \in [t_0, b)$, so that for $t_0 \leq t_1 \leq t_2 < b$ we have

$$\| x(t_2) - x(t_1) \| \leq \int_{t_1}^{t_2} \| f(s, x_s) \| \, ds \leq \int_{t_1}^{t_2} h(s, u(s)) \, ds \tag{4.4}$$
$$= u(t_2) - u(t_1) .$$

The boundedness and monotonicity of u imply that $\lim_{t \to b_-} u(t)$ exists and is finite, so that (4.4) implies the existence of the limit $\lim_{t \to b_-} x(t)$ (cf. Ladas-Lakshmikantham [2] p. 163.

The inequality (4.3) estimates in a sense the distance of the solution of (1) from the initial mapping. Next we shall derive a similar estimate for the norm of the solution of (1).

Theorem 4.2 — Let $f : I \times C \to X$ satisfy the hypotheses of Theorem 3.1 with $I = [t_0, b); t_0 < b \leq \infty$, and $A = C$.
Suppose moreover that for each $(t, \varphi) \in I \times C$

$$\| f(t, \varphi) \| \leq q(t, \| \varphi \|_0) \tag{4.5}$$

where $q \in E$ and the initial value problem

$$w'(t) = q(t, w(t)), \quad w(t_0) = r_0 \tag{4.6}$$

has for each $r_0 \geq 0$ a solution on I. Then for each $\varphi_0 \in C$ the solution x of the functional differential equation (1) with $x_{t_0} = \varphi_0$ satisfies for all $t \in I$ the inequality

$$\| x_t \|_0 \leq w(t) \tag{4.7}$$

where w is the minimal solution of the initial value problem (4.6) with $r_0 = \| \varphi_0 \|_0$. If this solution w is bounded on I, then the solution x is bounded on $[t_0 - \tau, b)$ and $\lim_{t \to b_-} x(t)$ exists in X.

Proof. — Let $\varphi_0 \in C$ be given and let T be the operator defined by (2.1) (p. 4) with $t_0 + \alpha = b$. If (x^n) is the sequence defined by (3.4) (p. 6), it is easy to see that

$$\| T x_t^n \|_0 \leq \| \varphi_0 \|_0 + \int_{t_0}^{t} q(s, \| x_s^n \|_0) \, ds . \tag{4.8}$$

The sequence (w_n), defined inductively by $w_1(t) \equiv \| \varphi_0 \|_0$ and

$$w_{n+1}(t) = \| \varphi_0 \|_0 + \int_{t_0}^{t} q(s, w_n(s)) \, ds , \quad t \in I, n \in N ,$$

converges by Lemma 2.1 on I to the minmal solution w of (4.6) with $r_0 = \| \varphi_0 \|_0$. Using (4.8) one can show by induction that

$$\| x_t^n \|_0 \leq w_n(t) \tag{4.9}$$

for all $t \in I$ and $n \in N$. The inequality (4.7) is then obtained from (4.9) as $n \to \infty$.

From (4.5) and (4.7) it follows (cf. the proof of (4.4)) that

$$\| x(t_2) - x(t_1) \| \leq w(t_2) - w(t_1) \tag{4.10}$$

whenever $t_0 \leq t_1 \leq t_2 < b$. Hence, if the solution w is bounded, then the inequality (4.7) implies the boundedness of x and the inequality (4.10) the existence of $\lim_{t \to b_-} x(t)$.

5. - Dependence on the initial mapping

As another consequence from the use of the method of successive approximations we obtain.

Theorem 5.1. — Assume that the hypotheses of Theorem 3.1 hold, and let x and $\bar{x}$ denote the solutions of (1) with the given $\varphi_0, \bar{\varphi}_0 \in A$ as the initial mappings, respectively. Then there is $\alpha > 0$ such that for all $t \in [t_0, t_0 + \alpha)$

$$\| x_t - \bar{x}_t \|_0 \leq v(t) \tag{5.1}$$

where v is the minimal solution of (2.5) with $r_0 = \| \varphi_0 - \bar{\varphi}_0 \|_0$.

Moreover, the solution x of (1) depends continuously on the initial mapping φ_0.

PROOF. — Let T and $\bar{T}$ be the operators defined by (2.1) (p. 4) with φ_0 and $\bar{\varphi}_0$ as the initial mappings, respectively, and let (x^n) and $(\bar{x}^n)$ be the corresponding sequences given by (3.4) (p. 6). By Theorem 3.1 there is $\alpha > 0$ such that these sequences converge on $[t_0 - \tau, t_0 + \alpha)$ uniformly to the solutions x and $\bar{x}$ of (1), respectively. From the inequalities (2.2) and (3.1) we deduce that for all $t \in [t_0, t_0 + \alpha)$

$$\| T x_t^n - \bar{T} \bar{x}_t^n \|_0 \leq \| \varphi_0 - \bar{\varphi}_0 \|_0 + \int_{t_0}^{t} g(s, \| x_s^n - \bar{x}_s^n \|_0)\, ds . \tag{5.2}$$

Define the sequence (v_n) inductively by $v_1(t) = \| \varphi_0 - \bar{\varphi}_0 \|_0$ and

$$v_{n+1}(t) = \| \varphi_0 - \bar{\varphi}_0 \|_0 + \int_{t_0}^{t} g(s, v_n(s))\, ds , \quad t \in I, n \in N . \tag{5.3}$$

From (5.2) and (5.3) it follows by induction that

$$\| x_t^n - \bar{x}_t^n \|_0 \leq v_n(t) \tag{5.4}$$

for all $n \in N$ and $t \in [t_0, t_0 + \alpha)$. The inequality (5.1) follows then by Theoren. 3.1 and by Lemma 2.1 from (5.4) as $n \to \infty$.

The continuity assertion is an immediate consequence of the inequality (5.1), v depends continuously on the initial value r_0 at $r_0 = 0$.

From the definition (2.1) of the operator T we see that if there is $\varphi_0 \in A$ such that $f(t, \varphi_t^0) = 0$ for all $t \in I$, then the mapping $x = \varphi^0$ is a fixed point of T, i. e. the functional differential equation (1) has a trivial solution

$$x(t) = \begin{cases} \varphi_0(t - t_0), & t_0 - \tau \leq t \leq t_0 \\ \varphi_0(0), & t > t_0 . \end{cases} \tag{5.5}$$

From (5.1) one can derive various stability criteria for this trivial solution, for example the following:

THEOREM 5.2. — Let $f : R_+ \times C \to X$ satisfy the hypotheses of Theorem 3.1 with $I = R_+$ and $A = C$. If there is $\varphi_0 \in C$ such that $f(t, \varphi_t^0) = 0$ for all $t \in R_+$, then the functional differential equation (1) has for each $t_0 \in R_+$ the trivial solution x given by (5.5) as the only solution with $x_{t_0} = \varphi_0$. If, moreover, the minimal solution $t \to v(t, t_0, r_0)$ of (2.5) has the property

$$\lim_{r_0 \to 0+} \sup \{ v(t, t_0, r_0) \mid t_0 \geq 0, t \geq t_0 \} = 0 , \tag{5.6}$$

then the trivial solution of (1) is uniformly stable in the sense that

$$\lim_{\bar{\varphi}_0 \to \varphi_0} \sup \{ \| \varphi_0(0) - x(t, t_0, \bar{\varphi}_0) \| \mid t_0 \geq 0, \ t \geq t_0 \} = 0 \tag{5.7}$$

where $t \to x(t, t_0, \bar{\varphi}_0)$ denotes the solution of (1) with $x_{t_0} = \bar{\varphi}_0$.

6. - Special cases

We shall first consider the case f satisfies an Osgood-condition.

Theorem 6.1. — Let f be a continuous mapping from $I \times A$ into X, and assume that for all $(t, \varphi), (t, \bar{\varphi}) \in I \times A$

$$\| f(t, \varphi) - f(t, \bar{\varphi}) \| \leq p(t) \psi(\| \varphi - \bar{\varphi} \|_0) \tag{6.1}$$

where p is a continuous function from I into R_+ and ψ is a continuous and nondecreasing function from R_+ into R_+ for which the integrals $\int_0^1 \frac{dr}{\psi(r)}$ and $\int_1^\infty \frac{dr}{\psi(r)}$ diverge.

Then for each $\varphi_0 \in A$ there is $\alpha > 0$ such that the functional differential equation (1) has on $[t_0 - \tau, t_0 + \alpha)$ a unique solution with φ_0 as the initial mapping, and this solution depends continuously on φ_0.

Proof. — The initial value problem

$$v'(t) = p(t) \psi(v(t)), \quad v(t_0) = r_0 \tag{6.2}$$

has for each $r_0 \geq 0$ a unique solution v on I which vanishes identically when $r_0 = 0$, and which for $r_0 > 0$ satisfies the integral equation

$$\int_{r_0}^{v(t)} \frac{dr}{\psi(r)} = \int_{t_0}^{t} p(s)\, ds, \quad t \in I. \tag{6.3}$$

This shows that the hypotheses of Theorem 3.1 hold with

$$g(t, r) = p(t) \psi(r),$$

and thus proves the existence and uniqueness of the solution x of (1) with the given initial mapping $\varphi_0 \in A$.

The continuous dependence of x on φ_0 follows from Theorem 5.1, because by (6.3) the solution v depends continuously on the initial value r_0.

Theorem 6.2. — Assume that $I = R_+$ and $A = C$ in Theorem 6.1, and that the integral $\int_0^\infty p(s)\, ds$ converges. If there is $\varphi_0 \in C$ such that $f(t, \varphi_t^0) = 0$ for all $t \in R_+$, then the trivial solution $x = \varphi^0$ of (1) is uniformly stable. If the integral $\int_{t_0}^\infty \| f(s, \varphi_s^0) \|\, ds$ converges for some $(t_0, \varphi_0) \in R_+ \times C$, then the solution x of (1) with $x_{t_0} = \varphi_0$ is bounded on $[t_0 - \tau, \infty)$.

PROOF. — From the equation (6.3) it follows by the convergence of the integral $\int_0^\infty p(s)\,ds$, the divergence of the integral $\int_0^1 \frac{dr}{\psi(r)}$ and the monotonicity of ψ that the solution $v = v(t, t_0, r_0)$ of the initial value problem (6.2) satisfies the condition (5.6) (p. 11). Also the other hypotheses of Theorem 5.2. hold with $g(t, r) = p(t)\psi(r)$, which proves the stability assertion.

From the convergence of the integral $\beta = \int_{t_0}^\infty \| f(s, \varphi_s^0) \| \, ds$ it follows that the solution u of the initial value problem

(6.4) $$u'(t) = \| f(t, \varphi_t^0) \| + p(t)\psi(u(t)), \quad u(t_0) = 0$$

can be bounded above by the solution of the initial value problem (6.2) with $r_0 = \beta$. But this solution is bounded above on $[t_0, \infty)$ by the constant c given by

$$\int_\beta^c \frac{dr}{\psi(r)} = \int_{t_0}^\infty p(s)\,ds \, ,$$

so that also u is bounded on $[t_0, \infty)$. The boundedness of the solution x of (1) with $x_{t_0} = \varphi_0$ follows then from the inequality (3.9), where now u is the solution of (6.4).

The hypotheses of Theorem 6.1 hold particularly when f is a continuous mapping from $I \times A$ into X satisfying for all $(t, \varphi), (t, \bar{\varphi}) \in I \times A$ a Lipschitz-condition

(6.5) $$\| f(t, \varphi) - f(t, \bar{\varphi}) \| \leq p(t) \| \varphi - \bar{\varphi} \|_0$$

where p is a continuous function from I into R_+. In this case we can take

$$g(t, r) = p(t)\, r$$

in the previous considerations. For example, the inequality (3.9) (p. 7) takes the form

$$\| x_t - \varphi_t^0 \|_0 \leq \int_{t_0}^t \left(\| f(s, \varphi_s^0) \| \exp\left(\int_s^t p(z)\,dz \right) \right) ds \, ,$$

and the inequality (5.1) (p. 10) the form

$$\| x_t - \bar{x}_t \|_0 \leq \| \varphi_0 - \bar{\varphi}_0 \|_0 \exp\left(\int_{t_0}^t p(s)\,ds \right) .$$

If, moreover,

$$K = \sup_{s \in I} p(s) < \infty \quad \text{and} \quad M = \sup_{s \in I} \| f(s, \varphi_s^0) \| < \infty \, ,$$

it follows from (3.11) an estimate

$$\| x_t - x_t^m \|_0 \leq \frac{M}{K} \exp(K(t - t_0)) \frac{(K(t - t_0))^m}{m!}$$

for the rapidity of the convergence of the successive approximations given by (3.4) (p. 6).

In case $\tau = 0$ the equation (1) is reduced to the ordinary differential equation

$$(1)' \qquad x'(t) = f(t, x(t)),$$

and the initial condition $x_{t_0} = \varphi_0$ to

$$x(t_0) = x_0 \; (= \varphi_0(0)).$$

The estimates (4.3), (4.7) and (5.1) turn into

$$(4.3)' \qquad \| x(t) - x_0 \| \leq u(t),$$

$$(4.7)' \qquad \| x(t) \| \leq w(t) \text{ and}$$

$$(5.1)' \qquad \| x(t) - \bar{x}(t) \| \leq v(t)$$

where u, w and v are the minimal solutions of (4.2), of (4.6) with $r_0 = \| x_0 \|$ and of (2.5) with $r_0 = \| x_0 - \bar{x}_0 \|$, respectively.

The use of Dini derivatives yields the same inequalities with corresponding maximal solutions as estimators (see for ex. LAKSHMIKANTHAM-LEELA [3], Chapters 2 and 6).

To study differences between these minimal and maximal solutions, define for given $r_1 > 0$ a function g by

$$(6.6) \qquad g(t, r) = \begin{cases} r, & 0 \leq r \leq r_1 e^{-t}, t \geq 0, \\ r_1 e^{-t}, & r_1 e^{-t} \leq r \leq r_1 (2 - e^{-t}), t \geq 0, \\ r_1 e^{-t} + \sqrt{r - r_1 (2 - e^{-t})}, & r \geq r_1 (2 - e^{-t}), t \geq 0. \end{cases}$$

One can see immediately that $g \in E$ and that (2.5) has $v(t) \equiv 0$ as the only solution with $r_0 = 0$, $t_0 \geq 0$. Let now $t_0 \geq 0$ and $r_0 > 0$ be given. If we choose

$$r_1 = r_0 (2 - e^{-t_0})^{-1}$$

in (6.6), then

$$v_*(t) = r_1 (2 - e^{-t}) \text{ and } v^*(t) = r_1 (2 - e^{-t}) + \frac{(t - t_0)^2}{4}$$

are the minimal and maximal solutions of (2.5), respectively. If g is replaced by q and v by w, we obtain similar results for the initial value problem (4.6). Moreover, the equation

$$h(t, r) = g(t, r + r_0)$$

defines a function $h \in E$ such that

$$u_*(t) = v_*(t) - r_0 \text{ and } u^*(t) = v^*(t) - r_0$$

are the minimal and maximal solutions of (4.2), respectively.

Besides, of being more accurate estimators, the minimal solutions constructed above are bounded, whereas the maximal ones are unbounded. Thus the boundedness of a minimal solution (cf. Corollary 4.1, Theorem 4.2) is really weaker condition than the boundedness of the corresponding maximal solution (cf. LADAS-LAKSHMIKANTHAM [2], Theorem 5.6.1). Moreover, the above example shows that (2.5) may have many solutions even when $g(t_0, r_0) > 0$ and $g(t, r)$ is nondecreasing in r (cf. WOUK [5], p. 6).

REMARK. — By Peano's existence theorem the hypothesis that (2.5) has a solution on I for all $r_0 > 0$ can be excluded in Theorems 3.1 and 5.1 Similarly, the existence of the solution of (4.2) and the divergence of the integral $\int_1^\infty \frac{dr}{\psi(r)}$ need not be assumed in Theorems 4.1 and 6.1, respectively. These hypoteses are needed to obtain the global results of Corollaries 3.1 and Theorems 4.2, 5.2 and 6.2.

ACKNOWLEDGEMENTS. — I am deeply indebted to Mr. S. Seikkala for his cooperation during the preparation of this work. I also wish to express my gratitude to Professor V. Lakshmikantham, Chairman of the Department of Mathematics in the University of Arlington, for his encouragement and comments.

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