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Projectively Related Lorentzian and Riemannian Metrics (*)

Summary: An investigation of the circumstances under which a Riemannian metric and a normal-hyperbolic metric on a four-dimensional differentiable manifold can have the same geodesic paths is carried out in this paper with a view toward applications in general relativity.

I. Introduction

Two metrics on a differentiable manifold are said to be projectively related provided the two metrics have the same geodesic paths. The problem of a normal hyperbolic metric and a Riemannian metric projectively related on a four-dimensional differentiable manifold is considered in this paper. There are three separate cases to be considered, depending on the algebric classification of the normal hyperbolic metric with respect to the Riemannian metric at a point of the differentiable manifold.

Section II gives the general formulation of the problem and Section III gives the formulation with respect to an adapted basis in the tangent space. Section IV gives the solution in the three distinct cases that occur and Section V a discussion of the applications is given.

Given a four-dimensional differentiable manifold M on which is defined a C^r ($r \geq 2$) normal hyperbolic metric $\tilde{g}$ and a positive definite (Riemannian) metric g, a vector field X on M is geodesic with respect to g if

$$D_x X = \lambda X, \tag{2.1}$$

where D_x is the covariant derivative in the direction of the vector field X with respect to g and λ is a real valued function. Also X is a geodesic vector field with respect to $\tilde{g}$ if

$$\tilde{D}_x X = \tilde{\lambda} X, \tag{2.2}$$

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where $\widetilde{D}_X$ is the covariant derivative in the direction of the vector field X with respect to $\widetilde{g}$ and $\widetilde{\lambda}$ is a real-valued function. If, in an open subset U of M, a basis for the tangent vectors $\{X_a\}$ is chosen, then the vector field X may be written as $X = \xi^a X_a$, where the ξ^a are C^{r-1} functions on U. Then equation (2.1) may be written as

$$\xi^a_{;b}\, \xi^b = \xi^a_{,b}\, \xi^b + \omega^a_{bc}\, \xi^b \xi^c = \lambda\, \xi^a \tag{2.3}$$

and equation (2.2) as

$$\xi^a_{\widetilde{;}b}\, \xi^b = \xi^a_{,b}\, \xi^b + \widetilde{\omega}^a_{bc}\, \xi^b \xi^c = \widetilde{\lambda}\, \xi^a \tag{2.4}$$

where ; denotes covariant differentation with respect to g, where $\widetilde{;}$ denotes covariant differentiation with respect to $\widetilde{g}$, ω^a_{bc} and $\widetilde{\omega}^a_{bc}$ are the components of the connection forms of g and $\widetilde{g}$ respectively with respect to the basis $\{X_a\}$ and the dual basis $\{\theta^a\}$. The requirement that X be geodesic with respect to both $\widetilde{g}$ and g may be written as

$$(\widetilde{\omega}^a_{bc} - \omega^a_{bc})\, \xi^b \xi^c = (\widetilde{\lambda} - \lambda)\, \xi^a. \tag{2.5}$$

If g and $\widetilde{g}$ have the same geodesic paths, the above equation must hold at every point of U and for all tangent vectors at any point of U since the assignment of a tangent vector at a point generates a geodesic through that point with tangent, vector, the assigned tangent vector. Thus we must have

$$(\widetilde{\omega}^{[a}_{bc} - \omega^{[a}_{bc})\, \xi^{d]} \xi^b \xi^c = 0\,, \tag{2.6}$$

where [ ] denotes skew-symmetrization. If

$$\mu^a_{bc} = (\widetilde{\omega}^a_{bc} + \widetilde{\omega}^a_{cb}) - (\omega^a_{bc} + \omega^a_{cb}),$$

then (2.6) may be written as

$$(\delta^d_e\, \mu^a_{bc} - \delta^a_e\, \mu^d_{bc})\, \xi^e \xi^b \xi^c = 0$$

which says that the totally symmetric part on the indices ebc of the above expression vanishes. Thus

$$\delta^d_e\, \mu^a_{bc} + \delta^d_c\, \mu^a_{eb} + \delta^d_b\, \mu^a_{ce} = \delta^a_e\, \mu^b_{dc} + \delta^a_c\, \mu^d_{eb} + \delta^a_b\, \mu^d_{ce}\,.$$

Contracting on d and e,

$$5\, \mu^a_{bc} = \delta^a_c\, \mu^d_{bd} + \delta^a_b\, \mu^d_{cd}\,.$$

The condition that g and $\tilde{g}$ have the same geodesic paths is that

$$\tilde{\omega}^{a}_{bc} - \omega^{a}_{bc} = \delta^{a}_{b}\,\psi_{c} + \delta^{a}_{c}\,\psi_{b} \tag{2.7}$$

where

$$\psi_{b} = \frac{1}{5}\,(\tilde{\omega}^{a}_{ba} - \omega^{a}_{ba}).$$

III. On U a basis for the one-forms $\{\theta^{a}\}$ can be found such that

$$g = \sum_{a=1}^{4} \theta^{a} \otimes \theta^{a}, \tag{3.1}$$

$$\tilde{g} = \sum_{a=1}^{4} \rho(a)\,\theta^{a} \otimes \theta^{a}, \tag{3.2}$$

where the $\rho(a)$ are four real-valued function on U which are nonvanishing. If $\tilde{g}$ is to have signature $+2$, θ^{1} can be chosen such that $\rho(1) < 0$ and $\rho(\alpha) > 0$, $\alpha = 2, 3, 4$.

In a coordinate neighborhood, $\theta^{a} = h^{a}_{\alpha}\,d\,x^{\alpha}$ and

$$C^{a}_{bc} \equiv \tfrac{1}{2}(\omega^{a}_{bc} - \omega^{a}_{cb}) = \tfrac{1}{2}(h^{a}_{\alpha,\beta} - h^{a}_{\beta,\alpha})\,h^{\alpha}_{c}\,h^{\beta}_{b},$$

where h^{α}_{a} are components of the inverse matrix of h^{a}_{α}. Now

$$\omega_{abc} = g_{ae}\,C^{e}_{bc} + g_{ce}\,C^{e}_{ba} + g_{be}\,C^{e}_{ca}, \quad \text{and}$$

$$\tilde{\omega}_{abc} = \tilde{g}_{ae}\,C^{e}_{bc} + \tilde{g}_{ce}\,C^{e}_{ba} + \tilde{g}_{be}\,C^{e}_{ca} - \tfrac{1}{2}(\tilde{g}_{ab/c} + \tilde{g}_{ac/b} - \tilde{g}_{bc/a}),$$

where g_{ab} and $\tilde{g}_{ab}$ are the expressions for g and $\tilde{g}$ respectively in the basis $\{\theta^{a}\}$. Also if $\psi = -\tfrac{1}{2}\,n\,\mu$ where $\mu = C\left(\frac{G}{\tilde{G}}\right)^{1/3}$, C costant and G and $\tilde{G}$ the determinants of g_{ab} and $\tilde{g}_{ab}$ respectively than $\psi_{c} = \psi_{/c}$.

The equations expressing the fact that g and $\tilde{g}$ have the same geodesic paths can be derived from $\tilde{g}_{ab\tilde{;}c} = 0$ and $g_{ab;c} = 0$. We may write $g_{ab} = \delta_{ab}$ and $\tilde{g}_{ab} = \rho(a)\,\delta_{ab}$ for the above basis. Then, if $f_{/a} = f_{,\alpha}\,h^{\alpha}_{a}$,

$$\tilde{g}_{ab\tilde{;}c} = \tilde{g}_{ab/c} - \tilde{\omega}^{d}_{bc}\,\tilde{g}_{ad} - \tilde{\omega}^{d}_{ac}\,\tilde{g}_{db}$$

$$= \rho(a)_{/c}\,\delta_{ab} - \omega^{d}_{bc}\,\delta_{ad}\,\rho(a) - \omega^{d}_{ac}\,\delta_{db}\,\rho(b) -$$

$$(\delta^{d}_{b}\,\psi_{c} + \delta^{d}_{c}\,\psi_{b})\,\delta_{ad}\,\rho(a) - (\delta^{d}_{a}\,\psi_{c} + \delta^{d}_{c}\,\psi_{a})\,\delta_{db}\,\rho(b) = 0 \tag{3.3}$$

using the expression for g_{ab} and $\widetilde{g}_{ab}$ in this basis and (2.7). If (3.3) is examined in the various instance of the equality of the indices a, b, c, we have

$$(\rho(b) - \rho(c))\, \omega_{abc} = 0 \quad a, b, c \neq , \tag{3.4}$$

$$2\mu(\rho(a) - \rho(b))\, \omega_{abb} = \rho(b)\, \mu_{/a} \quad a \neq b = c, \tag{3.5}$$

$$(\mu \rho(b))_{/c} = 0 \quad a = b \neq c, \tag{3.6}$$

$$(\mu^2 \rho(a))_{/a} = 0 \quad a = b = c. \tag{3.7}$$

IV. There are three distinct cases to examine for the solution of the equations (3.4) - (3.7).

Case I : $\rho(2) = \rho(3) = \rho(4) = \rho$.

In this case, equations (3.4) and (3.5) give conditions only in the case in which one of the indices is 1. If Greek indices are used for the indices 2, 3, 4, we obtain,

$$\omega_{1\alpha\beta} = 0 \qquad \alpha \neq \beta,$$

or

$$d\theta^1 \wedge \theta^1 = 0. \tag{4.1}$$

Thus θ^1 is proportional to a gradient, say x^1, so

$$\theta^1 = e^{\sigma}\, d x^1.$$

Then (3.6) gives

$$\mu_{/\alpha} = 0 \text{ or } \mu = \mu(x^1).$$

Also (3.6) and (3.7) yield

$$\rho(b)_{/\alpha} = 0 \text{ or } \rho(b) = \rho(b)(x^1);$$

and

$$\rho(\alpha) = \frac{c}{\mu},\ \rho(1) = \frac{d}{\mu^2}, \tag{4.2}$$

where c and d are constants. Equation (3.5) yields

$$2\mu(\rho(\alpha) - \rho(1))\, \omega_{\alpha 11} = \rho(1)\, \mu_{/\alpha} = 0 \quad \text{or} \quad \omega_{\alpha 11} = 0 \quad \text{or}$$

$$d\theta^1 = 0, \quad \text{ie} \quad \theta^1 = d x^1 .$$

Again from (3.5)

$$\omega_{1\alpha\alpha} = -\tfrac{1}{2} n (d - c\mu)_{,1} . \tag{4.3}$$

Thus θ^1 must be geodesic, exact, shear-free, and the expansion is a function only of x^1 referred to the metric g. Summarizing,

Theorem : If a space-time admits a time-like covector field which is exact, $\Phi = d\,x^1$, has vanishing shear, and expansion a function only of x^1, then the space time has the same geodesic paths as a Riemmanian metric.

Corollary : If a space-time admits a time-like covariantly-constant vector field, then it has the same geodesic paths as a Riemannian metric.

Corollary : There exist no nontrivial vacum solutions of this type.

Proof : The covector field $\Phi = \Phi_a\,\theta^a$ must satisfy $\Phi_a = f_{/a}$ and we may take $\Phi_a\,\Phi^a = -1$. Also the expansion $\theta\,(f)$ is given by

$$\Phi_{a\,;\,b} = (g_{ab} + \Phi_a\,\Phi_b)\,\theta\,(f)\,.$$

and $\Phi_{a\,;\,[bc]} = R^d_{abc}\,\Phi_d = g_{a[b}\,\Phi_{c}\,(\theta_{,\,t} + \theta)$. If the space-time is vacum then $R^d_b\,\Phi_d = -3\,\Phi_b\,(\theta_{,\,t} + \theta) = 0$ and thus $R^d_{abc}\,\xi_d = 0$. This is inconsistent with the vacum condition since Φ is normal to hypersurfaces and the condition $R^d_{abc}\,\xi_d = 0$ gives the Ricci tensor of the three-dimensional submanifolds to be zero, hance the Riemann tensor of these submanifolds is zero, and then $R^d_{abc}\,\xi_d = 0$ gives the Riemann tensor as vanishing.

Case II. The $\rho\,(a)$ all different.

This case is treated in [1]. Equation (3.4) immediately gives

$$\omega_{abc} = 0, \ a, b, c \neq \,, \tag{4.4}$$

which implies $d\,\theta^a \wedge \theta^a = 0$ or locally $\theta^a = e^{\sigma(a)} d\,x^a$. Then (3.6) yields

$$(\mu\,\rho\,(a))_{,\,a} = 0, \ a \neq b \quad \text{or} \quad \mu\,\rho\,(b) = \frac{1}{\Phi\,(b)}$$

where $\Phi\,(b)$ is a function of x^b alone. Equation (3.7) gives $\frac{\mu}{\Phi\,(b)}$ to be independent of x^b and thus

$$\mu = c\,\Phi\,(1)\,\Phi\,(2)\,\Phi\,(3)\,\Phi\,(4)$$

where c is a constant. Equation (3.5) given $\frac{e^{\Sigma\sigma(a)}}{\pi_{b'}\,(\Phi\,(b) - \Phi\,(a))}$ to be a function of x^a alone where $\pi_{b'}\,(\Phi\,(b) - \Phi\,(a))$ excludes from the product the term with $b \neq a$. Thus coordinates, $x'^a = f\,(x^a)$, may be chosen such that the metric in these holonomic coordinates become

$$g_{ab} = \delta_{ab}\,|\,\pi_{b'}\,(\Phi\,(b) - \Phi\,(a))\,|\,. \tag{4.5}$$

Then

$$\widetilde{g}_{ab} = \delta_{ab} \frac{1}{\mu \Phi(a)} \left| \pi_{b'}, (\Phi(b) - \Phi(a)) \right| . \tag{4.6}$$

Thus locally both g and $\widetilde{g}$ admit simultaneously a set of orthogonal holonomic coordinates.

Case III. Two of the $\rho(\alpha)$ equal, say $\rho(2) = \rho(3) = \rho \neq \rho(4)$

Immediately from equation (3.4) $d\theta^1{}_\wedge \theta^1{}_\wedge = 0$, $d\theta^4{}_\wedge \theta^4 = 0$, $d\theta^2{}_\wedge \theta^2{}_\wedge \theta^3 = 0$, and $d\theta^3{}_\wedge \theta^2{}_\wedge \theta^3 = 0$. Thus $\theta^1 = e^{\sigma(1)}\, d x^1$ and $\theta^4 = e^{\sigma(4)}\, d x^4$. Equation (3.5) yields $\mu = \mu(x^1, x^4)$ and (3.6), (3.7) then yield

$\mu \rho(b) = \dfrac{1}{\Phi(b)}$ where $\Phi(1) = \Phi(1)(x^1), \Phi(4) = \Phi(4)(x^4), \Phi(2) = \Phi(3) = \Phi =$ const

and $\mu = c\,\Phi(1)\,\Phi(4)$ where c is a constant. A holonomic basis can be found such that

$$\begin{aligned} \theta^1 &= e^{\sigma(1)}\, d x^1 \\ \theta^2 &= e^{\mu(2)}\, d x^2 + e^{\mu(3)}\, d x^3 \\ \theta^3 &= e^{\lambda(2)}\, d x^2 + e^{\lambda(3)}\, d x^3 \\ \theta^4 &= e^{\sigma(4)}\, d x^4 . \end{aligned} \tag{4.7}$$

Equation (3.5) yields

$\dfrac{e^{2\sigma(1)}}{\Phi(4) - \Phi(1)}$ a function of x^1 and $\dfrac{e^{2\sigma(4)}}{\Phi(4) - \Phi(1)}$ a function of x^4, and by a change of coordinates we can make

$$e^{2\sigma(1)} = \Phi(4) - \Phi(1),\ e^{2\sigma(4)} = \Phi(4) - \Phi(1) \tag{4.8}$$

Equation (3.5) yields

$$(\Phi - \Phi(1))(\Phi - \Phi(4)) = g(x^2, x^3)\left(e^{\mu(3) + \lambda(2)} - e^{\mu(2) + \lambda(3)}\right),$$

and a change of coordinates will give

$$\theta^2{}_\wedge\, \theta^3 = (\Phi - \Phi(1))(\Phi - (4))\; d x^2{}_\wedge\, d x^3 . \tag{4.9}$$

Thus in these holonomic coordinates,

$$d s^2 = g_{ab}\, d x^a\, d x^b = |\Phi(4) - \Phi(1)|\,(d x^1)^2 + |\Phi(4) - \Phi(1)|\,(d x^4)^2 + (\theta^2)^2 + (\theta^3)^2 . \tag{4.10}$$

The further requirements are that shear of θ^1 and θ^4 projected into $\theta^2{}_\wedge\, \theta^3$ vanish, or that

$$\omega_{123} = \omega_{423} = 0 \,, \quad \omega_{122} = \omega_{133} \,, \quad \omega_{422} = \omega_{433} \tag{4.11}$$

These conditions are also sufficient. Then the line element of $\widetilde{g}$ given by

$$d\, s^2 = \widetilde{g}_{ab}\, d\, x^a\, d\, x^b = \frac{|\Phi(4) - \Phi(1)|}{e\, \Phi(1)^2\, \Phi(4)} (dx^1)^2 + \frac{|\Phi(4) - \Phi(1)|}{e\, \Phi(1)\, \Phi(4)^2} (d\, x^4)^2 +$$

$$\frac{1}{e\, \Theta(1)\, \Phi(4)\, \Phi} \left[(\theta^2)^2 + (\theta^3)^2\right] . \tag{4.12}$$

V. In Section IV, the various cases in which a normal hyperbolic metric and a Riemannian metric have the same geodesic paths were considered. It is seen that the restrictions on the space-time are quite severe, and that a general space-time will not fit into any of three classes. However, the space-time metrics in this category can be completed by using the correspondence and the fact any space on wich there is a Riemannian metric can be completed in the sense of Cauchy completion.

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(1) L. P. Eisenhart, *Riemannian Geometry*, Princeton (1926).