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On a mixed problem for a polyvibrating equation of Mangeron

Riassunto: In questo lavoro si studia un nuovo problema misto per un'equazione polivibrante d'ordine superiore di Mangeron e si perviene tra l'altro ad un'estensione delle equazioni integrali di Picone stabilite nelle Sue classiche Memorie del 1911.

1. In the present paper we deal with the solution of the polyvibrating equation of D. Mangeron

$$\frac{\partial^{2n} u}{\partial x^n \partial y^n} = \lambda c(x, y) u + f(x, y) \tag{1}$$

subject to the conditions

$$\begin{aligned} &\left.\frac{\partial^{2k} u}{\partial x^k \partial y^k}\right|_{y=0} = P_k(x) \;, \quad \left.\frac{\partial^{2k} u}{\partial x^k \partial y^k}\right|_{x=0} = Q_k(y) \quad (k = 0,1,\dots, n-2) \\ &\left.\frac{\partial^{2n-2} u}{\partial x^{n-1} \partial y^{n-1}}\right|_{y=0} = P_{n-1}(x) \;, \quad \left.\frac{\partial^{2n-2} u}{\partial x^{n-1} \partial y^{n-1}}\right|_{x=\alpha} = Q_{n-1}(y) \end{aligned} \tag{2}$$

where $n \geqslant 2$, α is a constant, λ is a parameter, $c(x, y)$, $f(x, y)$, $P_k(x)$ and $Q_k(y)$ are certain functions analytic for all finite values of their arguments. We assume that $P_k(0) = Q_k(0)$ for $k = 0,1,\dots, n-2$ and $P_{n-1}(\alpha) = Q_{n-1}(0)$. We wish to find a function $u = u(x, y)$ of class C^{2n-1} in the xy-plane with a continuous derivative $\frac{\partial^{2n} u}{\partial x^n \partial y^n}$, and satisfying the equation (1) and the conditions (2).

Clearly, the above formulated mixed problem becomes a Goursat problem for $\alpha = 0$. In the next section we shall establish the solution of (1) - (2).

(*) Paper introduced by the Academician MAURO PICONE.

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2. We begin with the integration of Eq. (1). We then obtain the relationships

$$\frac{\partial^{2n-2} u}{\partial x^{n-1} \partial y^{n-1}} = \lambda \int_0^x \int_0^y c(\xi, \eta) \, u(\xi, \eta) \, d\xi \, d\eta + \int_0^x \int_0^y f(\xi, \eta) \, d\xi \, d\eta + [o_{n-1}(x) + \psi_{n-1}(y)] \, ,$$

(3)

$$\frac{\partial^{2n-2m} u}{\partial u^{n-m} \partial y^{n-m}} = \lambda \int_0^x \int_0^y \frac{(x-\xi)^{m-1}}{(m-1)!} \frac{(y-\eta)^{m-1}}{(m-1)!} c(\xi, \eta) \, u(\xi, \eta) \, d\xi \, d\eta +$$

$$+ \int_0^x \int_0^y \frac{(x-\xi)^{m-1}}{(m-1)!} \frac{(y-\eta)^{m-1}}{(m-1)!} f(\xi, \eta) \, d\xi \, d\eta$$

$$+ \sum_{i=1}^{m-1} \int_0^x \int_0^y \frac{(x-\xi)^{m-i-1}}{(m-i-1)!} \frac{(y-\eta)^{m-i-1}}{(m-i-1)!} [o_{n-i}(\xi) + \psi_{n-i}(\eta)] \, d\xi \, d\eta$$

$$+ [o_{n-m}(x) + \psi_{n-m}(y)] \, ,$$

$$u(x, y) = \lambda \int_0^x \int_0^y \frac{(x-\xi^n)^{-1}}{(n-1)!} \frac{(y-\eta)^{n-1}}{(n-1)!} c(\xi, \eta) \, u(\xi, \eta) \, d\xi \, d\eta +$$

$$+ \int_0^x \int_0^y \frac{(x-\xi)^{n-1}}{(n-1)!} \frac{(y-\eta)^{n-1}}{(n-1)!} f(\xi, \eta) \, d\xi \, d\eta$$

$$+ \sum_{i=1}^{n-1} \int_0^x \int_0^y \frac{(x-\xi)^{n-i-1}}{(n-i-1)!} \frac{(y-\eta)^{n-i-1}}{(n-i-1)!} [o_{n-i}(\xi) + \psi_{n-i}(\eta)] \, d\xi \, d\eta$$

$$+ [o(x) + \psi_0(y)] \, ,$$

where $o_k(x)$ and $\psi_k(y)$ are certain sufficiently smooth functions which are to be determined.

By virtue of the equations (3) and the conditions (2), we have

(4) $\quad o_k(0) + \psi_k(y) = Q_k(y) \, , \; o_k(x) + \psi_k(0) = P_k(x) \quad (k = 0, 1, \ldots, n-2)$

and

(5)

$$o_{n-1}(x) + \psi_{n-1}(0) = P_{n-1}(x)$$

$$\psi_{n-1}(\alpha) + \psi_{n-1}(y) = Q_{n-1}(y) - \lambda \int_0^x \int_0^y c(\xi, \eta) \, u(\xi, \eta) \, d\xi \, d\eta - \int_0^x \int_0^y f(\xi, \eta) \, d\xi \, d\eta .$$

Hence

(6) $\quad o_k(x) + \psi_k(y) = P_k(x) + Q_k(y) - P_k(0) \qquad (k = 0, 1, \ldots, n-2)$

and

(7)

$$o_{n-1}(x) + \psi_{n-1}(y) = P_{n-1}(x) + Q_{n-1}(y) - P_{n-1}(0) - \int_0^x \int_0^y f(\xi, \eta) \, d\xi \, d\eta$$

$$- \lambda \int_0^x \int_0^y c(\xi, \eta) \, u(\xi, \eta) \, d\xi \, d\eta \quad .$$

Thus, combining Eqs. (6) and (7) with the last of the equations in (3), we obtain the following Volterra type integral equation

$$u(x, y) = (A u)(x, y) + F(x, y) , \tag{8}$$

where

$$\begin{aligned}(A u)(x, y) = &\int_0^x \int_0^y \frac{(x-\xi)^{n-1}}{(n-1)!} \frac{(y-\eta)^{n-1}}{(n-1)!} c(\xi, \eta)\, u(\xi, \eta)\, d\xi\, d\eta \\ &- \int_0^x \int_0^y \frac{(x-\xi)^{n-2}}{(n-2)!} \frac{(y-\eta)^{n-2}}{(n-2)!}\, d\xi\, d\eta \int_0^{\xi} \int_0^{\eta} c(\sigma, \tau)\, u(\sigma, \tau)\, d\sigma\, d\tau\end{aligned} \tag{9}$$

and

$$\begin{aligned}F(x,y) = &\int_0^x \int_0^y \frac{(x-\xi)^{n-1}}{(n-1)!} \frac{(y-\eta)^{n-1}}{(n-1)!} f(\xi, \eta)\, d\xi\, d\eta \\ &- \int_0^x \int_0^y \frac{(x-\xi)^{n-2}}{(n-2)!} \frac{(y-\eta)^{n-2}}{(n-2)!}\, d\xi\, d\eta \int_0^{\xi} \int_0^{\eta} f(\sigma, \tau)\, d\sigma\, d\tau \\ &+ \sum_{i=1}^{n-1} \int_0^x \int_0^y \frac{(x-\xi)^{n-i-1}}{(n-i-1)!} \frac{(y-\eta)^{n-i-1}}{(n-i-1)!} [P_{n-i}(\xi) + Q_{n-i}(\eta) - P_{n-i}(0)]\, d\xi\, d\eta \\ &+ [P_0(x) + Q_0(y) - P_0(0)] \quad .\end{aligned} \tag{10}$$

Obviously, the function $F(x, y)$ is analytic for all finite values of x and y, and $(A u)(x, y)$ is of class C^{2n} for each given continuous $u(x, y)$.

We can show without any difficulty, by the method of successive approximations, that the functional equation (8) admits a unique solution $U(x, y, \lambda)$ of class C^{2n} for all finite values of x and y, and for each λ. Note that $U(x, y ; \lambda)$ is analytic in λ. We can easily verify that $U(x, y ; \lambda)$ is the unique solution of the mixed problem (1) - (2).

Clearly, Eq. (8) is a Volterra integral equation of the second kind for $\alpha = 0$. For $\alpha \neq 0$, Eq. (8) can be considered as an extension of M. Picone's integral equations given in his comprehensive Memoires [2 b, 2 c] of 1910 and 1911.

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